ENTRAINMENT OF OSCILLATORS

BY IMPOSING AN EXTERNAL FORCING ON A PROBLEM EXHIBITING LIMIT CYCLE BEHAVIOR, IT IS POSSIBLE TO LOCK, QUENCH, OR ENTRAIN THE OSCILLATION SO THAT WE OBSERVE A SIGNAL OSCILLATING AT THE SAME FREQUENCY AS THE EXTERNAL FORCING.

- pacer for synchronizing cardiac nerve cells.
- synchronization of coupled oscillators (fireflies)
- Van der pol oscillator etc.. $x'' - \varepsilon(1-x^2)x' + x = f\cos(wt)$

Entrainment can be observed in the forced Van der Pol equation:

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = \varepsilon(1-x^2)y - x + f\cos(\omega t)$$

The limit cycle of this system appears similar to others, but by examining the time series we can confirm that the system is oscillating at the drive frequency.

ENTRAINMENT

WHEN $w \approx 1$

F Big.

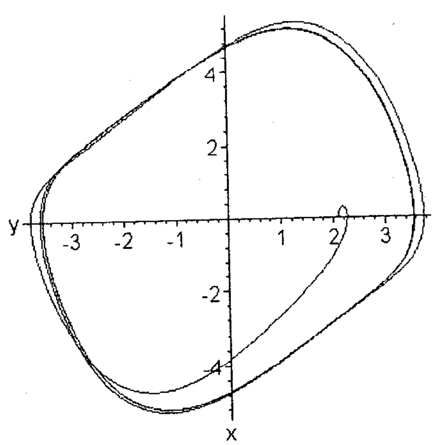


Figure 2: Entrainment: Limit Cycle for driven Van der Pol Oscillator with ε=0.25, f=3, ω=1.2

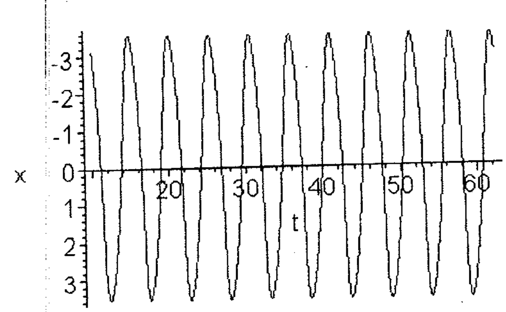


Figure 3: Entrainment: Time series for driven Van der Pol Oscillator with ε=0.25, f=3, ω=1.2

In figure 3, there are ten cycles in 52.4 time units, giving an angular frequency of ω=1.2, confirming that the system is entrained at the driving frequency.

QUASI-PERIODICITY

WHEN $\omega \simeq 1$ f too small

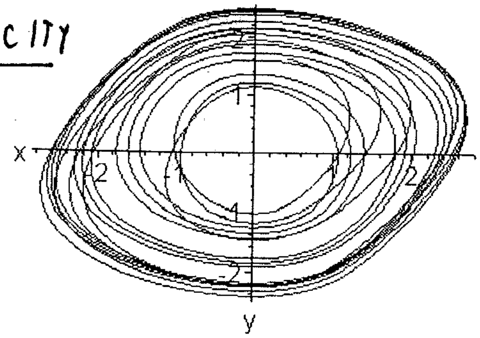


Figure 4: Quasiperiodicity: Phase trajectory for driven Van der Pol Oscillator with ε=0.25, f=0.4, ω=1.2

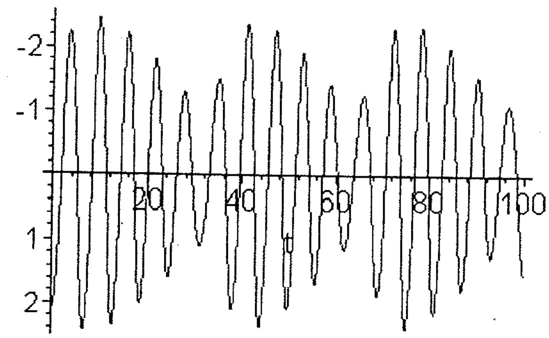


Figure 5: Quasiperiodicity: Time series for driven Van der Pol Oscillator with ε=0.25, f=0.4, ω=1.2

It is plain that the amplitude is changing, but to see the evidence for competing frequencies, it is best to look at a power spectrum.

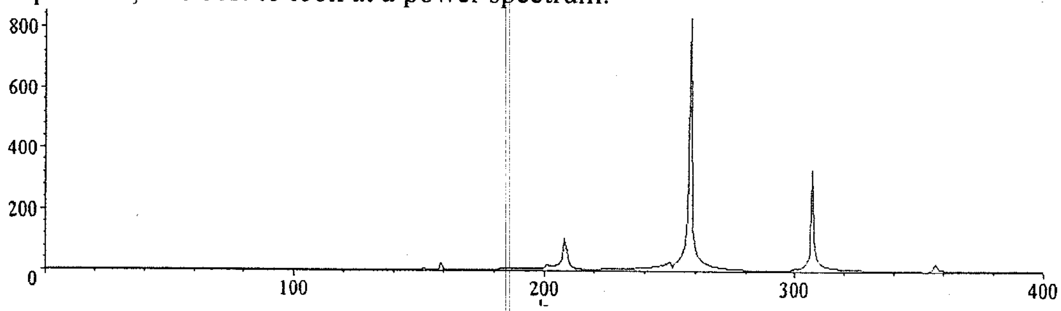


Figure 6: Quasiperiodicity: Power spectrum for driven Van der Pol Oscillator with ε=0.25, f=3, ω=1.2. On x-axis, 256 corresponds to a frequency of 1.

