Chapter 3

Basic Floquet Theory

3.1 General Results

If we have a problem of the form

$$\mathbf{x}' = \mathbf{A}(t)\mathbf{x} \tag{3.1}$$

where $\mathbf{A}(t)$ is periodic with period T, then $\mathbf{x}$ need not be periodic, however it must be of the form

$$e^{\mu t}\mathbf{p}(t) \tag{3.2}$$

where $\mathbf{p}(t)$ has period T. Additionally, it has n such μ_j and together they satisfy

$$e^{\mu_1 T} e^{\mu_2 T} \cdots e^{\mu_n T} = \exp\left(\int_0^T \operatorname{tr}\left(\mathbf{A}(s)\right)\, ds\right). \tag{3.3}$$

The following theorems prove those results. We follow Ward [28].

Definition (Fundamental Matrix)**.** Let $\mathbf{x}^1(t), \ldots, \mathbf{x}^n(t)$ be n solutions of $\mathbf{x}' = \mathbf{A}(t)\mathbf{x}$. Let

$$\mathbf{X}(t) = \begin{bmatrix} \begin{bmatrix} \\ \mathbf{x}^1 \\ \\ \end{bmatrix} \cdots \begin{bmatrix} \\ \mathbf{x}^n \\ \\ \end{bmatrix} \end{bmatrix} \tag{3.4}$$

so that $\mathbf{X}(t)$ is an $n \times n$ matrix solution of $\mathbf{X}' = \mathbf{A}\mathbf{X}$.

If $\mathbf{x}^1(t), \ldots, \mathbf{x}^n(t)$ are linearly independent, then $\mathbf{X}(t)$ is non-singular and is called a *fundamental matrix*. If $\mathbf{X}(t_0) = \mathbf{I}$, then $\mathbf{X}(t)$ is the *principal fundamental matrix*.

Lemma 3.1. *If* $\mathbf{X}(t)$ *is a fundamental matrix then so is* $\mathbf{Y}(t) = \mathbf{X}(t)\mathbf{B}$ *for any non-singular constant matrix* $\mathbf{B}$.

Proof. Since $\mathbf{X}(t)$ and $\mathbf{B}$ are non-singular then the inverse of $\mathbf{Y}(t)$ is $\mathbf{B}^{-1}\mathbf{X}^{-1}(t)$ and so $\mathbf{Y}(t)$ is non-singular. Also,

$$\mathbf{Y}' = \mathbf{X}'\mathbf{B} = \mathbf{A}\mathbf{X}\mathbf{B} = \mathbf{A}\mathbf{Y} \tag{3.5}$$

so that $\mathbf{Y}'(t) = \mathbf{A}\mathbf{Y}(t)$. □

Lemma 3.2. *Let the* Wronskian $W(t)$ *of* $\mathbf{X}(t)$ *be the determinant of* $\mathbf{X}(t)$. *Then*

$$W(t) = W(t_0) \exp\left(\int_{t_0}^{t} \operatorname{tr}(\mathbf{A}(s))\, ds\right). \tag{3.6}$$

Proof. Let t_0 be some time. Expanding in a Taylor series,

$$\mathbf{X}(t) = \mathbf{X}(t_0) + (t-t_0)\,\mathbf{X}'(t_0) + O\left((t-t_0)^2\right) \tag{3.7}$$

$$= \mathbf{X}(t_0) + (t-t_0)\,\mathbf{A}(t_0)\mathbf{X}(t_0) + O\left((t-t_0)^2\right) \tag{3.8}$$

$$= [\mathbf{I} + (t-t_0)\,\mathbf{A}(t_0)]\,\mathbf{X}(t_0) + O\left((t-t_0)^2\right) \tag{3.9}$$

so that

$$\det(\mathbf{X}(t)) = \det[\mathbf{I} + (t-t_0)\,\mathbf{A}(t_0)]\det(\mathbf{X}(t_0)) \tag{3.10}$$

$$W(t) = \det[\mathbf{I} + (t-t_0)\,\mathbf{A}(t_0)]\,W(t_0). \tag{3.11}$$

Now since

$$\det(\mathbf{I} + \epsilon\mathbf{C}) = 1 + \epsilon \operatorname{tr}(\mathbf{C}) + O\left(\epsilon^2\right), \tag{3.12}$$

we have that

$$W(t) = W(t_0)\left(1 + (t-t_0)\operatorname{tr}(\mathbf{A}(t_0))\right). \tag{3.13}$$

Now by expanding $W(t)$ in a Taylor series, we obtain that

$$W(t) = W(t_0) + (t-t_0)\,W'(t_0) + O\left((t-t_0)^2\right) \tag{3.14}$$

so that

$$W'(t_0) = W(t_0)\operatorname{tr}(\mathbf{A}(t_0)). \tag{3.15}$$

Since we have not made any assumptions about t_0, we can the write

$$W'(t) = W(t)\operatorname{tr}(\mathbf{A}(t)). \tag{3.16}$$

We know that the solution to this equation is

$$W(t) = W(t_0) \exp\left(\int_{t_0}^{t} \operatorname{tr}(\mathbf{A}(s))\, ds\right) \tag{3.17}$$

□

Theorem 3.3. *Let* $\mathbf{A}(t)$ *be a* T*-periodic matrix. If* $\mathbf{X}(t)$ *is a fundamental matrix then so is* $\mathbf{X}(t+T)$ *and there exists a non-singular constant matrix* $\mathbf{B}$ *such that*

i. $\mathbf{X}(t+T) = \mathbf{X}(t)\mathbf{B}$ *for all* t

ii. $\det(\mathbf{B}) = \exp\left(\int_0^T \operatorname{tr}(\mathbf{A}(s))\, ds\right)$

Proof. Begin by showing that $\mathbf{X}(t+T)$ is also a fundamental matrix. Let $\mathbf{Y}(t) = \mathbf{X}(t+T)$. Then

$$\mathbf{Y}'(t) = \mathbf{X}'(t+T) = \mathbf{A}(t+T)\mathbf{X}(t+T) = \mathbf{A}(t)\mathbf{X}(t+T) = \mathbf{A}(t)\mathbf{Y}(t) \tag{3.18}$$

and so $\mathbf{X}(t+T)$ is a fundamental matrix.

i. Let $\mathbf{B}(t) = \mathbf{X}^{-1}(t)\mathbf{Y}(t)$. Then

$$\mathbf{Y}(t) = \mathbf{X}(t)\mathbf{X}^{-1}(t)\mathbf{Y}(t) \tag{3.19}$$

$$= \mathbf{X}(t)\mathbf{B}(t) \tag{3.20}$$

Let $\mathbf{B}_0 = \mathbf{B}(t_0)$. We know by lemma 3.1 that $\mathbf{Y}_0(t) = \mathbf{X}(t)\mathbf{B}_0$ is a fundamental matrix, where, by definition, $\mathbf{Y_0}(t_0) = \mathbf{Y}(t_0)$. Since these are both solutions to $\mathbf{X}' = \mathbf{AX}$, by the uniqueness of the solution, we must then have $\mathbf{Y}_0(t) = \mathbf{Y}(t)$ for all time. As a result, $\mathbf{B}_0 = \mathbf{B}(t)$ and so $\mathbf{B}$ is time-independent.

ii. From lemma 3.2, we have that

$$W(t) = W(t_0)\exp\left(\int_{t_0}^{t} \operatorname{tr}(\mathbf{A}(s))\, ds\right) \tag{3.21}$$

$$W(t+T) = W(t_0)\exp\left(\int_{t_0}^{t} \operatorname{tr}(\mathbf{A}(s))\, ds + \int_{t}^{t+T} \operatorname{tr}(\mathbf{A}(s))\, ds\right) \tag{3.22}$$

$$W(t+T) = W(t)\exp\left(\int_{t}^{t+T} \operatorname{tr}(\mathbf{A}(s))\, ds\right) \tag{3.23}$$

$$W(t+T) = W(t)\exp\left(\int_{0}^{T} \operatorname{tr}(\mathbf{A}(s))\, ds\right). \tag{3.24}$$

We also know that

$$\mathbf{X}(t+T) = \mathbf{X}(t)\mathbf{B} \tag{3.25}$$

$$\det(\mathbf{X}(t+T)) = \det(\mathbf{X}(t))\det(\mathbf{B}) \tag{3.26}$$

$$W(t+T) = W(t)\det(\mathbf{B}) \tag{3.27}$$

and so

$$\det(\mathbf{B}) = \exp\left(\int_{0}^{T} \operatorname{tr}(\mathbf{A}(s))\, ds\right) \tag{3.28}$$

□

Remark. Since $\mathbf{B}$ is time-independent, it can be computed by setting $t = 0$, so that $\mathbf{B} = \mathbf{X}^{-1}(0)\mathbf{X}(T)$. If we took the initial conditions $\mathbf{X}(0) = \mathbf{I}$, then $\mathbf{B} = \mathbf{X}(T)$.

Definition (Characteristic Multipliers and Exponents)**.** The eigenvalues $\rho_1, \ldots, \rho_n$ of $\mathbf{B}$ are called the *characteristic multipliers* for $\mathbf{X}'(t) = \mathbf{A}(t)\mathbf{X}(t)$. The *characteristic exponents* or *Floquet exponents* are $\mu_1, \ldots, \mu_n$ satisfying

$$\rho_1 = e^{\mu_1 T}, \qquad \rho_2 = e^{\mu_2 T}, \qquad \ldots \qquad \rho_n = e^{\mu_n T}. \tag{3.29}$$

Note that μ_j for $j \in \mathbb{N}$ may be complex.

