DELAYED LOGISTIC EQUATION

WE BEGIN WITH

$$(*) \quad y'(t) = K\, y(t)\,[1 - y(t-\tau)]. \qquad K > 0.$$

IF THERE IS NO DELAY THEN $y=0$ IS unstable, while $y=1$ is stable.

NOW ASSUME THAT $\tau > 0$ AND CHANGE VARIABLES: $t \mapsto t\tau$.

THEN WE CAN WRITE $(*)$ EQUIVALENTLY AS

$$y'(t) = r\, y(t)\,[1 - y(t-1)] \qquad r = K\tau.$$

NOW LINEARIZE AROUND $y=1$ EQUILIBRIUM. WE PUT $y = 1 + u$ WITH $u \ll 1$

TO OBTAIN $$u' = r(1+u) - r[1+u][1+u(t-1)] = r + ru - r - ru - ru(t-1)$$ + quadratic terms.

THUS $u'(t) = -r\,u(t-1)$. IS LINEARIZED PROBLEM

WE PUT $u = e^{\lambda t}\phi$ TO OBTAIN

$$e^{\lambda t}\lambda\phi = -r\,e^{\lambda(t-1)}\phi \longrightarrow \lambda + r e^{-\lambda} = 0.$$

NOW WITH $\lambda = i\lambda_I$ AND $\lambda_I > 0 \longrightarrow i\lambda_I + r[\cos(\lambda_I) - i\sin(\lambda_I)] = 0.$

THUS $r\cos\lambda_I = 0$, $\lambda_I = r\sin(\lambda_I)$.

WE CONCLUDE THAT $\lambda_I = \pi/2$ IS FIRST CROSSING $\longrightarrow r = \pi/2 = K\tau$.

THUS THE SMALLEST VALUE OF τ FOR A HOPF BIFURCATION

IS AT $\tau_H = \pi/2K$ WITH $\lambda_I = \pi/2$.

WE HAVE THE PERIODIC SOLUTION $u = Ae^{i\pi/2\, t} = A[\cos(\pi t/2) + i\sin(\pi t/2)]$.

HENCE DELAY INDUCES OSCILLATIONS ABOUT $y=1$ WHEN $\tau = \tau_H = \pi/2K$.

TO DERIVE THE LIMITING FORM OF BIFURCATION BRANCH NEAR

THE HB POINT WE PUT

$$u(t) = y(t) - 1.$$

WE OBTAIN THAT $u'(t) = r\,[u(t)+1]\,[-u(t-1)] = -r\,u(t-1) - r\,u(t)\,u(t-1)$.

$$u'(t) = -r\,u(t-1) - r\,u(t)\,u(t-1).$$

NOW NEAR THE HB POINT WE PUT

$$r = \frac{\pi}{2} + \varepsilon^2 r_1$$

AND $\quad u = \varepsilon U.$

THIS IS CLASSIC SCALING FOR A HB. WE OBTAIN THAT

$$U'(t) = -\left(\pi/_2 + \varepsilon^2 r_1\right) U(t-1) - \varepsilon\left(\frac{\pi}{2} + \varepsilon^2 r_1\right) U(t)\,U(t-1).$$

FROM THIS EQUATION WE SEE THAT WE MUST EXPAND $U = U_0 + \varepsilon U_1 + \varepsilon^2 U_2 + \cdots$

HOWEVER, AS IN POINCARÉ-LINDSTEDT METHOD WE REQUIRE A FREQUENCY SHIFT WE PUT

$$V(s) = U\left(s/_\omega\right) \qquad \text{WITH} \quad s = \omega t, \qquad \omega = \pi/_2 + \varepsilon^2 \omega_1 + \cdots$$

THUS $\quad V(s-\omega) = U\left((s-\omega)/_\omega\right) = U\left(s/_\omega - 1\right) = U(t-1).$

THUS,

$$\omega V'(s) = -\left[\pi/_2 + \varepsilon^2 r_1\right] V(s-\omega) - \varepsilon\left[\frac{\pi}{2} + \varepsilon^2 r_1\right] V(s)\,V(s-\omega).$$

NOW WE EXPAND

$$V(s) = V_0(s) + \varepsilon V_1(s) + \varepsilon^2 V_2(s) + \cdots$$

$$\omega = \pi/_2 + \varepsilon^2 \omega_1 + \cdots \qquad \text{WITH} \quad \omega_0 = \pi/_2$$

THIS YIELDS THAT

$$\left(\pi/_2 + \varepsilon^2 \omega_1\right)\left[V_0' + \varepsilon V_1' + \varepsilon^2 V_2'\right] = -\left(\frac{\pi}{2} + \varepsilon^2 r_1\right)\left[V_0(s-\omega) + \varepsilon V_1(s-\omega) + \varepsilon^2 V_2(s-\omega)\right]$$
$$- \varepsilon\left(\frac{\pi}{2} + \varepsilon^2 r_1\right)\left[V_0 + \varepsilon V_1\right]\left[V_0(s-\omega) + \varepsilon V_1(s-\omega)\right].$$

BUT $\bar{Y}_0(s-w) = \bar{Y}_0(s - w_0 - \varepsilon^2 w_1) = \bar{Y}_0(s-w_0) - \varepsilon^2 w_1 \bar{Y}_0'(s-w_0) + \dots$