PERTURBATION ANALYSIS OF ENTRAINEMENT IN VDP.

$$x'' + x - \varepsilon(1-x^2)x' = \varepsilon F \cos(\omega t).$$

(i) IF $F = 0 \longrightarrow \exists$ LIMIT-CYCLE OR SELF-EXITED OSCILLATION WITH FREQUENCY $= 1 + O(\varepsilon^2)$. (HW 1)

(ii) CHOOSE

$$w = 1 + \varepsilon K_1$$

WITH K_1 FIXED.

THIS MEANS WE PUT EXTERNAL FORCING AT CLOSE TO FREQUENCY OF LIMIT CYCLE.

WE CONSIDER THE FORCED PROBLEM,

$$x'' + x - \varepsilon(1-x^2)x' = \varepsilon F \cos(\omega t)$$

WHERE WE LET $\omega = 1 + K_1 \varepsilon$.

<u>KEY POINT</u>: IF FORCING IS STRONG ENOUGH (F LARGE ENOUGH) AND IF K_1 IS SMALL ENOUGH (i.e. INTERNAL FREQUENCY AND DRIVING FREQUENCY ARE CLOSE), THEN THE RESPONSE MAY OCCUR ONLY AT THE DRIVING FREQUENCY. IF THIS OCCURS WE SAY THAT THE UNFORCED OSCILLATION IS QUENCHED OR ENTRAINED TO THE FORCING FUNCTION. WHERE DOES THIS OCCUR IN F VERSUS K_1 PARAMETER PLANE AS $\varepsilon \to 0$?

<u>ANALYSIS</u> WE LET $T = t$, $\tau = \varepsilon t$ AND WRITE $x = X(T,\tau)$.

THEN,

$$X_{tt} + 2\varepsilon X_{t\tau} + \varepsilon^2 X_{\tau\tau} + X - \varepsilon(1-X^2)(X_T + \varepsilon X_\tau) = \varepsilon F \cos(T + \underbrace{\varepsilon K_1 t}_{K_1\tau})$$

WE EXPAND $\quad X = X_0 + \varepsilon X_1 + \cdots$

TO OBTAIN

$$O(1):\ X_{0TT} + X_0 = 0$$

$$O(\varepsilon):\ X_{1TT} + X_1 = -2X_{0T\tau} + X_{0T}(1-X_0^2) + F\cos(T + K_1\tau)$$

THE LEADING ORDER SOLUTION IS

$$X_0 = R\cos(T+\phi) \quad \text{WITH} \quad R = R(\tau),\ \Phi = \Phi(\tau).$$

WE NOW WRITE

$$\begin{aligned}\cos(T + K_1\tau) &= \cos(T + \phi + (K_1\tau - \phi)) \\ &= \cos(T+\phi)\cos(\phi - K_1\tau) + \sin(T+\phi)\sin(\phi - K_1\tau)\end{aligned}$$

WE GET

$$\begin{aligned}X_{1TT} + X_1 = -2X_{0T\tau} &- R\sin(T+\phi)\left[1 - R^2\cos^2(T+\phi)\right] \\ &+ F\cos(T+\phi)\cos(\phi - K_1\tau) \\ &+ F\sin(T+\phi)\sin(\phi - K_1\tau).\end{aligned}$$

BUT $X_{0T} = -R\sin(T+\phi)$ $\quad X_{0T\tau} = -R'\sin(T+\phi) - R\phi'\cos(T+\phi)$ (4)

USING $\cos^2\varphi\sin\varphi = \frac{1}{4}(\sin 3\varphi + \sin\varphi)$

WE GET

$$X_{1TT} + X_1 = 2R'\sin(T+\phi) + 2R\phi'\cos(T+\phi) - R\sin(T+\phi) + \frac{R^3}{4}\sin(T+\phi) + \frac{R^3}{4}\sin(3(T+\phi))$$
$$+ F\cos(T+\phi)\cos(\phi - K_1\tau) + F\sin(T+\phi)\sin(\phi - K_1\tau)$$

NOW WE OBTAIN

$$X_{1TT} + X_1 = \left(2R' - R + \frac{R^3}{4} + F\sin(\phi - K_1\tau)\right)\sin(T+\phi)$$
$$+ \left(2R\phi' + F\cos(\phi - K_1\tau)\right)\cos(T+\phi) + \frac{R^3}{4}\sin(3(T+\phi)).$$

TO SUPPRESS RESONANCE WE ELIMINATE $\sin(T+\phi)$ AND $\cos(T+\phi)$ TERMS TO OBTAIN ODE'S:

$$R' = \frac{R}{8}(4 - R^2) - \frac{F}{2}\sin(\phi - K_1\tau)$$
$$\phi' = -\frac{F}{2R}\cos(\phi - K_1\tau)$$

NOW DEFINE $\quad \psi = \phi - K_1\tau$

SO THAT

$$(*)\begin{cases} R' = \dfrac{R}{2} - \dfrac{R^3}{8} - \dfrac{F}{2}\sin\psi \equiv f(R,\psi) \\[2ex] \psi' = -K_1 - \dfrac{F}{2R}\cos\psi \equiv g(R,\psi) \end{cases}$$

SO THAT $\quad X \sim R\cos(T + K_1\tau + \psi) = R\cos(\omega t + \psi)$ WITH $\omega = 1 + K_1\varepsilon$.