Properties.

i. The characteristic multipliers (eigenvalues) $\rho_1, \ldots, \rho_n$ of $\mathbf{B} = \mathbf{X}(T)$ with $\mathbf{X}(0) = \mathbf{I}$ satisfy

$$\det(\mathbf{B}) = \rho_1 \rho_2 \cdots \rho_n = \exp\left(\int_0^T \operatorname{tr}(\mathbf{A}(s))\, ds \right). \tag{3.30}$$

This follows from theorem 3.3ii.

ii. Since the trace is the sum of the eigenvalues, we also have

$$\operatorname{tr}(\mathbf{B}) = \rho_1 + \rho_2 + \cdots + \rho_n. \tag{3.31}$$

iii. The characteristic exponents are not unique since if $\rho_j = e^{\mu_j T}$, then $\rho_j = e^{(\mu_j + 2\pi i/T)T}$.

iv. The characteristic multipliers ρ_j are an intrinsic property of the equation $\mathbf{X}'(t) = \mathbf{A}\mathbf{X}$ and do not depend on the choice of the fundamental matrix.

Proof. Suppose $\hat{\mathbf{X}}(t)$ is another fundamental matrix. Then

$$\hat{\mathbf{X}}(t+T) = \hat{\mathbf{X}}(t)\hat{\mathbf{B}}. \tag{3.32}$$

We have showed in the proof of theorem 3.3 that since $\mathbf{X}(t)$ and $\hat{\mathbf{X}}(t)$ are fundamental matrices then there is a constant non-singular matrix $\mathbf{C}$ such that

$$\hat{\mathbf{X}}(t) = \mathbf{X}(t)\mathbf{C} \tag{3.33}$$

so that

$$\hat{\mathbf{X}}(t+T) = \mathbf{X}(t+T)\mathbf{C} \tag{3.34}$$

$$\left(\hat{\mathbf{X}}(t)\hat{\mathbf{B}}\right) = \left(\mathbf{X}(t)\mathbf{B}\right)\mathbf{C} \tag{3.35}$$

$$\mathbf{X}(t)\mathbf{C}\hat{\mathbf{B}} = \mathbf{X}(t)\mathbf{B}\mathbf{C} \tag{3.36}$$

$$\mathbf{C}\hat{\mathbf{B}} = \mathbf{B}\mathbf{C} \tag{3.37}$$

$$\mathbf{C}\hat{\mathbf{B}}\mathbf{C}^{-1} = \mathbf{B} \tag{3.38}$$

so the eigenvalues of $\mathbf{B}$ and $\hat{\mathbf{B}}$ are the same. □

Theorem 3.4. *Let ρ be a characteristic multiplier and let μ be the corresponding characteristic exponent so that $\rho = e^{\mu T}$. Then there exists a solution $\mathbf{x}(t)$ of $\mathbf{x}' = \mathbf{A}(t)\mathbf{x}$ such that*

i. $\mathbf{x}(t+T) = \rho\mathbf{x}(t)$

ii. There exists a periodic solution $\mathbf{p}(t)$ with period T such that $\mathbf{x}(t) = e^{\mu t}\mathbf{p}(t)$.

Proof.

i. Let $\mathbf{b}$ be an eigenvector of $\mathbf{B}$ corresponding to eigenvalue ρ. Let $\mathbf{x}(t) = \mathbf{X}(t)\mathbf{b}$. Then $\mathbf{x}' = \mathbf{A}\mathbf{x}$ and

$$\mathbf{x}(t+T) = \mathbf{X}(t+T)\mathbf{b} \tag{3.39}$$
$$= \mathbf{X}(t)\mathbf{B}\mathbf{b} \tag{3.40}$$
$$= \rho\mathbf{X}(t)\mathbf{b} \tag{3.41}$$
$$= \rho\mathbf{x}(t) \tag{3.42}$$

so that $\mathbf{x}(t+T) = \rho\mathbf{x}(t)$.

ii. Let $\mathbf{p}(t) = \mathbf{x}(t)e^{-\mu t}$. We now need to show that $\mathbf{p}(t)$ is T-periodic.

$$\mathbf{p}(t+T) = \mathbf{x}(t+T)e^{-\mu(t+T)} \tag{3.43}$$
$$= \rho\mathbf{x}(t)e^{-\mu(t+T)} \tag{3.44}$$
$$= \frac{\rho}{e^{\mu T}}\mathbf{x}(t)e^{-\mu t} \tag{3.45}$$
$$= \mathbf{x}(t)e^{-\mu t} \tag{3.46}$$
$$= \mathbf{p}(t) \tag{3.47}$$

As a result, we have a solution of the form $\mathbf{x}(t) = e^{\mu t}\mathbf{p}(t)$ where $\mathbf{p}(t)$ is periodic with period T. □

Remarks.

i. If μ is replaced by $\mu + 2\pi i/T$, then we get

$$\mathbf{x}(t) = e^{\mu t}\mathbf{p}(t)e^{2\pi i t/T} \tag{3.48}$$

where $\mathbf{p}(t)e^{2\pi i t/T}$ is still periodic with period T. As a result, the fact that μ is not unique does not alter our results.

ii. We have that

$$\mathbf{x}_j(t+T) = \rho_j\mathbf{x}_j(t) \tag{3.49}$$
$$\mathbf{x}_j(t+NT) = \rho_j^N\mathbf{x}_j(t). \tag{3.50}$$

Each characteristic multipliers falls into one of the following categories:

(a) If $|\rho| < 1$, then $\mathrm{Re}\,(\mu) < 0$ and so $\mathbf{x}(t) \xrightarrow{t\to\infty} 0$.

(b) If $|\rho| = 1$, then $\mathrm{Re}\,(\mu) = 0$ and so we have a pseudo-periodic solution. If $\rho = \pm 1$, then the solution is periodic with period T.

(c) If $|\rho| > 1$, then $\mathrm{Re}\,(\mu) > 0$ and so $\mathbf{x}(t) \rightsquigarrow \infty$ as $t \to \infty$.

The entire solution is stable if all the characteristic multipliers satisfy $|\rho_j| \leq 1$.

iii. As for the general solution, suppose that $\mathbf{b}_1, \ldots, \mathbf{b}_n$ are n linearly independent eigenvectors of $\mathbf{B}$ corresponding to distinct eigenvalues $\rho_1, \ldots, \rho_n$. Then there are n linearly independent solutions to $\mathbf{x}' = \mathbf{A}\mathbf{x}$, which by the above theorem are given by

$$\mathbf{x}_j(t) = e^{\mu_j t}\mathbf{p}_j(t) \tag{3.51}$$

where $\mathbf{p}_j(t)$ is T-periodic. As a result, we can define

$$\mathbf{X}_0(t) = \begin{bmatrix} \begin{bmatrix} \\ \mathbf{x}_1 \\ \\ \end{bmatrix} \cdots \begin{bmatrix} \\ \mathbf{x}_n \\ \\ \end{bmatrix} \end{bmatrix}, \quad \mathbf{P}_0(t) = \begin{bmatrix} \begin{bmatrix} \\ \mathbf{p}_1 \\ \\ \end{bmatrix} \cdots \begin{bmatrix} \\ \mathbf{p}_n \\ \\ \end{bmatrix} \end{bmatrix}, \tag{3.52}$$

$$\mathbf{D}_0(t) = \begin{bmatrix} \mu_1 & & 0 \\ & \ddots & \\ 0 & & \mu_n \end{bmatrix}, \quad \mathbf{Y}_0(t) = \begin{bmatrix} e^{\mu_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\mu_n t} \end{bmatrix}, \tag{3.53}$$

such that

$$\mathbf{X}_0 = \mathbf{P}_0\mathbf{Y}_0, \qquad \mathbf{Y}_0' = \mathbf{D}_0\mathbf{Y}_0. \tag{3.54}$$

iv. Now consider what happens if $\rho < 0$. Suppose $\rho < 0$ real, so that we can write

$$\rho = e^{(\nu + i\pi/T)T} \tag{3.55}$$

where

$$\rho = -e^{\nu T}. \tag{3.56}$$

Then we obtain

$$\mathbf{x}(t) = e^{\mu t}\mathbf{p}(t) \tag{3.57}$$

$$= e^{\nu t}e^{i\pi t/T}\mathbf{p}(t) \tag{3.58}$$

$$= e^{\nu t}\mathbf{q}(t), \tag{3.59}$$

where $\mathbf{q}(t)$ has period T since $\mathbf{p}(t)$ has period T. Since we can choose $\mathbf{x}$ to be real, without loss of generality, we can also choose $\mathbf{q}$ to be real. For the general solution, if $\rho_j < 0$, we can replace $\mathbf{p}_j$ with $\mathbf{q}_j$ and μ_j with ν_j so that