THEN UP TO $O(\varepsilon^2)$ TERMS:

$$\left(\pi/2 + \varepsilon^2 w_1\right)\left[\bar{Y}_0' + \varepsilon \bar{Y}_1' + \varepsilon^2 Y_2'\right] = -\left(\pi/2 + \varepsilon^2 \Gamma_1\right)\left[\bar{Y}_0(s-w_0) + \varepsilon \bar{Y}_1(s-w_0) + \varepsilon^2\left[\bar{Y}_2(s-w_0) - w_1 \bar{Y}_0'(s-w_0)\right]\right]$$
$$- \varepsilon\left(\pi/2 + \varepsilon^2\Gamma_1\right)\left(\bar{Y}_0 + \varepsilon \bar{Y}_1\right)\left[Y_0(s-w_0) + \varepsilon Y_1(s-w_0)\right]$$

$$\bar{Y}_0' \frac{\pi}{2} + \varepsilon\left[\frac{\pi}{2}\bar{Y}_1'\right] + \varepsilon^2\left[\frac{\pi}{2}\bar{Y}_2' + w_1 \bar{Y}_0'\right] = -\frac{\pi}{2}\bar{Y}_0(s-w_0) + \varepsilon\left[-\frac{\pi}{2}\bar{Y}_1(s-w_0) - \frac{\pi}{2}\bar{Y}_0\, Y_0(s-w_0)\right]$$
$$+ \varepsilon^2\left[-\frac{\pi}{2}\bar{Y}_2(s-w_0) + \frac{w_1 \pi}{2}\bar{Y}_0'(s-w_0) - \Gamma_1 \bar{Y}_0(s-w_0)\right.$$
$$\left. - \frac{\pi}{2}\bar{Y}_0 \bar{Y}_1(s-w_0) - \frac{\pi}{2} Y_1 Y_0(s-w_0)\right)$$

THIS YIELDS THE FOLLOWING PROBLEMS

$O(1)$ $\quad \bar{Y}_0'(s) = -\bar{Y}_0(s-w_0) \qquad w_0 = \pi/2.$

$O(\varepsilon)$ $\quad \bar{Y}_1'(s) + \bar{Y}_1(s-w_0) = -\bar{Y}_0(s)\bar{Y}_0(s-w_0)$

$O(\varepsilon^2)$ $\quad \bar{Y}_2'(s) + \bar{Y}_2(s-w_0) = -\dfrac{w_1}{w_0}\bar{Y}_0'(s) + w_1 \bar{Y}_0'(s-w_0) - \dfrac{\Gamma_1}{w_0} Y_0(s-w_0)$
$$-\bar{Y}_0(s)\bar{Y}_1(s-w_0) - \bar{Y}_1(s)\bar{Y}_0(s-w_0)$$

NOW THE IDEA IS TO CHOOSE w_1 SO THAT WE OBTAIN A PERIODIC SOLUTION.

<u>METHOD I</u> WE SOLVE FOR $\bar{Y}_0$ AS

$\bar{Y}_0 = M \cos s$

NOW $\bar{Y}_0(s-w_0) = M\cos(s-\pi/2) = M \sin s.$

THIS YIELDS $\quad \bar{Y}_1'(s) + \bar{Y}_1(s-\pi/2) = -M^2 \cos s \sin s = -\dfrac{M^2}{2}\sin(2s)$

NOW WRITE

$$\hat{\tilde{Y}}_1'(s) + \hat{Y}_1(s - \pi/2) = -\frac{M^2}{2} e^{2is}$$

WE LET $Y_1 = \mathrm{IM}[\tilde{\hat{Y}}_1]$ AND PUT $\hat{Y}_1 = e^{2is} B$.

WE DERIVE $2iB - B = -M^2/2$.

THUS $$B = \frac{M^2}{2(1-2i)}$$

$$Y_1 = \mathrm{IM}[\tilde{\hat{Y}}_1] = \frac{M^2}{2}\,\mathrm{IM}\left[\frac{1}{1-2i} e^{2is}\right] = \frac{M^2}{2}\left[\mathrm{IM}\left(\left(\frac{1+2i}{5}\right)(\cos 2s + i \sin 2s)\right)\right]$$

$$Y_1 = \frac{M^2}{10}\left[2\cos(2s) + \sin(2s)\right].$$

NOW WE MUST LOOK AT $O(\varepsilon^2)$ EQUATION

$$Y_2'(s) + Y_2(s - w_0) = -\frac{w_1}{w_0} Y_0'(s) + w_1 Y_0'(s - w_0) - \frac{\Gamma_1}{w_0} Y_0(s - w_0)$$
$$- Y_0(s) Y_1(s - w_0) - Y_1(s) Y_0(s - w_0).$$

WE CALCULATE:

$$Y_0(s) = M\cos s, \quad Y_0'(s) = -M\sin s, \quad Y_0(s - w_0) = M\sin s, \quad Y_0'(s - w_0) = M\cos s.$$

NOW

$$Y_0(s) Y_1(s - w_0) = M\cos s\left[\frac{M^2}{10}\right]\left[2\cos(2s - \pi) + \sin(2s - \pi)\right] = \frac{M^3}{10}\left[-2\cos s \cos 2s - \sin 2s \cos s\right]$$
$$= -\frac{M^3}{10}\left[2\cos s\cos 2s + \sin 2s \cos s\right] = -\frac{M^3}{10}\left[\cos s + \cos 3s + \frac{1}{2}\sin s + \frac{1}{2}\sin 3s\right]$$

HERE WE USED

$$\cos A \cos B = \frac{1}{2}\left[\cos(A-B) + \cos(A+B)\right], \quad \sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right],$$
$$\sin A \cos B = \frac{1}{2}\left[\sin(A-B) + \sin(A+B)\right].$$

IN ADDITION $$Y_1(s) Y_0(s - w_0) = \frac{M^3}{10}\left[2\cos(2s)\sin s + \sin 2s \sin s\right]$$
$$= \frac{M^3}{10}\left[-\sin s + \sin 3s + \frac{1}{2}\cos s - \frac{1}{2}\cos 3s\right]$$

THE EQUATION FOR $\bar{Y}_2$ IS

$$\bar{Y}_2'(s) + \bar{Y}_2(s \cdot w_0) = \frac{M w_1}{w_0} \sin S + M w_1 \cos S - \frac{\Gamma_1 M}{w_0} \sin S$$
$$+ \frac{M^3}{10}\left[\cos S + \cos 3S + \frac{1}{2} \sin S + \frac{1}{2} \sin 3S \right]$$
$$+ \frac{M^3}{10}\left[\sin S - \sin 3S - \frac{1}{2} \cos S + \frac{1}{2} \cos 3S \right]$$
$$= \sin S \left(\frac{M w_1}{w_0} - \frac{\Gamma_1 M}{w_0} + \frac{M^3}{20} + \frac{M^3}{10} \right) + \cos S \left(M w_1 + \frac{M^3}{10} - \frac{M^3}{20} \right)$$