<u>REMARK</u> (i) EQUILIBRIA OF SLOW ODE'S (*) $\longrightarrow$ PERIODIC SOLUTIONS $\to$ ENTRAINED AT FREQUENCY ω.

(ii) WHEN EQUILIBRIA OF SLOW ODE ARE STABLE $\longrightarrow$ ENTRAINEMENT OCCURS AND LIMIT CYCLE IS SAID TO BE "QUENCHED" BY FORCING.

NOW LET $R = R_e$, $\psi = \psi_e$ BE AN EQUILIBRIUM POINT OF (*). THEN

$$\frac{R_e}{2} - \frac{R_e^3}{8} = -\frac{F}{2}\sin\psi_e, \qquad -\nu_1 R_e = +\frac{F}{2}\cos\psi_e$$

THUS

$$F\cos\psi_e = -2\nu_1 R_e, \qquad F\sin\psi_e = -R_e + R_e^3/4.$$

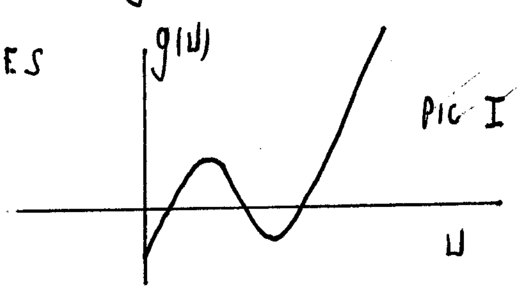
WE SQUARE AND ADD TO OBTAIN

$$F^2 = R_e^2\left(1 - R_e^2/4\right)^2 + 4\nu_1^2 R_e^2$$

NOW DEFINE $u = R_e^2$ SO THAT u IS A ROOT OF THE CUBIC POLYNOMIAL $g(u) = 0$, DEFINED BY

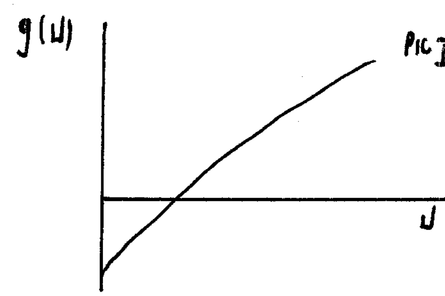
$$g(u) \equiv u^3/16 - u^2/2 + (4\nu_1^2 + 1)\,u - F^2.$$

OBSERVE THAT $g(0) = -F^2 < 0$ AND $g(u) \to +\infty$ AS $u \to +\infty$.
ALSO $g'(0) = 4\nu_1^2 + 1 > 0$.
THEREFORE, WE CONCLUDE THAT $\exists$ EITHER ONE OR THREE ROOTS TO $g(u) = 0$. THUS WE EITHER GET THE FOLLOWING TWO PICTURES



PIC I

OR



PIC II

THE TRANSITION BETWEEN PICS I AND II IS WHEN WE HAVE A REPEATED ROOT u_c TO $g'(u_c) = 0$ WHERE $g(u_c) = 0$.

SETTING $g'(U_C)=0$ GIVES

$$\frac{3U_c^2}{16} - U_c + (4u_i^2+1) = 0.$$ IF THIS HAS TWO ROOTS $U_{C\pm}$ THEN $U_{C\pm}>0$ OR $U_{C\pm}<0$.

NOW WE MUST COMBINE THIS WITH SETTING $g(U_C)=0$ TO OBTAIN A CURVE IN THE F VS u_1 PLANE.

NOW SET $g(U_c)=0$ AND $g'(U_c)=0$ TO OBTAIN

$$g(U_c)=0 \rightarrow \frac{U_c^3}{16} - \frac{U_c^2}{2} + B\,U_c = F^2 \qquad \text{WHERE } B = 4u_1^2+1$$

$$g'(U_c)=0 \rightarrow \frac{U_c^3}{16} - \frac{U_c^2}{3} + \frac{B\,U_c}{3} = 0$$

WE SUBTRACT TO OBTAIN THAT

$$-\frac{U_c^2}{6} + \frac{2U_c B}{3} = F^2 \longrightarrow U_c^2 = -6F^2 + 4U_cB$$

NOW PUT U_c^2 INTO $g'(U_c)=0$. THIS GIVES:

$$\frac{3}{16}\left(-6F^2+4U_cB\right) + B = U_c.$$

WE SOLVE FOR U_c TO OBTAIN $U_c\left(1-\frac{3B}{4}\right) = B - \frac{9}{8}F^2.$

FINALLY, WE SOLVE FOR U_c AND SUBSTITUTE BACK INTO

$$\frac{3}{16}U_c^2 - U_c + B = 0$$

THIS GIVES $$\frac{3}{16}\left(B-\frac{9}{8}F^2\right)^2 - \left(B-\frac{9}{8}F^2\right)\left(1-\frac{3B}{4}\right) + B\left(1-\frac{3B}{4}\right)^2 = 0.$$