$$\mathbf{P}_0 = \begin{bmatrix} \begin{bmatrix} \\ \mathbf{p}_1 \\ \\ \end{bmatrix} \cdots \begin{bmatrix} \\ \mathbf{q}_j \\ \\ \end{bmatrix} \cdots \begin{bmatrix} \\ \mathbf{p}_n \\ \\ \end{bmatrix} \end{bmatrix}, \quad \mathbf{Y}_0 = \begin{bmatrix} e^{\mu_1 T} & & & & 0 \\ & \ddots & & & \\ & & e^{\nu_j T} & & \\ & & & \ddots & \\ 0 & & & & e^{\mu_n T} \end{bmatrix} \tag{3.60}$$

and

$$\mathbf{X}_0(t) = \mathbf{P}_0(t)\mathbf{Y}_0(t). \tag{3.61}$$

v. Suppose now that ρ is complex. Then since ρ is an eigenvalue of the real matrix $\mathbf{B}$, $\overline{\rho}$ is as well. The characteristic exponents are μ and $\overline{\mu}$. Let

$$\mu = \nu + i\sigma, \qquad \mathbf{p}(t) = \mathbf{q}(t) + i\mathbf{r}(t) \tag{3.62}$$

where $\mathbf{q}(t)$ and $\mathbf{r}(t)$ must both have period T since $\mathbf{p}(t)$ does. Since $\mathbf{x}(t) = e^{\mu t}\mathbf{p}(t)$ is a solution to $\mathbf{x}' = \mathbf{A}(t)\mathbf{x}$, then by taking the complex conjugate, so is $\overline{\mathbf{x}}(t) = e^{\overline{\mu}t}\overline{\mathbf{p}}(t)$. We can write these as

$$\mathbf{x}(t) = e^{(\nu+i\sigma)t}\left(\mathbf{q}(t) + i\mathbf{r}(t)\right) \tag{3.63}$$
$$= e^{\nu t}\left[\left(\mathbf{q}\cos(\sigma t) - \mathbf{r}\sin(\sigma t)\right) + i\left(\mathbf{r}\cos(\sigma t) + \mathbf{q}\sin(\sigma t)\right)\right] \tag{3.64}$$

and

$$\overline{\mathbf{x}}(t) = e^{(\nu-i\sigma)t}\left(\mathbf{q}(t) - i\mathbf{r}(t)\right) \tag{3.65}$$
$$= e^{\nu t}\left[\left(\mathbf{q}\cos(\sigma t) - \mathbf{r}\sin(\sigma t)\right) - i\left(\mathbf{r}\cos(\sigma t) + \mathbf{q}\sin(\sigma t)\right)\right]. \tag{3.66}$$

We can alternately write the linearly independent real solutions

$$\mathbf{x}_R = \mathrm{Re}\left[e^{\mu t}\mathbf{p}(t)\right] = e^{\nu t}\left[\cos(\sigma t)\,\mathbf{q}(t) - \sin(\sigma t)\,\mathbf{r}(t)\right], \tag{3.67}$$
$$\mathbf{x}_I = \mathrm{Im}\left[e^{\mu t}\mathbf{p}(t)\right] = e^{\nu t}\left[\sin(\sigma t)\,\mathbf{q}(t) + \cos(\sigma t)\,\mathbf{r}(t)\right], \tag{3.68}$$

so that

$$\mathbf{X}_0 = \left[\begin{bmatrix} \\ \mathbf{x}_1 \\ \\ \end{bmatrix} \cdots \begin{bmatrix} \\ \mathbf{x}_R \\ \\ \end{bmatrix} \begin{bmatrix} \\ \mathbf{x}_I \\ \\ \end{bmatrix} \cdots \begin{bmatrix} \\ \mathbf{x}_n \\ \\ \end{bmatrix}\right], \tag{3.69}$$

$$\mathbf{P}_0 = \left[\begin{bmatrix} \\ \mathbf{p}_1 \\ \\ \end{bmatrix} \cdots \begin{bmatrix} \\ \mathbf{q} \\ \\ \end{bmatrix} \begin{bmatrix} \\ \mathbf{r} \\ \\ \end{bmatrix} \cdots \begin{bmatrix} \\ \mathbf{p}_n \\ \\ \end{bmatrix}\right], \tag{3.70}$$

$$\mathbf{Y}_0 = \begin{bmatrix} e^{\mu_1 T} & & & & 0 \\ & \ddots & & & \\ & & e^{\nu t}\cos(\sigma t) & e^{\nu t}\sin(\sigma t) & \\ & & -e^{\nu t}\sin(\sigma t) & e^{\nu t}\cos(\sigma t) & \\ & & & \ddots & \\ 0 & & & & e^{\mu_n T} \end{bmatrix} \tag{3.71}$$

and

$$\mathbf{X}_0(t) = \mathbf{P}_0(t)\mathbf{Y}_0(t). \tag{3.72}$$

3.1.1 Example

For example, consider

$$x_1' = \left(1 + \frac{\cos(t)}{2 + \sin(t)}\right)x_1 \tag{3.73}$$
$$x_2' = x_1 - x_2. \tag{3.74}$$

Here, we know that the solution is in general

$$x_1 = c_1 e^t \left(2 + \sin(t)\right) \tag{3.75}$$

$$x_2 = c_1 e^t \left(2 + \frac{1}{2}\sin(t) - \frac{1}{2}\cos(t)\right) + c_2 e^{-t} \tag{3.76}$$

which we can write as

$$\mathbf{x} = c_1 e^t \begin{bmatrix} 2 + \sin(t) \\ 2 + \frac{1}{2}\sin(t) - \frac{1}{2}\cos(t) \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{3.77}$$

Using all the above definitions, the fundamental matrix is

$$\mathbf{X}(t) = \begin{bmatrix} e^t \left(2 + \sin(t)\right) & 0 \\ e^t \left(2 + \frac{1}{2}\sin(t) - \frac{1}{2}\cos(t)\right) & e^{-t} \end{bmatrix} \tag{3.78}$$

so that

$$\mathbf{B} = \mathbf{X}^{-1}(0)\mathbf{X}(2\pi) \tag{3.79}$$

$$= \begin{bmatrix} 2 & 0 \\ \frac{3}{2} & 1 \end{bmatrix}^{-1} \begin{bmatrix} 2e^{2\pi} & 0 \\ \frac{3}{2}e^{2\pi} & e^{-2\pi} \end{bmatrix} \tag{3.80}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 0 \\ -\frac{3}{2} & 2 \end{bmatrix} \begin{bmatrix} 2e^{2\pi} & 0 \\ \frac{3}{2}e^{2\pi} & e^{-2\pi} \end{bmatrix} \tag{3.81}$$

$$= \begin{bmatrix} e^{2\pi} & 0 \\ 0 & e^{-2\pi} \end{bmatrix} \tag{3.82}$$

As a result $\rho_1 = e^{2\pi}$, $\rho_2 = e^{-2\pi}$ and so $\mu_1 = 1$ and $\mu_2 = -1$. Theorem 3.4 then tells us that there is a solution of the form

$$\mathbf{x}_1(t) = e^t \mathbf{p}_1(t), \qquad \mathbf{x}_2(t) = e^{-t}\mathbf{p}_2(t) \tag{3.83}$$

where $\mathbf{p}_1(t)$ and $\mathbf{p}_2(t)$ are periodic with period 2π. We know that in fact

$$\mathbf{p}_1(t) = \begin{bmatrix} 2 + \sin(t) \\ 2 + \frac{1}{2}\sin(t) - \frac{1}{2}\cos(t) \end{bmatrix}, \qquad \mathbf{p}_2(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{3.84}$$

3.1.2 Periodic Solution

Consider a problem of the form $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ with $\mathbf{x} \in \mathbb{R}^n$ where there is a periodic solution $\mathbf{x}(t) = \boldsymbol{\phi}(t)$ with period T. Linearise the solution about $\boldsymbol{\phi}$ by writing $\mathbf{x} = \boldsymbol{\phi} + \mathbf{v}$. We then obtain

$$\mathbf{v}' = \mathbf{A}(t)\mathbf{v} \tag{3.85}$$

where $\mathbf{A}(t)$ is the Jacobian of f (so $A_{ij}(t) = \left.\frac{\partial f_i}{\partial x_j}\right|_{\boldsymbol{\phi}(t)}$). Since $\boldsymbol{\phi}(t)$ has period T, so does $\mathbf{A}(t)$. Now let $\mathbf{X}(t)$ be the principal fundamental matrix of $\mathbf{v}' = \mathbf{A}\mathbf{v}$ (so that $\mathbf{X}(0) = \mathbf{I}$). Then $\mathbf{B} = \mathbf{X}(T)$.

Now by definition,

$$\boldsymbol{\phi}'(t) = \mathbf{f}(\boldsymbol{\phi}(t)) \tag{3.86}$$

so

$$\boldsymbol{\phi}''(t) = \left.\frac{\partial f_i}{\partial x_j}\right|_{\boldsymbol{\phi}(t)} \boldsymbol{\phi}'(t) \tag{3.87}$$

$$\boldsymbol{\phi}''(t) = \mathbf{A}(t)\boldsymbol{\phi}'(t) \tag{3.88}$$

If we let $\mathbf{v} = \boldsymbol{\phi}'$, then

$$\mathbf{v}'(t) = \mathbf{A}(t)\mathbf{v}(t) \tag{3.89}$$

where, since $\boldsymbol{\phi}(t)$ has period T by assumption, $\mathbf{v}(t)$ must also, and so the corresponding characteristic multiplier is 1. As a result, for a nonlinear system with a periodic solution, one characteristic multiplier is always $\rho = 1$.