+ higher harmonics

THUS $\quad M w_1 = - M^3/20 \quad$ ELIMINATES $\cos S$ TERM

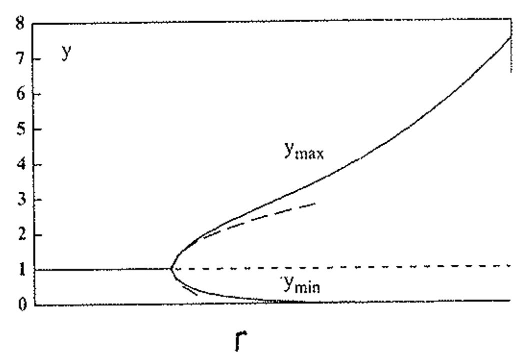
$$\frac{3 M^3}{20} + \frac{M w_1}{w_0} = \frac{\Gamma_1 M}{w_0} \qquad \Gamma_1 = w_1 + \frac{3}{20} w_0 M^2.$$

THIS YIELDS THAT $\quad w_1 = - M^2/20.$

$$\Gamma_1 = M^2 \left(-\frac{1}{20} + \frac{w_0 3}{20} \right)$$

WITH $w_0 = \pi/2$ WE OBTAIN

$$w_1 = - M^2/20$$
$$\Gamma_1 = + M^2 \left(\frac{3\pi}{40} - \frac{1}{20} \right)$$



NOW WE OBTAIN THAT FOR

$$\Gamma = \frac{\pi}{2} + \varepsilon^2 \left(\frac{3\pi}{40} - \frac{1}{20} \right) M^2$$

WE HAVE THAT $\quad y = 1 + \varepsilon M \cos(w t) \qquad w = \pi/2 - M^2/20 \; \varepsilon^2$

WLOG SET $\varepsilon M = \hat{\varepsilon}$.

THUS $\quad y \sim 1 + \hat{\varepsilon}\cos(wt) \qquad w = \pi/2 - \hat{\varepsilon}^2/20$

WITH $\quad \tilde{\varepsilon} = \sqrt{\left(\Gamma - \pi/2\right)\dfrac{40}{3\pi - 2}}$ IS THE LOCAL DESCRIPTION OF BIFURCATION BRANCH.

<u>METHOD 2</u> WE RETURN TO THE EQUATIONS FOR $\bar{y}_0, \bar{y}_1, \bar{y}_2$.

WE WRITE

$$\bar{y}_0 = e^{is} + \text{c.c.}$$

THEN $\quad \bar{y}_0(s - w_0) = e^{i(s-\pi/2)} = -ie^{is} + \text{c.c.}$

THE EQUATION FOR $\bar{y}_1$ IS

$$\bar{y}_1'(s) + \bar{y}_1(s-w_0) = -\bar{y}_0(s)\bar{y}_0(s-w_0) = -(e^{is} + e^{-is})(-ie^{is} + ie^{-is})$$

$$\bar{y}_1'(s) + \bar{y}_1(s-w_0) = -(-ie^{2is} + ie^{-2is}) = +ie^{2is} + \text{c.c.}$$

NOW PUT $\quad \bar{y}_1 = Be^{2is} \rightarrow \quad 2iB - B = i \quad \rightarrow \quad B = \dfrac{i}{-1+2i} = \dfrac{-i}{1-2i}$

HENCE $\quad \bar{y}_1(s) = \dfrac{-i}{1-2i}e^{2is} + \text{c.c.} \qquad$ c.c. = complex conjugate.

NOW $\quad \bar{y}_0'(s) = ie^{is} + \text{c.c.} \qquad \bar{y}_0'(s-w_0) = e^{is} + \text{c.c.} \qquad \bar{y}_0(s-w_0) = -ie^{is} + \text{c.c.}$

$$\bar{y}_0(s)\bar{y}_1(s-w_0) = \left[e^{is} + e^{-is}\right]\left[\frac{+i}{1-2i}e^{2is} - \frac{i}{1+2i}e^{-2is}\right]$$

$$= \frac{i}{1-2i}e^{3is} + \frac{i}{1-2i}e^{is} + \text{c.c.}$$

$$\bar{y}_1(s)\bar{y}_0(s-w_0) = \left(\frac{-i}{1-2i}e^{2is} + \frac{i}{1+2i}e^{-2is}\right)\left(-ie^{is} + ie^{-is}\right) = \frac{-1}{1-2i}e^{3is} + \frac{1}{1-2i}e^{is} + \text{c.c.}$$

NOW SUBSTITUTE INTO THE Y_2 EQUATION:

$$Y_2'(s) + Y_2(s\cdot w_0) = -\frac{w_1}{w_0}\left(ie^{is} + c.c.\right) + w_1\left(e^{is} + c.c.\right) - \frac{\Gamma_1}{w_0}\left(-ie^{is} + c.c.\right)$$
$$-\frac{i}{1-2i}e^{3is} - \frac{i}{1-2i}e^{is} + c.c. + \frac{1}{1-2i}e^{3is} - \frac{1}{1-2i}e^{is} + c.c.$$

THIS YIELDS THAT

$$Y_2'(s) + Y_2(s\cdot w_0) = \left[e^{is}\left[-\frac{iw_1}{w_0} + w_1 + \frac{\Gamma_1 i}{w_0} - \frac{i}{1-2i} - \frac{1}{1-2i}\right) + c.c.\right] - \frac{i}{1-2i}e^{3is} + c.c.$$
$$+ \frac{1}{1-2i}e^{3is} + c.c.$$

THUS TO ELIMINATE RESONANCE,

$$-\frac{iw_1}{w_0} + w_1 + \frac{\Gamma_1 i}{w_0} - \frac{i(1+2i)}{5} - \frac{(1+2i)}{5} = 0.$$