WE GET:

$$\left(B-\frac{9}{8}F^2\right)^2 - \frac{16}{3}\left(B-\frac{9}{8}F^2\right)\left(1-\frac{3B}{4}\right) + \frac{16B}{3}\left(1-\frac{3B}{4}\right)^2 = 0$$

$$\frac{81}{64} F^4 - \frac{9}{4} B F^2 + B^2 - \frac{16}{3}\left(B - \frac{9}{8} F^2 - \frac{3B^2}{4} + \frac{27}{32} B F^2 \right) + \frac{16B}{3}\left(1 - \frac{3B}{2} + \frac{9B^2}{16} \right) = 0$$

THUS,
$$\frac{81}{64} F^2 + F^2\left(6 - \frac{27}{4} B \right) + B^2 - \frac{16B}{3} + 4B^2 + \frac{16B}{3} - 8B^2 + 3B^3 = 0.$$

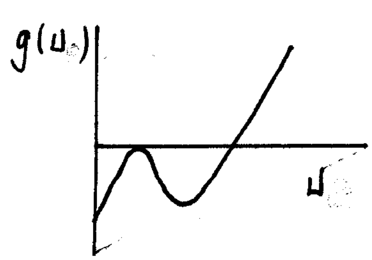
NOW THIS YIELDS
$$\frac{81}{64} F^4 + F^2\left(6 - \frac{27}{4} B \right) + 3B^3 - 3B^2 = 0.$$

WE OBTAIN
$$\frac{F^4}{64} + \frac{F^2}{81}\left(6 - \frac{27}{4} B \right) + \frac{1}{27} B^2 (B-1) = 0.$$

MULTIPLYING BY 4 AND RECALLING $B = 4u_1^2+1$, WE OBTAIN THAT
$$\frac{F^4}{16} + \frac{F^2}{81}\left(-3 - 108 u_1^2 \right) + \frac{16u_1^2}{27}\left(1 + 4u_1^2 \right)^2 = 0.$$

IN SUMMARY, WE CONCLUDE THAT $g(u_c) = g'(u_c) = 0$ ON THE CURVES
$$\frac{F^4}{16} - \frac{F^2}{27}\left(1 + 36 u_1^2 \right) + \frac{16}{27} u_1^2 \left(1 + 4u_1^2 \right)^2 = 0 \quad (+)$$

ON THIS CURVE WE HAVE A SADDLE-NODE BIFURCATION, i.e. A TRANSITION BETWEEN THREE AND ONLY ONE EQUILIBRIUM POINT, i.e. ONE POSITIVE EQUILIBRIUM AND ANOTHER POSITIVE ONE OF MULTIPLICITY TWO



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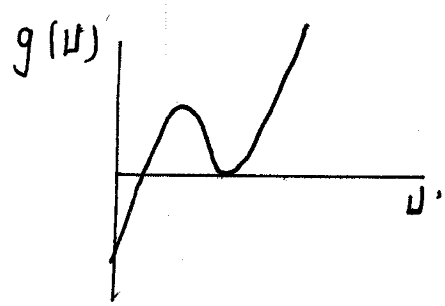


NOTICE THAT (+) IS A QUADRATIC EQUATION FOR F^2.

NOW $$F^4 - \frac{16}{27}(1+36u_1^2)F^2 + \frac{(16)^2}{27}u_1^2(1+4u_1^2)^2 = 0$$

WE OBTAIN $$F_\pm^2 = \frac{16}{54}(1+36u_1^2) \pm \frac{1}{2}\sqrt{\left(\frac{16}{27}\right)^2(1+36u_1^2)^2 - \frac{4(16)^2}{27}u_1^2(1+4u_1^2)^2}$$

THIS GIVES TWO CURVES $F_\pm > 0$ GIVEN BY:

$$F_\pm^2 = \frac{8}{27}(1+36u_1^2) \pm \frac{8}{27}\sqrt{(1+36u_1^2)^2 - 108u_1^2(1+4u_1^2)^2}$$

WE CAN WRITE THIS AS

$$F_\pm^2 = \frac{8}{27}(1+36u_1^2) \pm \frac{8}{27}(1+36u_1^2)\sqrt{1 - \frac{108u_1^2(1+4u_1^2)^2}{(1+36u_1^2)^2}} \quad (**)$$

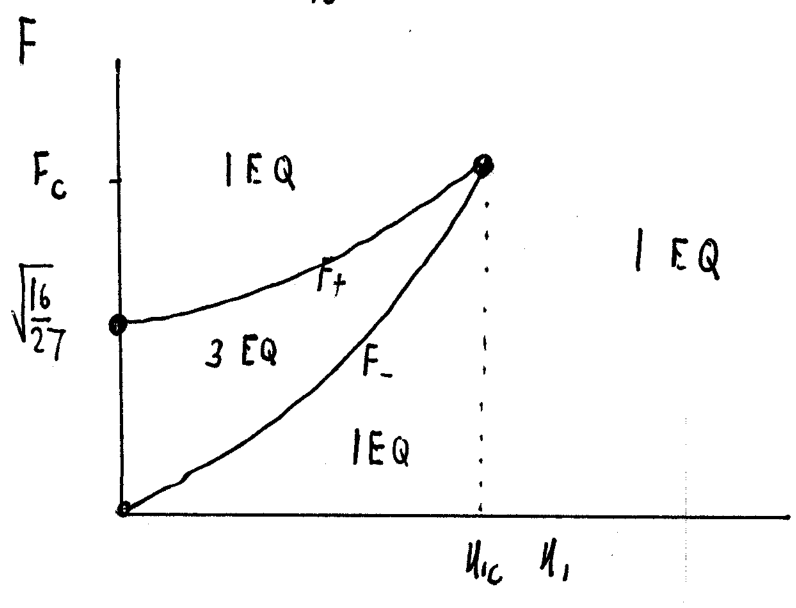
WE CAN VERIFY THAT

$$F_c \equiv F_\pm = \sqrt{\frac{32}{27}} \simeq 1.089 \text{ WHEN } u_1 = u_{1c} = \frac{1}{\sqrt{12}} \simeq .288.$$