3.2 General Results for n = 2

3.2.1 Stability of Periodic Solution

Consider a problem of the form $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ with $\mathbf{x} \in \mathbb{R}^2$ where there is a periodic solution $\mathbf{x}(t) = \boldsymbol{\phi}(t)$ with period T. We know from §3.1.2 that we must have $\rho_1 = 1$ and we know from theorem 3.3ii that

$$\rho_1\rho_2 = \exp\left(\int_0^T \operatorname{tr}(\mathbf{A}(s))\, ds\right) \tag{3.90}$$

$$\rho_2 = \exp\left(\int_0^T \operatorname{tr}(\mathbf{A}(s))\, ds\right). \tag{3.91}$$

From remark (ii) on page 53, we know that for the perturbation to be bounded and hence for the solution to be stable, we must have $\rho_1 \le 1$ and $\rho_2 \le 1$ and so, since we know $\rho_1 = 1$ and we wish ρ_1 and ρ_2 to be distinct, we must have

$$0 > \int_0^T \operatorname{tr}(\mathbf{A}(s))\, ds \tag{3.92}$$

$$0 > \int_0^T \operatorname{tr}\left(\left.\frac{\partial f_i}{\partial x_j}\right|_{\boldsymbol{\phi}(s)}\right) ds \tag{3.93}$$

$$0 > \int_0^T \left.\left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}\right)\right|_{\boldsymbol{\phi}(s)} ds \tag{3.94}$$

$$0 > \int_0^T \nabla \cdot \mathbf{f}|_{\mathbf{x}=\boldsymbol{\phi}}\, ds. \tag{3.95}$$

We get instability when

$$0 < \int_0^T \nabla \cdot \mathbf{f}|_{\mathbf{x}=\boldsymbol{\phi}}\, ds. \tag{3.96}$$

3.2.2 Example

Consider

$$x' = x - y - x\left(x^2 + y^2\right) \tag{3.97}$$
$$y' = x + y - y\left(x^2 + y^2\right). \tag{3.98}$$

Let

$$x = r(t)\cos(\theta(t)) \tag{3.99}$$
$$y = r(t)\sin(\theta(t)) \tag{3.100}$$

so that our problem becomes

$$\sin(\theta)(r - r\theta') = \cos(\theta)\left(r - r^3 - r'\right) \tag{3.101}$$
$$\cos(\theta)(r - r\theta') = -\sin(\theta)\left(r - r^3 - r'\right). \tag{3.102}$$

By squaring and adding these equations, we obtain that

$$(r - r\theta')^2 = \left(r - r^3 - r'\right)^2 \tag{3.103}$$

so we can write

$$a = r - r\theta' \tag{3.104}$$
$$sa = r - r^3 - r' \tag{3.105}$$

where $s = \pm 1$. Our equations then become

$$a\sin(\theta) = sa\cos(\theta) \tag{3.106}$$
$$a\cos(\theta) = -sa\sin(\theta) \tag{3.107}$$

which can be rewritten as

$$a\sin(\theta) = sa\cos(\theta) \tag{3.108}$$
$$-s^2 a\sin(\theta) = sa\cos(\theta) \tag{3.109}$$

so that we must have

$$a\sin(\theta) = -a\sin(\theta) \tag{3.110}$$
$$a\sin(\theta) = 0. \tag{3.111}$$

As a result, we have that

$$a\sin(\theta) = sa\cos(\theta) = 0 \tag{3.112}$$

so that we must have $a = 0$. This means that

$$r - r\theta' = r - r^3 - r' = 0. \tag{3.113}$$

We have that

$$r' = r\left(1 - r^3\right) \tag{3.114}$$

and so we have a solution of constant radius when $r = 0$ (the trivial case) and $r = \pm 1$. Without loss of generality, choose $r = 1$. Then since

$$r\theta' = r, \tag{3.115}$$

we have that $\theta' = 1$, so $\theta = t + C$. As a result, our solution has period $T = 2\pi$.

Now

$$\begin{aligned}
\nabla \cdot \mathbf{f}|_{r=1} &= \left[\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y}\right]_{r=1} && (3.116)\\
&= \left[\left(1 - 3x^2 - y^2\right) + \left(1 - x^2 - 3y^2\right)\right]_{r=1} && (3.117)\\
&= \left[2 - 4r^2\right]_{r=1} && (3.118)\\
&= -2 && (3.119)
\end{aligned}$$

so that

$$\begin{aligned}
\rho_2 &= \exp\left(\int_0^T \operatorname{tr}\left(\mathbf{A}(s)\right)\, ds\right) && (3.120)\\
&= \exp\left(\int_0^{2\pi} -2\, ds\right) && (3.121)\\
&= e^{-4\pi} && (3.122)\\
&< 1. && (3.123)
\end{aligned}$$

As a result, the limit cycle with radius $r = 1$ is stable.

3.2.3 Stability of Second-Order ODE

Consider the second-order ODE

$$x'' + a(t)x = 0 \tag{3.124}$$

where $a(t)$ is periodic with period T. Letting $x_1 = x$ and $x_2 = x_1'$, this can be rewritten as

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a(t) & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{3.125}$$

By choosing the initial condition

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{3.126}$$

we obtain a solution of the form

$$\begin{bmatrix} x_1^{(1)}(t) \\ x_1'^{(1)}(t) \end{bmatrix}. \tag{3.127}$$

Likewise by choosing the initial condition

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{3.128}$$

we obtain a solution of the form

$$\begin{bmatrix} x_1^{(2)}(t) \\ x_1'^{(2)}(t) \end{bmatrix}. \tag{3.129}$$

As a result, we have chosen $\mathbf{X}(0) = \mathbf{I}$ so that

$$\mathbf{B} = \mathbf{X}(T) = \begin{bmatrix} x_1^{(1)}(T) & x_1^{(2)}(T) \\ x_1'^{(1)}(T) & x_1'^{(2)}(T) \end{bmatrix}. \tag{3.130}$$

Now we have from property (i) on page 52 that

$$\rho_1\rho_2 = \exp\left(\int_0^T \operatorname{tr}\left(\mathbf{A}(s)\right)\, ds\right) \tag{3.131}$$

$$= \exp\left(\int_0^T 0\, ds\right) \tag{3.132}$$

$$= 1 \tag{3.133}$$

and from property (ii) that

$$\rho_1 + \rho_2 = \operatorname{tr}\left(\mathbf{B}\right) \tag{3.134}$$

$$= x_1^{(1)}(T) + x_1'^{(2)}(T). \tag{3.135}$$

Let $\phi = \operatorname{tr}\left(\mathbf{B}\right)/2$ so that

$$\rho_1\rho_2 = 1 \tag{3.136}$$

$$\rho_1 + \rho_2 = 2\phi. \tag{3.137}$$

Solving these, we obtain that

$$\rho = \phi \pm \sqrt{\phi^2 - 1}. \tag{3.138}$$

We can rewrite ρ_i as $\exp(\mu_i T)$, so that

$$\mu_1 + \mu_2 = 0 \tag{3.139}$$

and so

$$e^{\mu_1 T} + e^{\mu_2 T} = 2\phi \tag{3.140}$$

$$e^{\mu_1 T} + e^{-\mu_1 T} = 2\phi \tag{3.141}$$

$$\frac{e^{\mu_1 T} + e^{-\mu_1 T}}{2} = \phi \tag{3.142}$$

$$\cosh\left(\mu_1 T\right) = \phi. \tag{3.143}$$

Consider the following cases.

I. Let $-1 < \phi < 1$. We can then define σ by $\phi = \cos(\sigma T)$, where, without loss of generality, $0 < \sigma T < \pi$, so that

$$\rho = \phi \pm \sqrt{\phi^2 - 1} \tag{3.144}$$
$$= \cos(\sigma T) \pm i \sin(\sigma T) \tag{3.145}$$
$$= e^{\pm i\sigma T} \tag{3.146}$$

As in remark (v) on page 55, we can write the general solution as

$$\mathbf{x}(t) = c_1 \operatorname{Re}\left(e^{i\sigma t}\mathbf{p}(t)\right) + c_2 \operatorname{Im}\left(e^{i\sigma t}\mathbf{p}(t)\right) \tag{3.147}$$

and since $|\rho_1| = 1$ and $|\rho_2| = 1$, then from remark (ii) on page 53, the solution is stable and pseudo-periodic.

Now $e^{i\sigma t}$ has period $\hat{T} = \frac{2\pi}{\sigma}$. Now since $\phi \neq 1$ and $\phi \neq -1$, we must have

$$\sigma T \neq m\pi \tag{3.148}$$
$$\frac{2\pi}{\hat{T}} T \neq m\pi \tag{3.149}$$
$$\frac{2T}{m} \neq \hat{T} \tag{3.150}$$

so that $\hat{T} \neq 2T, T, \frac{2}{3}T, \ldots$

Note that for $\hat{T}$ to equal nT, we must have

$$\sigma = \frac{2\pi}{nT} \tag{3.151}$$

for $n \neq 1, 2$ from above.