FROM REAL AND IMAGINARY PARTS

Real $\quad w_1 + \frac{2}{5} - \frac{1}{5} = 0 \longrightarrow w_1 = -1/5.$

Imaginary $\quad -\frac{w_1}{w_0} + \frac{\Gamma_1}{w_0} - \frac{1}{5} - \frac{2}{5} = 0 \longrightarrow \frac{\Gamma_1}{w_0} = \frac{3}{5} + \frac{w_1}{w_0}.$

THUS $\quad w_1 = -1/5, \qquad \Gamma_1 = \frac{\pi}{2}\frac{3}{5} - \frac{1}{5} = \frac{3\pi}{10} - \frac{1}{5}.$

WE CONCLUDE THAT

$$y = 1 + \varepsilon\left(e^{iwt} + c.c.\right) = 1 + 2\varepsilon\sin(wt) \qquad w = \pi/2 - \varepsilon^2/5 + \cdots$$

WITH $\quad \lambda \sim \frac{\pi}{2} + \varepsilon^2\left(3\pi - 2\right)/10 \quad$ THIS IS EQUIVALENT TO RESULT EARLIER.

NEXT WE CONSIDER THE DELAY EQUATION

$$x'(t) = -x(t-T) - x^3(t)$$

WE FIRST NOTE THAT FOR LINEARIZED PROBLEM NEAR $x=0$, $x'(t) = -x(t-T)$ THAT A PERIODIC SOLUTION EXISTS WHEN $T = \pi/2$, WITH FREQUENCY $\omega_0 = 1$.

WE LOOK AT A NEIGHBORHOOD

$$T = \pi/2 + \varepsilon\mu \quad \text{AND} \quad x(t) = \sqrt{\varepsilon}\, u(t) \quad \text{WITH} \quad \varepsilon > 0.$$

THEN

$$x'(t) = -x(t-T) - x^3(t)$$

BECOMES

$$u'(t) = -u(t - \pi/2 - \varepsilon\mu) - \varepsilon u^3(t)$$

NOW WE EXPAND $u = U_0 + \varepsilon U_1 + \ldots$ AND LET $s = \omega t$, $\omega = 1 + \varepsilon K_1$

THEN $U(s) = u(s/\omega)$

$$\omega U'(s) = -u\left(\frac{s - \omega(\pi/2 + \varepsilon\mu)}{\omega}\right) - \varepsilon U^3(s)$$

$$= -U\left(s - \omega(\pi/2 + \varepsilon\mu)\right) - \varepsilon U^3(s)$$

$$= -U\left(s - \pi/2 - \varepsilon\left(\mu + K_1\pi/2\right)\right) - \varepsilon U^3(s)$$

$$\omega U'(s) = -U(s - \pi/2) + \varepsilon(\mu + K_1\pi/2)\, U'(s - \pi/2) - \varepsilon U^3(s)$$

NOW LET $U = U_0 + \varepsilon U_1 + \ldots$

$$(1 + \varepsilon K_1)\left[U_0'(s) + \varepsilon U_1'(s)\right] + \ldots = -U_0(s - \pi/2) - \varepsilon U_1(s - \pi/2) + \varepsilon\left(\mu + \frac{K_1\pi}{2}\right) U_0'\left(s - \frac{\pi}{2}\right) - \varepsilon U_0^3(s) + \ldots$$

THIS YIELDS THAT

$$U_0'(s) = -U_0(s - \pi/2)$$

$$U_1'(s) + U_1(s - \pi/2) = -K_1 U_0'(s) + \left(K_1\pi/2 + \mu\right) U_0'(s - \pi/2) - U_0^3(s).$$

WE WRITE THAT

$$U_0 = A\cos S. \qquad U_0'(S) = -A\sin S, \quad U_0'(S-\pi/2) = A\cos S$$

THEN,

$$U_1'(S) + U_1(S-\pi/2) = K_1 A\sin S + \left(K_1\pi/2 + \mu\right) A\cos S - A^3\cos^3 S$$

$$= K_1 A\sin S + \left(K_1\pi/2 + \mu\right) A\cos S - A^3\left(\frac{3}{4}\cos S + \frac{1}{4}\cos 3S\right)$$

NOW ELIMINATING SIN S AND COS S TERMS WE OBTAIN

$$K_1 A = 0, \qquad K_1\pi/2 + \mu = 3A^2/4.$$

THUS $K_1 = 0$ AND $A^2 = 4\mu/3 \longrightarrow A = 2\sqrt{\mu/3}$.

WE CONCLUDE THAT

$$X \sim \frac{2}{\sqrt{3}}\sqrt{\mu}\,\varepsilon\cos(S) \qquad \text{WHEN} \quad T = \pi/2 + \varepsilon\mu$$

$$\text{where} \qquad S = [1 + O(\varepsilon^2)]\,t$$

Mechanical vibrations

FROM "APPLIED DELAY DIFFERENTIAL EQUATION" BY T. ERNEUX SPRINGER 2009 SURVEYS AND TUTORIALS IN THE APPLIED MATH SCIENCES

10

6.1 Control engineering

Control engineering saw rapid development in many countries in the period immediately following the Second World War. Research groups were set up in industrial, academic, and government laboratories. Heretofore secret wartime work was widely disseminated, and new military, industrial, and other applications of the emerging discipline were identified. Alexandr Aleksandrovich Andronov (1901–1952) was a key figure in the development of control engineering in the former Soviet Union during this period, yet his name, and his contributions to control theory and nonlinear dynamics, are much less well known in the West than they deserve [30]. Major figures in the promulgation of his work in the English-speaking world include Princeton mathematician Solomon Lefschetz (e.g., in [4], [131]) and Nicholas Minorsky [164]. The latter performed an invaluable service by reporting the Soviet state of the art in a set of (at first classified) U.S. government reports, which formed the basis of later published texts. Lefschetz was convinced that the Soviet nonlinear oscillation research was a vital field of applied mathematics that had been neglected in the United States. In the context of heavy military support for basic research created by World War II and the Cold War, he set up a "project on differential equations" most specifically devoted to nonlinear equations [8], p.291.