THIS CORRESPONDS TO THE VALUE WHERE WE HAVE A TRIPLE ROOT,

i.e. $g(u_c) = g'(u_c) = g''(u_c) = 0 \rightarrow g''(u_c) = 0 \rightarrow u_c = 8/3.$

(NOTE: $g'(u_c) = 0 \rightarrow \frac{3}{16}\left(\frac{8}{3}\right)^2 - \frac{8}{3} + 4u_1^2 + 1 = 0 \rightarrow u_1^2 = 1/12 \rightarrow F_\pm = \sqrt{\frac{32}{27}}$.



NEXT, WE MUST ANALYZE THE LINEAR STABILITY OF THESE EQUILIBRIA. WE WRITE

$$R' = \frac{R}{2} - \frac{R^3}{8} - \frac{F}{2}\sin\psi = f(R,\psi)$$

$$\psi' = -\kappa_1 - \frac{F}{2R}\cos\psi = g(R,\psi)$$

NOW
$$M_e \equiv \begin{pmatrix} f_R & f_\psi \\ g_R & g_\psi \end{pmatrix} = \begin{pmatrix} \frac{1}{2} - \frac{3}{8}R_e^2 & -\frac{F}{2}\cos\psi_e \\ \frac{F}{2R_e^2}\cos\psi_e & \frac{F}{2A_e}\sin\psi_e \end{pmatrix}$$

BUT $\frac{F}{2}\cos\psi_e = -\kappa_1 R_e$, $\frac{F}{2}\sin\psi_e = \frac{R_e}{2} - \frac{R_e^3}{8}$.

THUS YIELDS
$$M_e = \begin{pmatrix} \frac{1}{2} - \frac{3}{8}R_e^2 & \kappa_1 R_e \\ -\kappa_1/R_e & \frac{1}{2} - \frac{R_e^2}{8} \end{pmatrix}$$

WE CONCLUDE THAT

$$\text{trace } M_e = 1 - R_e^2/2$$

$$\det M_e = \left(\frac{1}{2} - \frac{3}{8}R_e^2\right)\left(\frac{1}{2} - \frac{R_e^2}{8}\right) + \kappa_1^2 = \frac{1}{4}\left(\frac{3}{16}R_e^4 - R_e^2 + 4\kappa_1^2 + 1\right)$$

NOW RECALL $u_e = A_e^2$ FROM OUR ANALYSIS OF THE POLYNOMIAL

THUS, IF u_e IS A ROOT OF $g(u) = \frac{u^3}{16} - \frac{u^2}{2} + (4\kappa_1^2+1)u - F^2 = 0$

WE OBTAIN THAT

$$\text{trace } M_e = 1 - u_e/2$$

$$\det M_e = \frac{1}{4}g'(u_e) \quad \text{WITH} \quad g'(u) = \frac{3u^2}{16} - u + (4\kappa_1^2+1)$$

NOW FOR LINEAR STABILITY, WE MUST HAVE THAT THE FOLLOWING IS TRUE:

$$\text{trace } M_e < 0 \quad \underline{\text{AND}} \quad \det M_e > 0.$$

NOW WE FIND WHERE trace $M_e = 0$ FOR AN EQUILIBRIUM SOLUTION WITH $U_e = 2$. IMPOSING $g(U_e) = 0 \rightarrow g(2) = \frac{1}{2} - 2 + 2(4u_1^2 + 1) - F^2 = 0$ SO THAT

- FOR AN EQUILIBRIUM SOLUTION WHERE $U_e = 2$ WE HAVE THAT ON

$$F^2 = 8u_1^2 + \frac{1}{2} \longrightarrow \text{trace } M_e = 0.$$

WE NOW ASK WHERE THIS CURVE INTERSECTS WITH THE SADDLE NODE CONDITION THAT

$$F^4 - \frac{16}{27}(1 + 36u_1^2)F^2 + \frac{(16)^2}{27}u_1^2(1 + 4u_1^2)^2 = 0 \qquad (*)$$

$$\rightarrow \quad \frac{27}{16}\left(8u_1^2 + \frac{1}{2}\right)^2 - \left(8u_1^2 + \frac{1}{2}\right)(1 + 36u_1^2) + 16u_1^2(1 + 4u_1^2)^2 = 0$$

$$\rightarrow \quad \left(8u_1^2 + \frac{1}{2}\right)\left[\frac{27}{16}\left(8u_1^2 + \frac{1}{2}\right) - (1 + 36u_1^2)\right] + 16u_1^2(1 + 4u_1^2)^2 = 0$$

$$\rightarrow \quad \left(8u_1^2 + \frac{1}{2}\right)\left[-\frac{45u_1^2}{2} - \frac{5}{32}\right] + 16u_1^2(1 + 4u_1^2)^2 = 0$$

$$-\frac{5}{2}\left(8u_1^2 + \frac{1}{2}\right)\left(9u_1^2 + \frac{1}{16}\right) + 16u_1^2(1 + 4u_1^2)^2 = 0.$$