II. Let $\phi > 1$. Then since $\rho = \phi \pm \sqrt{\phi^2 - 1}$, we must have $\rho_1 > 1$ and since $\rho_1 \rho_2 = 1$, we must have $\rho_1 > 1 > \rho_2 > 0$ and $\rho_2 = \frac{1}{\rho_1}$ means $\mu_2 = -\mu_1$. Our solution must therefore be of the form

$$\mathbf{x}(t) = c_1 e^{\mu_1 t}\mathbf{p}_1(t) + c_2 e^{-\mu_1 t}\mathbf{p}_2(t) \tag{3.152}$$

where $\mathbf{p}_1(t)$ and $\mathbf{p}_2(t)$ are both periodic with period T. As a result, the solution is unstable.

III. Let $\phi = 1$. Then $\rho_1 = \rho_2 = 1$. Here, theorem 3.4 only guarantees that we will have *one* solution $\mathbf{x}(t)$ of the form $e^{\mu t}\mathbf{p}(t)$. If $\mathbf{B}$ has two linearly independent eigenvectors, we can find two linearly independent $\mathbf{p}_1(t)$ and $\mathbf{p}_2(t)$ so that the two solutions are both in the standard form. However, if $\mathbf{B}$ only has one eigenvector, we will end up with one solution of the form $\mathbf{p}_1(t)$ (since $\rho = 1$ in this case) and the other of the form $t\mathbf{p}_1(t) + \mathbf{p}_2(t)$. To see this, we replace

$$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \tag{3.153}$$

with the Jordan block

$$\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}. \tag{3.154}$$

As a result, instead of our solution being of the form

$$\mathbf{X}(t) = \mathbf{P}(t) \exp\left(\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} t \right) \tag{3.155}$$

$$= \mathbf{P}(t) \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} \tag{3.156}$$

$$= \begin{bmatrix} P_1 e^{\lambda_1 t} & P_2 e^{\lambda_2 t} \\ P_3 e^{\lambda_1 t} & P_4 e^{\lambda_2 t} \end{bmatrix}, \tag{3.157}$$

it will be of the form

$$\mathbf{X}(t) = \mathbf{P}(t) \exp\left(\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} t \right) \tag{3.158}$$

$$= \mathbf{P}(t) \begin{bmatrix} e^{\lambda t} & t e^{\lambda t} \\ 0 & e^{\lambda t} \end{bmatrix} \tag{3.159}$$

$$= \begin{bmatrix} P_1 e^{\lambda t} & P_1 t e^{\lambda t} + P_2 e^{\lambda t} \\ P_3 e^{\lambda t} & P_3 t e^{\lambda t} + P_4 e^{\lambda t} \end{bmatrix}. \tag{3.160}$$

See the papers by Akhmedov [1] and Wiesel and Pohlen [30].

IV. Let $\phi < -1$. Since $\rho = \phi \pm \sqrt{\phi^2 - 1}$, we must have $\rho_1 < -1$ and since $\rho_1 \rho_2 = 1$, we must have $\rho_1 < -1 < \rho_2 < 0$ and $\rho_2 = \frac{1}{\rho_1}$ means $\mu_2 = -\mu_1$. Now we can write $\mu_1 = \frac{i\pi}{T} + \gamma$ so that our solution must be of the form

$$\mathbf{x}(t) = c_1 e^{\gamma t} e^{i\pi t/T} \mathbf{p}_1(t) + c_2 e^{-\gamma t} e^{i\pi t/T} \mathbf{p}_2(t) \tag{3.161}$$

where $\mathbf{p}_1(t)$ and $\mathbf{p}_2(t)$ are both periodic with period T and so $e^{i\pi t/T}\mathbf{p}_1(t)$ and $e^{i\pi t/T}\mathbf{p}_2(t)$ are both periodic with period $2T$. As a result, the solution is unstable.

V. Let $\phi = -1$. Then $\rho_1 = \rho_2 = -1$. As in the case when $\phi = 1$, we have one solution which is periodic (this time with period $2T$),

$$\mathbf{x}_1(t) = e^{i\pi t/T} \mathbf{p}_1(t) \tag{3.162}$$

and the other which grows linearly with time,

$$\mathbf{x}_2(t) = t e^{i\pi t/T} \mathbf{p}_1(t) + e^{i\pi t/T} \mathbf{p}_2(t). \tag{3.163}$$

We summarise these results in figure 3.1. For $\phi > 1$, we have an unstable solution of the form

$$\mathbf{x}(t) = c_1 e^{\mu_1 t} \mathbf{p}_1(t) + c_2 e^{-\mu_1 t} \mathbf{p}_2(t). \tag{3.164}$$

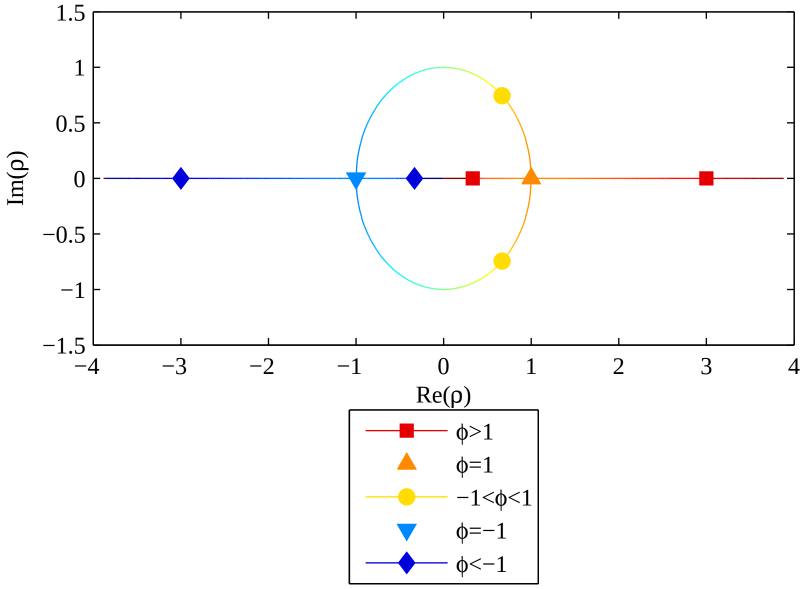

Figure 3.1: The range of ρ_1, ρ_2 for different values of ϕ real. In the region $\phi > 1$, the sample point has $\rho = 1/3, 3$; for $\phi = 1$, we have $\rho = 1$. In $1 < \phi < 1$, the sample point shown is $\rho = 2/3 \pm i\sqrt{5}/3$; for $\phi = -1$, we have $\rho = -1$ and in the region $\phi < -1$, we show $\rho = -1/3, -3$.

For $\phi = 1$, we have an unstable solution of the form

$$\mathbf{x}(t) = (c_1 + tc_2)\,\mathbf{p}_1(t) + c_2\mathbf{p}_2(t). \tag{3.165}$$

For $-1 < \phi < 1$, we have a stable pseudo-periodic solution of the form

$$\mathbf{x}(t) = c_1 \operatorname{Re}\left(e^{i\sigma t}\mathbf{p}(t)\right) + c_2 \operatorname{Im}\left(e^{i\sigma t}\mathbf{p}(t)\right). \tag{3.166}$$

For $\phi = -1$, we have an unstable solution of the form

$$\mathbf{x}(t) = (c_1 + tc_2)\,\mathbf{q}_1(t) + c_2\mathbf{q}_2(t). \tag{3.167}$$

Finally, for $\phi < -1$, we have an unstable solution of the form

$$\mathbf{x}(t) = c_1 e^{\gamma t}\mathbf{q}_1(t) + c_2 e^{-\gamma t}\mathbf{q}_2(t) \tag{3.168}$$

where $\mathbf{p}_i(t)$ represents a function that has period T and $\mathbf{q}_i(t)$ represents a function that has period $2T$.

3.2.4 Application to Hill's Equation

Consider Hill's equation

$$x'' + (\delta + \epsilon b(t)) = 0 \tag{3.169}$$

where $b(t)$ has period T. If $\epsilon = 0$, the solution is stable, however, there are some values of δ for which the solution is only marginally stable, according to the above criteria. As a result, we expect that for ϵ small but nonzero near those values of δ, we will get the beginning of a region of instability. We wish to find those values of δ.

For $\epsilon = 0$, if $\mathbf{X}(0) = \mathbf{I}$, then

$$\mathbf{X}(t) = \begin{bmatrix} \cos\left(\sqrt{\delta}t\right) & \frac{1}{\sqrt{\delta}}\sin\left(\sqrt{\delta}t\right) \\ -\sqrt{\delta}\sin\left(\sqrt{\delta}t\right) & \cos\left(\sqrt{\delta}t\right) \end{bmatrix} \tag{3.170}$$

and so

$$\mathbf{B} = \mathbf{X}(T) = \begin{bmatrix} \cos\left(\sqrt{\delta}T\right) & \frac{1}{\sqrt{\delta}}\sin\left(\sqrt{\delta}T\right) \\ -\sqrt{\delta}\sin\left(\sqrt{\delta}T\right) & \cos\left(\sqrt{\delta}T\right) \end{bmatrix}. \tag{3.171}$$

As a result,

$$\phi = \frac{\mathrm{tr}\,(\mathbf{B})}{2} = \cos\left(\sqrt{\delta}T\right). \tag{3.172}$$

If $\phi = 1$, then

$$\sqrt{\delta}T = 2m\pi \tag{3.173}$$

$$\delta = \left(2m\frac{\pi}{T}\right)^2 \tag{3.174}$$

where m is a positive integer since $\sqrt{\delta} > 0$. If $\phi = -1$, then

$$\sqrt{\delta}\pi = (2m+1)\,\pi \tag{3.175}$$

$$\delta = \left((2m+1)\,\frac{\pi}{T}\right)^2. \tag{3.176}$$

Now we have from the previous section that $\phi = 1$ corresponds to the existence of a periodic solution of period T and $\phi = -1$ corresponds to the existence of a periodic solution of period $2T$. As a result, we will have the border between stability and instability breaking off from $\epsilon = 0$ at

$$\delta = \left(2m\frac{\pi}{T}\right)^2 \tag{3.177}$$

corresponding to solutions with period T and breaking off from $\epsilon = 0$ at

$$\delta = \left((2m+1)\,\frac{\pi}{T}\right)^2 \tag{3.178}$$

corresponding to solutions with period $2T$.