DDE problems naturally appear in control engineering. Any system involving a feedback control will almost certainly involve time delays. These arise because a finite time is required to sense information and then react

T. Erneux, *Applied Delay Differential Equations*, Surveys and Tutorials in the Applied Mathematical Sciences 3, DOI 10.1007/978-0-387-74372-1_6,

to it. In the 1930s and 1940s, studies began of certain stabilization problems in which delay plays a role (Minorsky[1] [164]). Consider, for example, a system whose motion is described by the following second order linear equation

$$y'' + ay' + y = 0. \tag{6.1}$$

The solution of this equation with arbitrarily specified initial conditions is a function that decays exponentially toward zero. Let us assume that the solution is underdamped ($a^2 < 4$) and we wish to somehow increase the damping coefficient a in order to diminish the oscillations more rapidly. If our system is a spring-mass system then we might simply immerse the whole system in motor oil. However, if, as in Minorsky's case, our system is a ship rolling in the sea and y is the angle of tilt from the normal upright position [166] (Figure 6.1), we must be more ingenious. We might, for example, introduce ballast tanks, partially filled with water, on each side of the ship. We would also have a servomechanism designed to pump water from one tank to the other in an attempt to counteract the roll of the ship. It is hoped that this will introduce another term proportional to y' in the equation say, by' :

$$y'' + ay' + by' + y = 0. \tag{6.2}$$

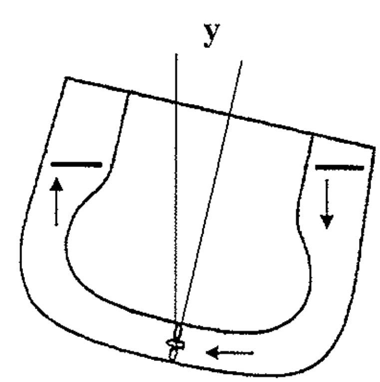


Figure 6.1: Ship rolling and its control. Two ballast tanks are partially filled with water on each side of the ship. A servomechanism pumps water from one tank to the other trying to counteract the roll of the ship (redrawn from Driver [164]).

[1] Nicholas Minorsky (1885–1970) contributed to several problems in ship navigation. In 1922 he presented a detailed analysis of a position feedback control system which today is refered to as Proportional-Integral-Derivative (PID) control [165]. Minorsky was born in Russia and studied at the Imperial Technical School in St. Petersburg. He joined the Russian Navy in 1917 but immigrated to the U.S. in 1918. His experiments

Thus instead of Eq. (6.2), we should consider

$$y'' + ay' + by'(t-\tau) + y = 0. \tag{6.3}$$

But what would happen if the delayed control was too slow? Could the force $by'(t-\tau)$ initiate some undesired instability? We analyze this problem in the next section by assuming small values of a and b.

6.2 The method of multiple scales

Equation (6.1) with $y(0) = 0$ and $y'(0) = 1$ has an exact solution. Its expression simplifies for small a as

$$y = \exp(-at/2)\sin(t) + O(a). \tag{6.4}$$

The expression (6.4) displays the product of two functions with distinct time-scales, namely a *fast time* t and a *slow time* $s \equiv at$. This suggests seeking a solution of the DDE (6.3) for small a and $b = O(a)$ that is a function of both t and s. Specifically, we introduce a small parameter ε defined by

$$\varepsilon \equiv a \tag{6.5}$$

and expand b as

$$b = \varepsilon b_1 + \dots. \tag{6.6}$$

We then seek a solution of Eq. (6.3) of the form

$$y = y_0(t,s) + \varepsilon y_1(t,s) + \dots, \tag{6.7}$$

where $s \equiv \varepsilon t$ is defined as a slow time variable. In the method of multiple scales [22, 124], the two times t and s are treated as independent variables. This implies the chain rules

$$\begin{aligned} y' &= y_t + \varepsilon y_s \\ y'' &= y_{tt} + 2\varepsilon y_{ts} + \varepsilon^2 y_{ss}, \end{aligned} \tag{6.8}$$

where the subscripts mean partial derivatives with respect to t or s. The two independent time variables also mean that $y(t-\tau)$ is now a function of both $t-\tau$ and $s-\varepsilon\tau$. Expanding for small ε, we have

$$y(t-\tau) = y(t-\tau, s-\varepsilon\tau) = y(t-\tau, s) - \varepsilon\tau y_s(t-\tau, s) + \dots. \tag{6.9}$$

MINORSKY'S EQUATION

THE MODEL PROBLEM FOR SHIP ROLL STABILITY IS

$$y'' + a y' + y = -b y'(t-\tau) + \varepsilon c y'^3(t-\tau)$$

WE CONSIDER THE SMALL DAMPING CASE $a = \varepsilon \ll 1$ AND ASSUME THAT THE FEEDBACK TERM b SATISFIES

$$b = \varepsilon b_1, \qquad c = O(1).$$

IF $c = 0$ AND $\tau = 0$ THEN FOR $b = \varepsilon b_1$ WE HAVE

$$y \sim \exp\left[-\frac{(\varepsilon + b_1 \varepsilon)t}{2}\right] \sin t$$

it enhances stability WITH NO delay.

WE WANT TO DERIVE A MULTIPLE-SCALES APPROXIMATION

$$y = Y(t, s) \quad \text{WITH} \quad s = \varepsilon t.$$

THEN

$$Y_{tt} + 2\varepsilon Y_{ts} + \varepsilon^2 Y_{ss} + a\left[Y_t + \varepsilon Y_s\right] + Y = -b_1 \varepsilon Y_t\left[t-\tau, \varepsilon(t-\tau)\right] + \varepsilon c Y_t^3\left[t-\tau, \varepsilon(t-\tau)\right]$$

NOW SETTING $a = \varepsilon$ WE OBTAIN

$$Y_{tt} + \varepsilon\left[2 Y_{ts}\right] + \varepsilon\left[Y_t\right] + Y = -b_1 \varepsilon\left[Y(t-\tau, s) + O(\varepsilon)\right] + \varepsilon c\left[Y_t^3(t-\tau, s) + \dots\right]$$