WE CONCLUDE THAT $\det M_E = 0$ MUST OCCUR ONLY ON CURVE WHERE $g(U_e) = g'(U_e) = 0$ (i.e. ON $(*)$) AND THAT THE trace $M_e = 0$ CURVE INTERSECTS THE $\det M_e = 0$ CURVE WHEN

$$\left(8u_1^2 + \frac{1}{2}\right)\left(9u_1^2 + \frac{1}{16}\right) = \frac{32}{5}u_1^2\left(1 + 4u_1^2\right)^2. \qquad (+)$$

IF WE DEFINE $\zeta = u_1^2$ $(+)$ YIELDS THAT

$$512\zeta^3 - 104\zeta^2 + 7\zeta - \frac{5}{32} = 0$$

WHICH CAN BE FACTORED AS $512\left(\zeta - \frac{1}{16}\right)^2\left(\zeta - \frac{5}{64}\right) = 0$

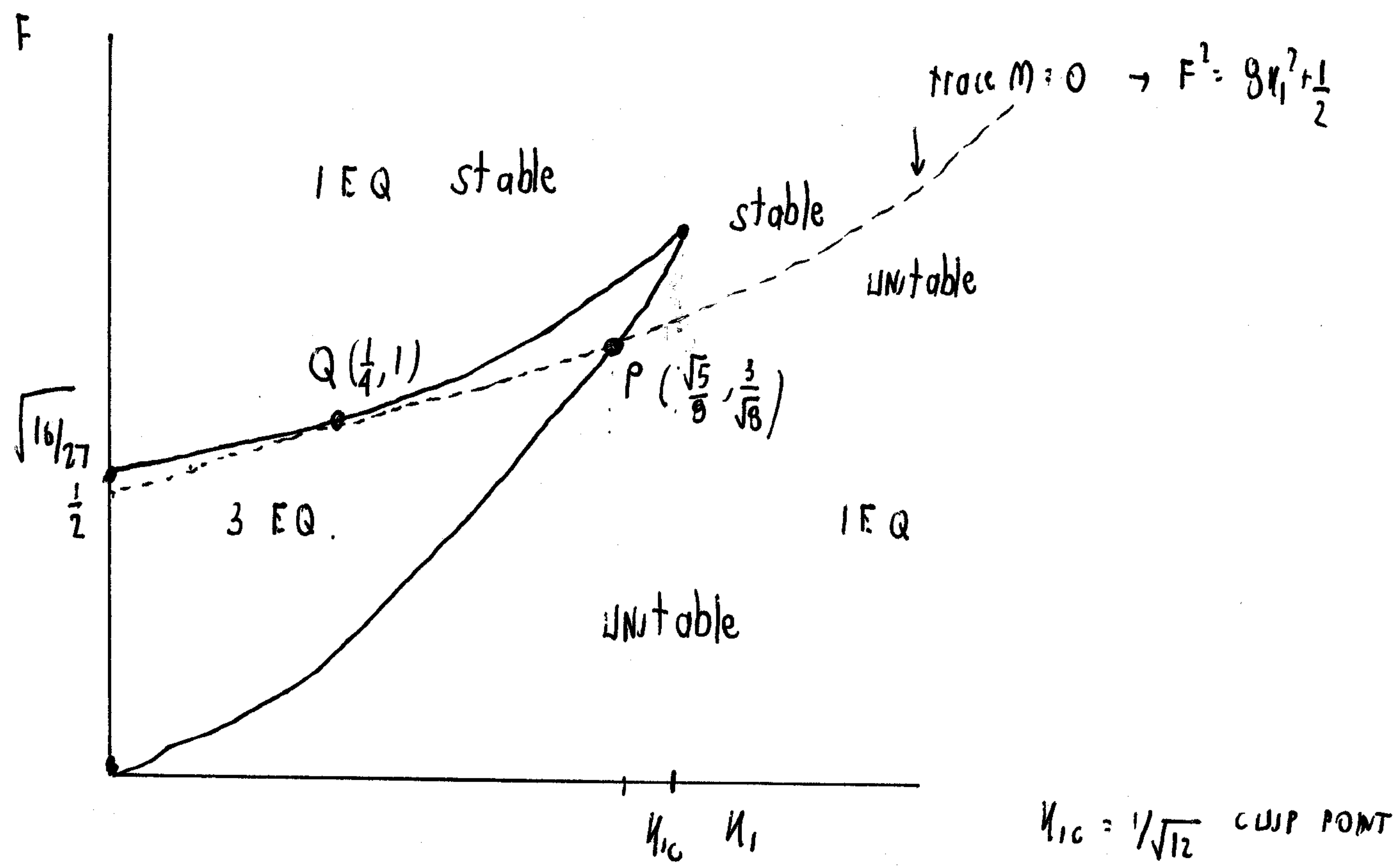
SO THAT $\zeta = 5/64 \longrightarrow u_1 = \sqrt{5}/8$ AND $\zeta = 1/16 \rightarrow u_1 = 1/4$.