3.3 Stability Boundary of Mathieu's Equation

3.3.1 Undamped Case

We have from §3.2.3 and §3.2.4 that on the edge of the region of stability, we have either $\phi = 1$ or $\phi = -1$. The former corresponds to the existence of a periodic solution with period T and the latter to a periodic solution with period $2T$. In order to determine the region of stability of the Mathieu equation in the δ-ϵ plane, we then need to determine the conditions on δ and ϵ required in order to have a solution which is periodic with either period π or 2π. We follow McLachlan [17] and Ward [28].

Functions of Period π

We can write a general function of period π as

$$x = \sum_{n=0}^{\infty} a_n \cos(2nt) + \sum_{n=1}^{\infty} b_n \sin(2nt) . \tag{3.179}$$

We then obtain

$$0 = x'' + (\delta + \epsilon \cos(2t))\, x \tag{3.180}$$

$$0 = \sum_{n=0}^{\infty} \left(\delta - 4n^2\right) a_n \cos(2nt) + \sum_{n=1}^{\infty} \left(\delta - 4n^2\right) b_n \sin(2nt) + \epsilon \sum_{n=0}^{\infty} a_n \cos(2nt)\cos(2t) + \epsilon \sum_{n=1}^{\infty} b_n \sin(2nt)\cos(2t) . \tag{3.181}$$

Using the identities

$$\cos(A)\cos(B) = \frac{1}{2}\left(\cos(A-B) + \cos(A+B)\right) \tag{3.182}$$

$$\sin(A)\cos(B) = \frac{1}{2}\left(\sin(A-B) + \sin(A+B)\right) \tag{3.183}$$

this becomes

$$0 = \sum_{n=0}^{\infty} \left(\delta - 4n^2\right) a_n \cos(2nt) + \sum_{n=1}^{\infty} \left(\delta - 4n^2\right) b_n \sin(2nt) + \frac{\epsilon}{2} \sum_{n=0}^{\infty} a_n \left(\cos(2(n+1)t) + \cos(2(n-1)t)\right) + \frac{\epsilon}{2} \sum_{n=1}^{\infty} b_n \left(\sin(2(n+1)t) + \sin(2(n-1)t)\right) \tag{3.184}$$

and so we must have

$$0 = \sum_{n=0}^{\infty} \left(\delta - 4n^2\right) a_n \cos(2nt) + \frac{\epsilon}{2} \sum_{n=0}^{\infty} a_n \left(\cos(2(n+1)t) + \cos(2(n-1)t)\right) \tag{3.185}$$

$$0 = \left(\delta a_0 + \frac{\epsilon}{2} a_1\right) \cos(0) + \left((\delta - 4) a_1 + \frac{\epsilon}{2} (2a_0 + a_2)\right) \cos(2t) + \sum_{n=2}^{\infty} \left(\left(\delta - 4n^2\right) a_n + \frac{\epsilon}{2} (a_{n-1} + a_{n+1})\right) \cos(2nt) \tag{3.186}$$

and

$$0 = \sum_{n=1}^{\infty} \left(\delta - 4n^2\right) b_n \sin(2nt) + \frac{\epsilon}{2} \sum_{n=1}^{\infty} b_n \left(\sin(2(n+1)t) + \sin(2(n-1)t)\right) \tag{3.187}$$

$$0 = \left((\delta - 4) b_1 + \frac{\epsilon}{2} b_2\right) \sin(2t) + \sum_{n=2}^{\infty} \left(\left(\delta - 4n^2\right) b_n + \frac{\epsilon}{2} (b_{n-1} + b_{n+1})\right) \sin(2nt). \tag{3.188}$$

By orthogonality of the sine and cosine, these can be rewritten as

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix} = \begin{bmatrix} \delta & \frac{\epsilon}{2} & & & & 0 \\ \epsilon & \delta - 4 \cdot 1^2 & \frac{\epsilon}{2} & & & \\ & \frac{\epsilon}{2} & \delta - 4 \cdot 2^2 & \frac{\epsilon}{2} & & \\ & & \frac{\epsilon}{2} & \delta - 4 \cdot 3^2 & \frac{\epsilon}{2} & \\ 0 & & & \ddots & \ddots & \ddots \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \end{bmatrix} \tag{3.189}$$

and

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix} = \begin{bmatrix} \delta - 4 \cdot 1^2 & \frac{\epsilon}{2} & & & 0 \\ \frac{\epsilon}{2} & \delta - 4 \cdot 2^2 & \frac{\epsilon}{2} & & \\ & \frac{\epsilon}{2} & \delta - 4 \cdot 3^2 & \frac{\epsilon}{2} & \\ 0 & & \ddots & \ddots & \ddots \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \end{bmatrix} \tag{3.190}$$

In order to have a non-zero solution, the determinant of at least one of these (infinite) matrices must be zero. This gives us the requirement that ϵ and δ must satisfy in order to be on the borderline between stability and instability. We can approximate the determinants of these matrices by the determinants of the finite $n \times n$ matrices of the same form. The resultant curves in the δ-ϵ plane for different values of n are shown in figure 3.2.

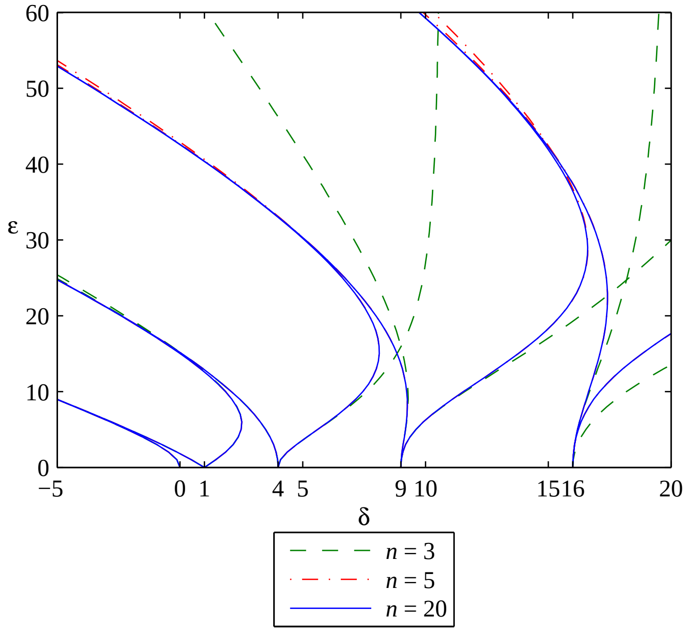

Figure 3.2: The approximation to the border of the region of stability of the Mathieu equation (determined by equations 3.189, 3.190, 3.198, 3.199) where each infinite matrix is approximated by its $n \times n$ counterpart.

Functions of Period 2π

We now perform a similar analysis for functions of period 2π. We can write a general function of period 2π as

$$x = \sum_{n=0}^{\infty} a_n \cos(nt) + \sum_{n=1}^{\infty} b_n \sin(nt) \,. \tag{3.191}$$

We then remove from this all the terms which also have period π since we have already dealt with those. If we included them, we would obtain the lines in the δ-ϵ plane where we obtain solutions that either have period π or have period 2π. As a result, we have

$$x = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} a_n \cos(nt) + \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} b_n \sin(nt) \,. \tag{3.192}$$

so that we obtain

$$0 = x'' + (\delta + \epsilon \cos(2t))\, x \tag{3.193}$$

$$\begin{aligned} 0 = &\sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \left(\delta - n^2\right) a_n \cos(nt) + \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \left(\delta - n^2\right) b_n \sin(nt) \\ &+ \epsilon \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} a_n \cos(nt)\cos(2t) + \epsilon \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} b_n \sin(nt)\cos(2t) \end{aligned} \tag{3.194}$$

$$\begin{aligned} 0 = &\sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \left(\delta - n^2\right) a_n \cos(nt) + \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \left(\delta - n^2\right) b_n \sin(nt) \\ &+ \frac{\epsilon}{2} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} a_n \left(\cos((n+2)t) + \cos((n-2)t)\right) \\ &+ \frac{\epsilon}{2} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} b_n \left(\sin((n+2)t) + \sin((n-2)t)\right) . \end{aligned} \tag{3.195}$$