NOW EXPAND $Y = Y_0 + \varepsilon Y_1 + \dots$

$$Y_{0tt} + Y_0 = 0$$

$$Y_{1tt} + Y_1 = -2 Y_{0ts} - Y_{0t} - b_1 Y_{0t}(t-\tau, s) + c Y_{0t}^3(t-\tau, s).$$

WE WRITE

$$Y_0 = A(s) e^{it} + \text{c.c.}$$

$$Y_{1tt} + Y_1 = -2A' i e^{it} - i A e^{it} - i b_1 A e^{-i\tau} e^{it} \dots + c\left(A e^{i(t-\tau)} + \text{c.c.}\right)^3$$

NOW WE CALCULATE

$$c\, Y_{0t}^{3}(t-\tau,s) = c\left[A e^{it-i\tau} + \bar{A} e^{-it+i\tau} \right]^{3}$$

$$= \left(c\, 3 A^{2} \bar{A} e^{it} e^{-i\tau} + c.c. \right) + \text{higher harmonics.}$$

THUS
$$Y_{1tt} + Y_{1} = \left(-2A' i e^{it} - iAe^{it} - ib_{1} A e^{-i\tau} e^{it} + 3c A^{2} \bar{A} e^{it} e^{-i\tau} + c.c. \right)$$
$$+ \text{higher harmonics.}$$

THUS
$$-2A' - A + Ab_{1} e^{-i\tau} + 3c A^{2} \bar{A} e^{-i\tau} = 0.$$

WE WRITE THIS AS A COMPLEX ODE.

$$2A' = -A - b_{1} A e^{-i\tau} + 3c A^{2} \bar{A} e^{-i\tau}$$

NOW IF WE LET $A = Re^{i\phi}$ SO THAT

$$Y_{0} = R e^{i(t+\phi)} + c.c. = 2R \sin(t+\phi)$$

WE OBTAIN

$$2R' e^{i\phi} + 2Ri\phi' e^{i\phi} = -Re^{i\phi} - b_{1} R e^{i\phi} e^{-i\tau} + 3cR^{3} e^{i\phi - i\tau}.$$

CANCELLING $e^{i\phi}$ GIVES

$$2R' + 2iR\phi' = -R - b_{1} R\left[\cos\tau - i\sin\tau\right] + 3cR^{3}\left[\cos\tau - i\sin\tau\right]$$

WE OBTAIN THE COUPLED ODE,

$$\begin{cases} 2R' = -\left[1 + b_{1}\cos\tau\right] R + 3cR^{3}\cos\tau \\ 2\phi' = b_{1}\sin\tau - 3cR^{2}\sin\tau \end{cases}$$

WITH $y \sim 2R\sin(t+\phi)$ $\quad R = R(\varepsilon t), \quad \phi = \phi(\varepsilon t)$

NOTICE THAT $R=0$ IS UNSTABLE IFF

$$1 + b_1 \cos \tau < 0.$$

SUPPOSE THAT $0 < b_1 < 1$, THEN $R=0$ IS STABLE FOR ALL DELAYS τ.

NEXT, SUPPOSE THAT $b_1 > 1$ AND $\cos \tau < 0$, SUCH AS WHEN $\pi/2 < \tau < 3\pi/2$ OR $5\pi/2 < \tau < 7\pi/2$. THEN THE HOPF BIFURCATION VALUE FOR b_1 IS

$$b_{1H} = -\cos^{-1}(\tau) \qquad \text{WITH } \cos\tau < 0.$$

NOW WE WRITE R' ODE AS

$$2R' = -\left[1 + b_{1H}\cos\tau + (b_1 - b_{1H})\cos\tau\right] R + 3cR^3 \cos\tau.$$

BUT SINCE $1 + b_{1H}\cos\tau = 0$ WE SET $R' = 0$ TO OBTAIN THAT

$$-(b_1 - b_{1H})\cos\tau \, R + 3cR^3\cos\tau = 0.$$

THIS LEADS TO THE EQUILIBRIUM EQUATION

$$R_{EQ} = \sqrt{\frac{b_1 - b_{1H}}{3c}} \qquad \text{FOR} \quad b_1 > b_{1H}.$$

SINCE THE COEFFICIENT OF THE R^3 TERM, NAMELY $3c\cos\tau$ SATISFIES $3c\cos\tau < 0$ WHEN $\cos\tau < 0$, WE CONCLUDE THAT R_{EQ} IS A STABLE FIXED POINT OF ODE FOR R $\longrightarrow$ SUPERCRITICAL HOPF BIFURCATION AT $b_1 = b_{1H}$.

WE THEN PLOT THE MAXIMUM/MINIMUM OBSERVED AMPLITUDES VS b, FOR THE PARAMETER VALUES

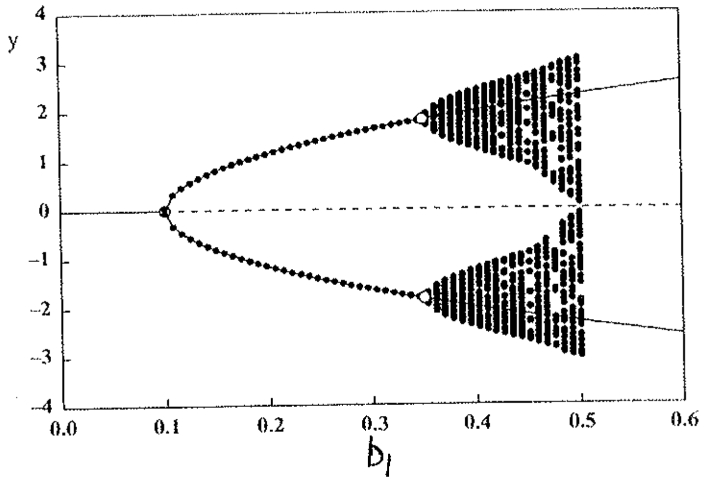
$$a = 0.1, \quad c = 1, \quad \varepsilon = 0.1, \quad \tau = 3\pi$$

NOTICE THAT A HOPF BIFURCATION OCCURS AT $b_1 = 1$.