THIS GIVES THE INTERSECTION POINTS

$$P \equiv (\kappa_1, F) = \left(\frac{\sqrt{5}}{8}, \frac{3}{\sqrt{8}}\right) \simeq (.2795, 1.06) \quad \text{(LOWER BRANCH)}$$

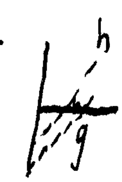
$$Q \equiv (\kappa_1, F) = \left(\frac{1}{4}, 1\right) = (.25, 1) \quad \text{(TANGENT TO UPPER BRANCH)}$$

THIS GIVES THE PICTURE



- NOW SUPPOSE WE ARE <u>OUTSIDE</u> WEDGE REGION BOUNDED BY SOLID CURVES SO THAT $\exists$ ONLY ONE EQUILIBRIUM POINT. THEN $g'(u_e) > 0$ SO $\det M_e > 0$. THEN IF $F^2 > 8\kappa_1^2 + 1/2$ (ABOVE trace $M = 0$ CURVE) WE HAVE

$$g(u) = \frac{u^3}{16} - \frac{u^2}{2} + (4\kappa_1^2 + 1)u - F^2 < \underbrace{\frac{u^3}{16} - \frac{u^2}{2} + (4\kappa_1^2+1)(u-2) + \left(u - \frac{1}{2}\right)}_{\equiv\, h(u)}$$



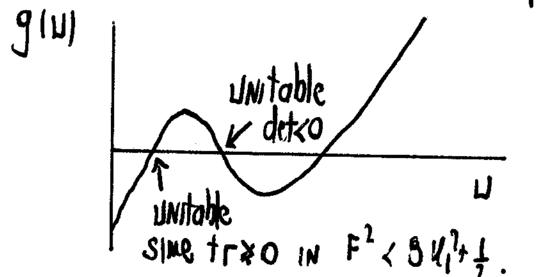
SINCE $h(2) = 0$ WE CONCLUDE THAT $g(2) < 0$. AS SUCH THE UNIQUE ROOT TO $g(u_e) = 0$ <u>MUST</u> SATISFY $u_e > 2$. BUT THIS MEANS

THAT FOR THIS ROOT WE MUST HAVE TRACE $M_e = 1 - u_e/2 < 0$.

AS SUCH WE HAVE THAT THE EQ SOLUTION IS STABLE. → <u>ENTRAINMENT!</u>

- IF $F^2 < 8K_1^2 + 1/2$ AND WE ARE IN REGION WITH A UNIQUE EQUILIBRIUM SOLUTION, THEN SINCE $\det M_e > 0$ AND $\operatorname{trace} M_e > 0$ → UNSTABLE.

NOW INSIDE THE WEDGE-SHAPED REGION ∃ THREE EQUILIBRIA TO $g(u) = 0$. WE MOREOVER HAVE $\det M_e = \frac{1}{4} g'(u_e)$



THE EQUILIBRIUM SHOWN WHERE $g'(u_e) < 0$ IS UNSTABLE. MUST HAVE ONE POSITIVE AND ONE NEGATIVE EIGENVALUE FOR M_e.

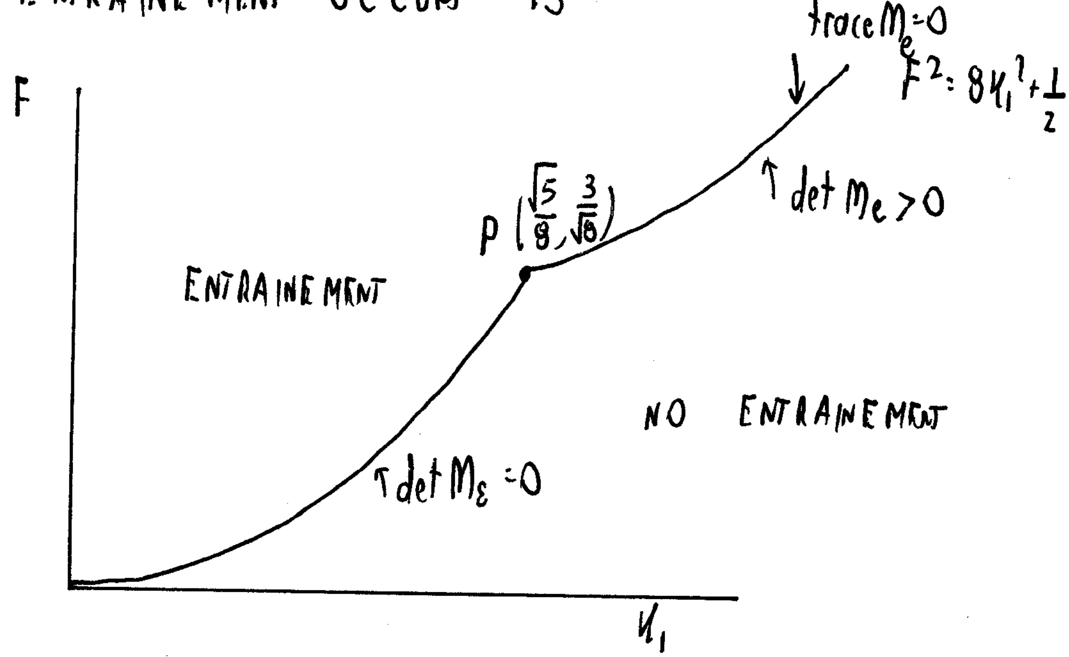
NOW IF $F^2 < 8K_1^2 + \frac{1}{2}$ SO THAT WE ARE BELOW HOPF-LINE THEN ANY EQUILIBRIUM SOLUTION WITH $u_e < 2$ SATISFIES $\operatorname{trace} M_e > 0$. SINCE

$$g(u) > h(u) = \frac{u^3}{16} - \frac{u^2}{2} + (4K_1^2+1)(u-2) + (u - 1/2) \quad \text{WITH } h(2) = 0$$

ON THIS RANGE, WE MUST HAVE THAT ∃ $u_e < 2$. IT IS UNSTABLE.