We must then have

$$\begin{aligned} 0 = &\sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \Big((\delta - 1)\, a_1 + \frac{\epsilon}{2} (a_1 + a_3) \Big) \cos(t) \\ &+ \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \Big(\left(\delta - n^2\right) a_n + \frac{\epsilon}{2} (a_{n-2} + a_{n+2}) \Big) \cos(nt) \end{aligned} \tag{3.196}$$

and

$$0 = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \left((\delta - 1)\, b_1 + \frac{\epsilon}{2} (-b_1 + b_3) \right) \sin(t)$$

$$+ \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \left(\left(\delta - n^2\right) b_n + \frac{\epsilon}{2} (b_{n-2} + b_{n+2}) \right) \sin(nt) \tag{3.197}$$

which we can write as

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix} = \begin{bmatrix} \delta - 1^2 + \frac{\epsilon}{2} & \frac{\epsilon}{2} & & & 0 \\ \frac{\epsilon}{2} & \delta - 3^2 & \frac{\epsilon}{2} & & \\ & \frac{\epsilon}{2} & \delta - 5^2 & \frac{\epsilon}{2} & \\ 0 & & \ddots & \ddots & \ddots \end{bmatrix} \begin{bmatrix} a_1 \\ a_3 \\ a_5 \\ \vdots \end{bmatrix} \tag{3.198}$$

and

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix} = \begin{bmatrix} \delta - 1^2 - \frac{\epsilon}{2} & \frac{\epsilon}{2} & & & 0 \\ \frac{\epsilon}{2} & \delta - 3^2 & \frac{\epsilon}{2} & & \\ & \frac{\epsilon}{2} & \delta - 5^2 & \frac{\epsilon}{2} & \\ 0 & & \ddots & \ddots & \ddots \end{bmatrix} \begin{bmatrix} b_1 \\ b_3 \\ b_5 \\ \vdots \end{bmatrix} \tag{3.199}$$

As before, in order to obtain a nonzero solution, we must have the determinant of at least one of the matrices being zero. This constrains δ and ϵ.

The resultant region of stability is shown in figure 3.3

3.3.2 Undamped Case with ϵ small

Consider now when ϵ is small. We have from §3.2.4 that for ϵ small, we will have the border between stability and instability near

$$\delta = (2m)^2 \tag{3.200}$$

and

$$\delta = (2m+1)^2 . \tag{3.201}$$

As a result, we seek periodic solutions near $\delta = n^2$ to the equation

$$x'' + (\delta + \epsilon \cos(2t))\, x = 0. \tag{3.202}$$

Let

$$x = x_0(t) + \epsilon x_1(t) + \epsilon^2 x_2(t) + \ldots, \tag{3.203}$$

$$\delta = n^2 + \epsilon\delta_1 + \epsilon^2 \delta_2 + \ldots \tag{3.204}$$

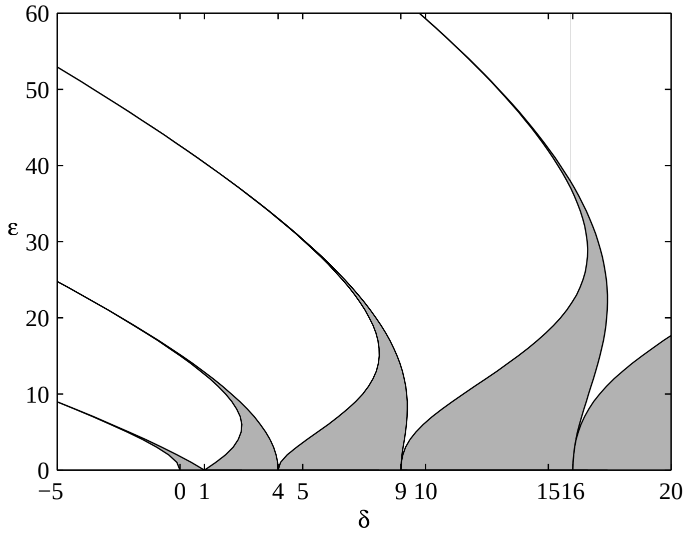

Figure 3.3: The region of stability of the Mathieu equation.

Substituting these into Mathieu's equation, we obtain

$$x_0'' + n^2 x_0 = 0 \tag{3.205}$$
$$x_1'' + n^2 x_1 = -\delta_1 x_0 - x_0 \cos(2t) \tag{3.206}$$
$$x_2'' + n^2 x_2 = -\delta_1 x_1 - \delta_2 x_0 - x_1 \cos(2t) . \tag{3.207}$$

For $n \neq 0$, the solution to equation 3.205 is

$$x_0 = a \cos(nt) + b \sin(nt) . \tag{3.208}$$

Inserting this into equation 3.206, we obtain

$$\begin{aligned} x_1'' + n^2 x_1 &= -\delta_1 x_0 - x_0 \cos(2t) && (3.209) \\ &= -\delta_1 (a \cos(nt) + b \sin(nt)) \\ &\qquad - (a \cos(nt) + b \sin(nt)) \cos(2t) && (3.210) \\ &= -\delta_1 a \cos(nt) - \delta_1 b \sin(nt) \\ &\qquad - \frac{a}{2} \cos((n+2)t) - \frac{a}{2} \cos((n-2)t) \\ &\qquad - \frac{b}{2} \sin((n+2)t) - \frac{b}{2} \sin((n-2)t) && (3.211) \end{aligned}$$

Under the assumption that $n \neq 1$, in order to eliminate secular terms, we must have

$$-\delta_1 a = 0, \qquad\qquad -\delta_1 b = 0. \tag{3.212}$$

As a result, in order to avoid x_0 being the zero solution, we must have $\delta_1 = 0$. We then have

$$x_1'' + n^2 x_1 = -\frac{a}{2}\cos\left(\left(n+2\right)t\right) - \frac{a}{2}\cos\left(\left(n-2\right)t\right) - \frac{b}{2}\sin\left(\left(n+2\right)t\right) - \frac{b}{2}\sin\left(\left(n-2\right)t\right). \tag{3.213}$$

Letting

$$x_1 = \sum_{i=0}^{\infty} c_i \sin\left(it\right) + d_i \cos\left(it\right), \tag{3.214}$$

this becomes

$$-\sum_{i=1}^{\infty} c_i i^2 \sin\left(it\right) + d_i i^2 \cos\left(it\right) + \sum_{i=0}^{\infty} c_i n^2 n^2 \sin\left(it\right) + d_i n^2 \cos\left(it\right) = -\frac{a}{2}\cos\left(\left(n+2\right)t\right) - \frac{a}{2}\cos\left(\left(n-2\right)t\right) - \frac{b}{2}\sin\left(\left(n+2\right)t\right) - \frac{b}{2}\sin\left(\left(n-2\right)t\right). \tag{3.215}$$

Equating coefficients of the sines and cosines, we obtain that

$$c_{n-2} = \frac{b}{8\left(-n+1\right)}, \qquad d_{n-2} = \frac{a}{8\left(-n+1\right)} \tag{3.216}$$

$$c_{n+2} = \frac{b}{8\left(n+1\right)}, \qquad d_{n+2} = \frac{a}{8\left(n+1\right)}. \tag{3.217}$$

We can assume that all the $\sin\left(nt\right)$ and $\cos\left(nt\right)$ component is already in x_0, so we can choose $c_n = 0$, $d_n = 0$. All remaining c_i and d_i are zero. As a result,

$$x_1 = \frac{b}{8\left(-n+1\right)}\sin\left(\left(n-2\right)t\right) + \frac{b}{8\left(n+1\right)}\sin\left(\left(n+2\right)t\right) + \frac{a}{8\left(-n+1\right)}\cos\left(\left(n-2\right)t\right) + \frac{a}{8\left(n+1\right)}\cos\left(\left(n+2\right)t\right). \tag{3.218}$$

Finally, inserting this into equation 3.207, we obtain that

$$\begin{aligned} x_2'' + n^2 x_2 = {} & -\delta_2\left(a\cos\left(nt\right) + b\sin\left(nt\right)\right) \\ & - \frac{b}{16\left(-n+1\right)}\left(\sin\left(nt\right) + \sin\left(\left(n-4\right)t\right)\right) \\ & - \frac{b}{16\left(n+1\right)}\left(\sin\left(\left(n+4\right)t\right) + \sin\left(nt\right)\right) \\ & - \frac{a}{16\left(-n+1\right)}\left(\cos\left(nt\right) + \cos\left(\left(n-4\right)t\right)\right) \\ & - \frac{a}{16\left(n+1\right)}\left(\cos\left(\left(n+4\right)t\right) + \cos\left(nt\right)\right). \end{aligned} \tag{3.219}$$