WE OBSERVE THAT THERE IS A SECONDARY BIFURCATION AT $b_s = 0.35$ (i.e. $b_1 = 3.5$)

THE AMPLITUDE OF THE RESPONSE IS

$$R_{EQ} = \pm\sqrt{\frac{b_1 - 1}{3}}$$



NEXT WE STUDY THIS SECONDARY INSTABILITY THAT OCCURS NUMERICALLY FOR $b \approx 0.35$ AND FOR $\tau = 3\pi$. WE TREAT τ AS $\tau >> 1$.

RECALL THAT WE DERIVED AMPLITUDE EQUATION BASED ON THE ASSUMPTION THAT $\tau = O(1)$.

WE BEGIN WITH

$$y'' + a y' + y = -b y'(t-\tau) + \varepsilon c y'^3(t-\tau).$$

$$b = \varepsilon b_1,\; c = O(1),\; a = \varepsilon$$

NOW LET $y = \tilde{Y}(t, s)$ $\quad s = \varepsilon t$.

WE LET $\tau = O(1/\varepsilon)$ THEN WE DEFINE $\varphi \equiv \varepsilon\tau = O(1)$.

NOW WE CALCULATE

$$\tilde{Y}_{tt} + 2\varepsilon \tilde{Y}_{ts} + \varepsilon^2 \tilde{Y}_{ss} + \varepsilon\left(\tilde{Y}_t + \varepsilon \tilde{Y}_s\right) + \tilde{Y}$$
$$= -b_1\varepsilon\left[Y_t(t-\tau,\; \varepsilon(t-\tau)) + O(\varepsilon)\right] + \varepsilon c\left[\tilde{Y}_t(t-\tau, \varepsilon(t-\tau))\right.$$
$$+ O($$

THEN

$$Y_{tt} + 2\varepsilon Y_{ts} + \varepsilon^2 Y_{ss} + \varepsilon (Y_t + \varepsilon Y_s) + Y$$
$$= - b_1 \varepsilon Y_t (t-\tau, s-\varphi) + \varepsilon c Y_t^3 (t-\tau, s-\varphi)$$

NOW WE EXPAND $Y = Y_0 + \varepsilon Y_1 + \cdots$

$$Y_{0_{tt}} + Y_0 = 0$$

$$Y_{1_{tt}} + Y_1 = -2 Y_{0_{ts}} - Y_{0_t} - b_1 Y_{0_t}(t-\tau, s-\varphi) + c\, Y_{0_t}^3 (t-\tau, s-\varphi).$$

WE OBTAIN THAT

$$Y_0 = A(s)\, e^{it} + c.c.$$

IF WE THEN CALCULATE AS BEFORE WE OBTAIN THE AMPLITUDE EQUATION

$$2\, dA/_{ds} = -A - b_1\, e^{-i\tau} A(s-\varphi) + 3c\, |A(s-\varphi)|^2 A(s-\varphi) e^{-i\tau}.$$

NOW FOR SIMPLICITY WE WILL SET

$\tau = (1 + 2n)\pi$. WHERE n = positive integer.

THEN WE OBTAIN

$$(*) \quad 2\, dA/_{ds} = -A + A(s-\varphi)\, b_1 - 3c\, |A(s-\varphi)|^2 A(s-\varphi)$$

<u>REMARK</u> (i) FIRST ORDER DDE. STEADY AND PERIODIC SOLUTIONS OF (*) CORRESPOND TO PERIODIC AND QUASI-PERIODIC SOLUTIONS OF THE FULL SYSTEM.

NOW WE LET $A = Re^{i\phi}$ WE SUBSTITUTE AND COLLECT REAL AND IMAGINARY PARTS.

$$A(s-\varphi) = R(s-\varphi)\, e^{i\phi(s-\varphi)}$$

$$2\left(R' e^{i\phi(s)} + iR\phi' e^{i\phi(s)}\right) = + R(s-\varphi)\, e^{i\phi(s-\varphi)}\, b_1 - R(s)\, e^{i\phi(s)} - 3c\, R^2(s-\varphi)\, R(s-\varphi)\, e^{i\phi(s-\varphi)}$$

THIS YIELDS

$$2R' + 2iR\phi' = -R + R(s-\varphi)\, e^{i\phi(s-\varphi) - i\phi(s)}\, b_1 - 3c\, R^3(s-\varphi)\, e^{i\phi(s-\varphi) - i\phi(s)}$$

THE REAL AND IMAGINARY PARTS GIVE

$$2R' = -R + R(s-\varphi)\left[b_1 - 3cR^2(s-\varphi)\right] \cos\left[\phi(s-\varphi) - \phi(s)\right]$$

$$2R\phi' = R(s-\varphi)\left[b_1 - 3cR^2(s-\varphi)\right] \sin\left[\phi(s-\varphi) - \phi(s)\right]$$

ONE SOLUTION IS THE HB VALUE

$$R = \sqrt{\frac{b_1 - 1}{3c}} \qquad \phi = \phi_0$$

WE NOW EXAMINE THE STABILITY OF THIS. THE LINEARIZED EQUATION IS $R = R_{EQ} + u$

$$2u' = -u + u(s-\varphi)\left[b_1 - 9cR^2\right]$$

$$2v' = (b_1 - 3cR^2)\left(v(s-\varphi) - v(s)\right)$$

NOW SUBSTITUTE $u = e^{\sigma s} u_0, \quad v = e^{\sigma s} v_0$

THUS YIELDS

$$2\sigma = -1 + e^{-\sigma\varphi}(b_1 - 9cR^2)$$

$$2\sigma = (b_1 - 3cR^2)(e^{-\sigma\varphi} - 1)$$

BUT $R^2 = \dfrac{b_1 - 1}{3c}$. HENCE $\quad b_1 - 3cR^2 = b_1 - (b_1 - 1) = 1$

$$b_1 - 9cR^2 = b_1 - 3(b_1 - 1) = 3 - 2b_1$$

$$\begin{cases} 2\sigma = -1 + e^{-\sigma\varphi}(3 - 2b_1) \\ 2\sigma = e^{-\sigma\varphi} - 1. \end{cases}$$

NOW SUPPOSE THAT σ IS REAL. THE SECOND EQUATION YIELDS THAT $\sigma = 0$ IS THE ONLY POSSIBILITY.