A LITTLE MORE WORK SHOWS THAT THE THIRD ROOT HAS $u_e > 2$ EVEN IF $F^2 < 8K_1^2 + \frac{1}{2}$. HENCE, IN WEDGE-SHAPED REGION THERE IS ALWAYS A STABLE EQ. SOLUTION. (BUT TWO UNSTABLE ONES).

IN SUMMARY, THE REGION IN F-K_1 PLANE WHERE $\exists$ LINEARLY STABLE EQUILIBRIA OF SLOW-FLOW ODE'S AND CONSEQUENTLY WHERE ENTRAINEMENT OCCURS IS



OBSERVE

(i) CONDITION FOR ENTRAINEMENT DEPENDS ON K_1^2 (NOT K_1) HENCE SAME AS TO WHETHER $\omega - 1 > 0$ OR $\omega - 1 < 0$ (WITH $\omega = 1 + \varepsilon K_1$)

(ii) AS $K_1 \uparrow$, i.e. THE FARTHER WE GET AWAY FROM NATURAL BUILT-IN FREQUENCY $\omega = 1$, THE LARGER WE MUST TAKE F TO OBTAIN ENTRAINED SOLUTION.

(iii) ON CURVE STARTING FROM P WE HAVE A UNIQUE STEADY-STATE OF SLOW FLOW EQUATIONS FOR WHICH trace $M_e = 0$ AND det $M_e > 0$. THIS IS A HOPF BIFURCATION AND CORRESPONDS IN THE FULL SYSTEM TO A QUASI-PERIODIC OSCILLATION WITH PERIODS $t = O(1)$ AND PERIOD $t = O(1/\varepsilon) \rightarrow$ BEAT LIKE

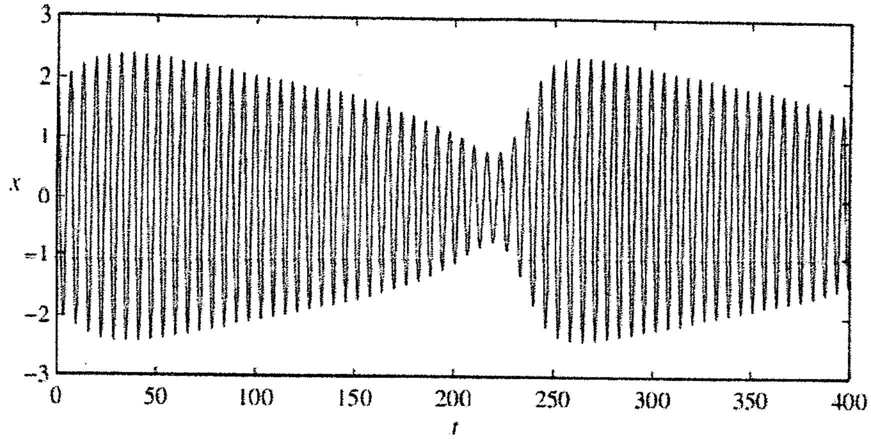


Figure 1.9: The solution to equation 1.147 with $\epsilon = 0.1$, $k_1 = 0.4$, $F = 1.2$ and with initial conditions $x(0) = 2$, $x'(0) = 0$.

F = 1.2
k_1 = 0.4
NEAR HOPF-
LINE
BEYOND P
"BEATING"
OR QUASI-PERI

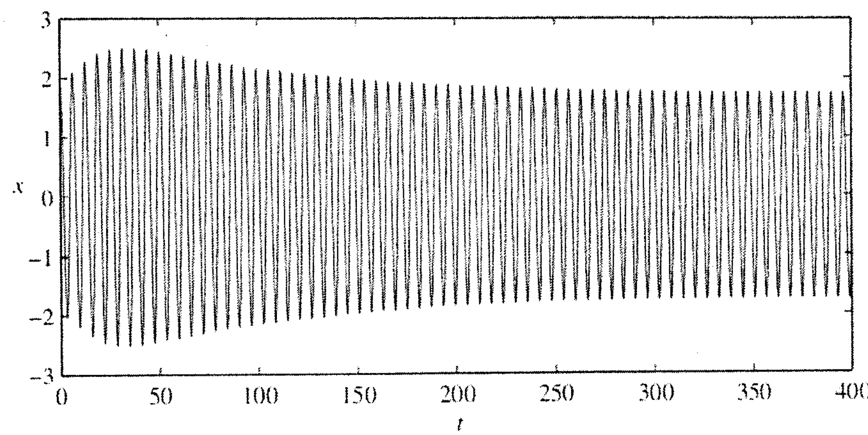


Figure 1.10: The solution to equation 1.147 with $\epsilon = 0.1$, $k_1 = 0.4$, $F = 1.5$ and with initial conditions $x(0) = 2$, $x'(0) = 0$.

k_1 = 0.4
F = 1.5
ABOVE
HOPF-LINE
"STABLE"
ENTRAINED
SOLUTION