NOW LOOK FOR A HB POINT AT SOME VALUE OF b_1.

WE PUT $\sigma = i\omega$ WITH $\omega > 0$ AND OBTAIN FROM THE AMPLITUDE EQUATION FOR R THAT

$$2i\omega = -1 + e^{-i\omega\varphi}(3 - 2b_1)$$

FROM REAL AND IMAGINARY PART WE OBTAIN

$$0 = -1 + \cos(\omega\varphi)(-2b_1 + 3)$$

$$2\omega = -\sin(\omega\varphi)(-2b_1 + 3).$$

NOW DIVIDING WE GET

$$(3 - 2b_1)\cos\omega\varphi = 1$$

$$(3 - 2b_1)\sin(\omega\varphi) = -2\omega.$$

THUS $\quad (3 - 2b_1)^2 = 1 + 4\omega^2.$

THIS YIELDS THAT w SATISFIES

$$\tan(w\varphi) = -2w$$

AND $$2b_1 = 3 - 1/\cos(w\varphi)$$

FOR THE PARAMETERS GIVEN WE HAVE $\tau = 3\pi$, $\varepsilon = .1$ SO THAT

$$\varphi \approx .94$$

WE SOLVE TO GET $w\varphi \approx 1.8$ AND $b_1 \approx 3.5$.

THUS $b = \varepsilon b_1 \approx .35$ IS THE PREDICTION FOR WHERE THE SECONDARY INSTABILITY OCCURS.

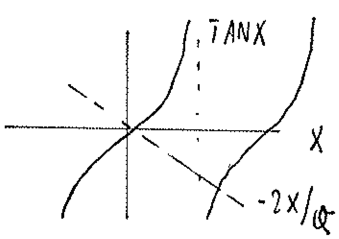
NOW CONSIDER THE AMPLITUDE EQUATION WITH $\tau = (1+2n)\pi$

$$2\, dA/ds = -A + b_1 A(s-\varphi) - 3c|A(s-\varphi)|^2 A(s-\varphi)$$

NOW WE ASSUME THAT $\varphi >> 1$. (i.e. EVEN LONGER DELAYS)

FROM $\tan(w\varphi) = -\dfrac{2(w\varphi)}{\varphi}$ IT FOLLOWS THAT $w\varphi \to \pi$

AND CONSEQUENTLY $b_1 \to 2$ FROM $2b_1 = 3 - 1/\cos(w\varphi) \to 3+1 = 4$.



$\varphi >> 1$ small slope.

THEN DEFINE $s' = s/\varphi$ SINCE TIME SCALE $\omega = O(1/\varphi) << 1$.

(i.e. short time scale).

then $dA/ds = dA/ds' \cdot \frac{1}{\varphi} = -A + b_1 A(\varphi s' - \varphi) - 3c A^2(\varphi s' - \varphi) A^*(\varphi s' - \varphi)$,

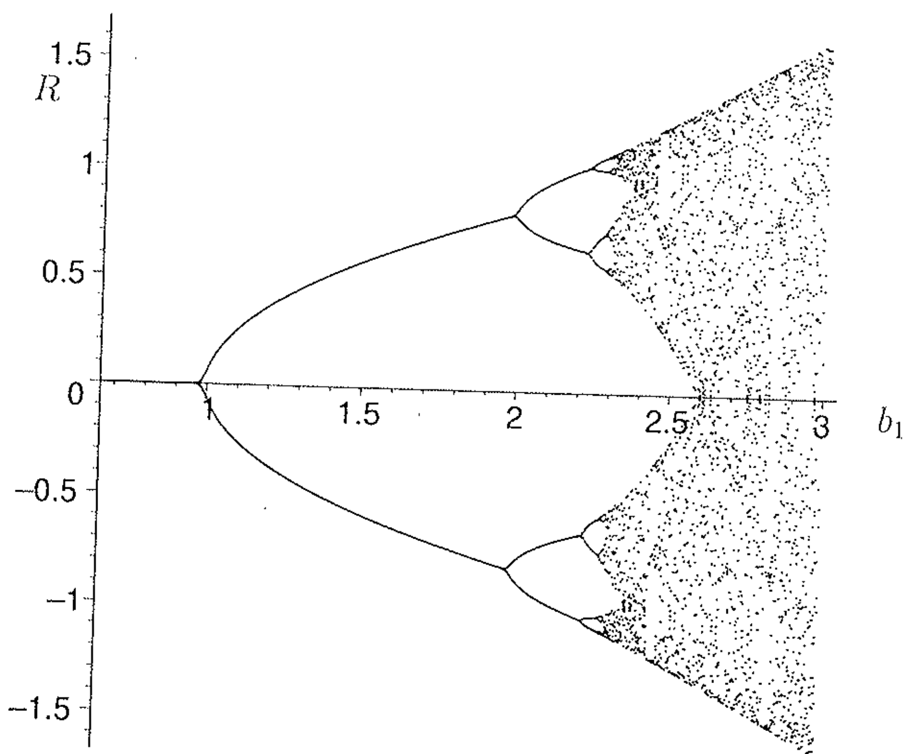
SINCE $\varphi >> 1$ THEN DEFINE $B(s') = A(\varphi s')$ WE OBTAIN

THE DISCRETE MAP $B(s') = b_1 B(s'-1) - 3c B^2(s'-1) B^*(s'-1)$.

$B_n = b_1 B_{n-1} - 3c B_{n-1}^3$ WHICH IS A DISCRETE MAP.

THIS DISCRETE MAP AS A PERIODIC DOUBLING ROUTE TO CHAOS.

THE FIRST PERIOD DOUBLING IS WHEN $B_2 = B_0$ ETC..



Period doubling and chaos for large φ. Note that the first two bifurcations occur at $b_1 = 1$ and $b_1 = 2$. Here, $c = 0.5$, AND $\tau = \varphi\pi$.