Under the assumption that $n \neq 2$, in order to eliminate the secular terms, we must have

$$0 = -\delta_2 a - \frac{a}{16\,(-n+1)} - \frac{a}{16\,(n+1)}, \tag{3.220}$$

$$0 = -\delta_2 b - \frac{b}{16\,(-n+1)} - \frac{b}{16\,(n+1)}, \tag{3.221}$$

which can be rewritten as

$$0 = -a\left(\delta_2 - \frac{1}{8\,(n^2-1)}\right), \tag{3.222}$$

$$0 = -b\left(\delta_2 - \frac{1}{8\,(n^2-1)}\right). \tag{3.223}$$

As a result, in order to avoid a nonzero x_0 (i.e., making sure that we don't simultaneously have $a = 0$ and $b = 0$), we must have

$$\delta_2 = \frac{1}{8\,(n^2-1)}. \tag{3.224}$$

Case n = 2

In the case $n = 2$, eliminating the secular terms in equation 3.219 tells us that

$$0 = -\delta_2 a + \frac{a}{8} - \frac{a}{48}, \tag{3.225}$$

$$0 = -\delta_2 b - 0 - \frac{b}{48}, \tag{3.226}$$

which become

$$0 = -a\left(\delta_2 - \frac{5}{48}\right), \tag{3.227}$$

$$0 = -b\left(\delta_2 + \frac{1}{48}\right). \tag{3.228}$$

As a result, for $n = 2$ we must have either

$$a = 0, \qquad \delta_2 = -\frac{1}{48} \tag{3.229}$$

or

$$b = 0, \qquad \delta_2 = \frac{5}{48}. \tag{3.230}$$

As a result, for $n = 2$, we either have

$$\delta = 4 - \epsilon^2 \frac{1}{48} + O\left(\epsilon^3\right) \tag{3.231}$$

or

$$\delta = 4 + \epsilon^2 \frac{5}{48} + O\left(\epsilon^3\right). \tag{3.232}$$

We also have either

$$x = b \sin(2t) + \epsilon \frac{b}{24} \sin(4t) + O\left(\epsilon^2\right) \tag{3.233}$$

or

$$x = a \cos(2t) + \epsilon \left(-\frac{a}{8} + \frac{a}{24} \cos(4t)\right) + O\left(\epsilon^2\right), \tag{3.234}$$

which both have period π, as expected.

Case $\mathbf{n = 1}$

In the case $n = 1$, eliminating the secular terms in equation 3.211 tells us that

$$0 = -\delta_1 a - \frac{a}{2}, \tag{3.235}$$

$$0 = -\delta_1 b + \frac{b}{2} \tag{3.236}$$

and so we must either have

$$\delta_1 = -\frac{1}{2}, \qquad b = 0 \tag{3.237}$$

or

$$\delta_1 = \frac{1}{2}, \qquad a = 0. \tag{3.238}$$

In either of these cases, equation 3.211 becomes

$$x_1'' + x_1 = -\frac{a}{2} \cos(3t) - \frac{b}{2} \sin(3t). \tag{3.239}$$

As before, we let

$$x_1 = \sum_{i=1}^{\infty} c_i \sin(it) + d_i \cos(it) \tag{3.240}$$

and find that

$$c_3 = \frac{b}{16}, \qquad d_3 = \frac{a}{16}. \tag{3.241}$$

As a result,

$$x_1 = \frac{b}{16} \sin(3t) + \frac{a}{16} \cos(3t). \tag{3.242}$$

Then equation 3.207 becomes

$$x_2'' + x_2 = -\delta_1 \left(\frac{b}{16} \sin(3t) + \frac{a}{16} \cos(3t) \right) - \delta_2 \left(a \cos(t) + b \sin(t) \right) - \left(\frac{b}{16} \sin(3t) + \frac{a}{16} \cos(3t) \right) \cos(2t) \tag{3.243}$$

$$= -\delta_1 \left(\frac{b}{16} \sin(3t) + \frac{a}{16} \cos(3t) \right) - \delta_2 a \cos(t) - \delta_2 b \sin(t) - \frac{b}{32} \sin(t) - \frac{b}{32} \sin(5t) - \frac{a}{32} \cos(t) - \frac{a}{32} \cos(5t). \tag{3.244}$$

In order to eliminate the secular terms, we must have

$$0 = -a \left(\delta_2 + \frac{1}{32} \right) \tag{3.245}$$

$$0 = -b \left(\delta_2 + \frac{1}{32} \right). \tag{3.246}$$

As a result, $\delta_2 = -1/32$, so that either

$$\delta = 1 - \epsilon \frac{1}{2} - \epsilon^2 \frac{1}{32} + O\left(\epsilon^3\right) \tag{3.247}$$

or

$$\delta = 1 + \epsilon \frac{1}{2} - \epsilon^2 \frac{1}{32} + O\left(\epsilon^3\right). \tag{3.248}$$

We also have either

$$x = a \cos(t) + \epsilon \frac{a}{16} \cos(3t) + O\left(\epsilon^2\right) \tag{3.249}$$

or

$$x = b \sin(t) + \epsilon \frac{b}{16} \sin(3t) + O\left(\epsilon^2\right), \tag{3.250}$$

which are periodic with period 2π, as expected.

Case $\mathbf{n = 0}$

In the case $n = 0$, we get

$$x_0 = a + bt. \tag{3.251}$$

Now we expect a periodic solution, so $b = 0$. As a result, equation 3.206 becomes

$$x_1'' = -\delta_1 a - a \cos(2t). \tag{3.252}$$

In analogy with before, when we eliminated secular terms, we must have $\delta_1 = 0$. As a result, we have

$$x_1'' = -a \cos(2t) \tag{3.253}$$

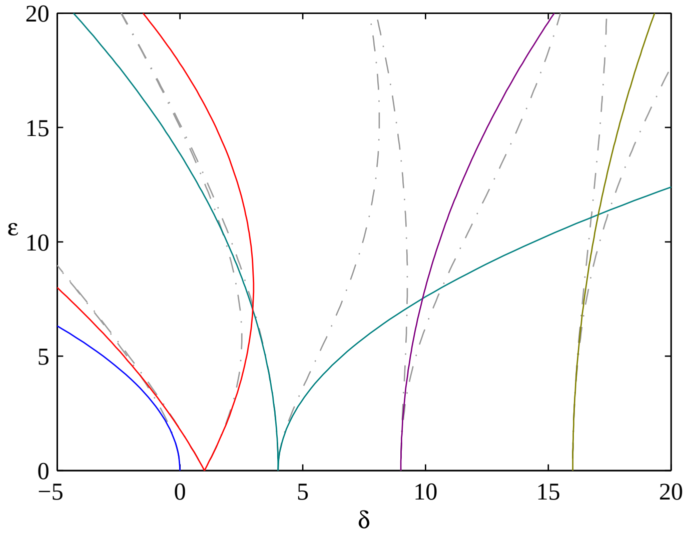

Figure 3.4: The quadratic approximations to the boundary between stability and instability of the Mathieu equation, in comparison with the approximation from §3.3.1, with $n = 20$.

so that

$$x_1 = \frac{a}{4}\cos(2t) \tag{3.254}$$

and equation 3.207 becomes

$$x_2'' = -\delta_2 a - \frac{a}{4}\cos(2t)\cos(2t) \tag{3.255}$$

$$= -\delta_2 a - \frac{a}{8} - \frac{a}{8}\cos(4t) \tag{3.256}$$

so that we must have

$$0 = -a\left(\delta_2 + \frac{1}{8}\right). \tag{3.257}$$

so that $\delta_2 = -1/8$ and

$$\delta = 0 - \epsilon^2 \frac{1}{8} \tag{3.258}$$

with

$$x = a + \epsilon\frac{a}{4}\cos(2t) + O\left(\epsilon^2\right), \tag{3.259}$$

which is again periodic with period π, as expected.

These approximations to $\delta(\epsilon)$ for ϵ small are compared to the approximation in the previous section (which is valid for both small and large ϵ) in figure 3.4.

3.3.3 Damped Case

We follow Richards [24]. Our equation is

$$x'' + kx' + (\delta + \epsilon \cos(2t))\, x = 0. \tag{3.260}$$

If we let

$$y(t) = e^{\frac{k}{2}t} x(t), \tag{3.261}$$

we obtain that

$$y'' + (a + \epsilon \cos(2t))\, y = 0 \tag{3.262}$$

where

$$a = \delta - \frac{k^2}{4}. \tag{3.263}$$

Now equation 3.260 isn't of the form of equation 3.124 (§3.2.3), but equation 3.262 is. As a result, we know that the solution to equation 3.262 is of the form

$$y(t) = e^{\mu_1 t} p_1(t) + e^{\mu_2 t} p_2(t) \tag{3.264}$$

where μ_1 and μ_2 satisfy

$$e^{\mu\pi} = \rho = \phi \pm \sqrt{\phi^2 - 1} \tag{3.265}$$

where ϕ is half of the trace of $\mathbf{B}$ for $y(t)$ above when we use the initial conditions $\mathbf{X}(0) = \mathbf{I}$. As a result, the largest μ (the one most likely to cause instability) satisfies

$$e^{\mu\pi} = \rho = \phi + \sqrt{\phi^2 - 1} \tag{3.266}$$

so that

$$\mu\pi = \ln\left(\phi + \sqrt{\phi^2 - 1}\right) \tag{3.267}$$

$$\mu\pi = \cosh^{-1}(\phi) \tag{3.268}$$

$$\mu = \frac{\cosh^{-1}(\phi)}{\pi}. \tag{3.269}$$

Now in order for $x(t)$ to be stable, we must have

$$0 \geq \mathrm{Re}\left(\mu - \frac{k}{2}\right) \tag{3.270}$$

$$\frac{k}{2} \geq \mathrm{Re}(\mu) \tag{3.271}$$

with μ as above. This can be used to numerically determine the stability of the damped equation. The result for $k = 0.2$ is shown in figure 3.5.

3.3.4 Damped Case with ϵ small

Consider the damped Mathieu equation

$$x'' + kx' + (\delta + \epsilon \cos(2t))\, x = 0. \tag{3.272}$$

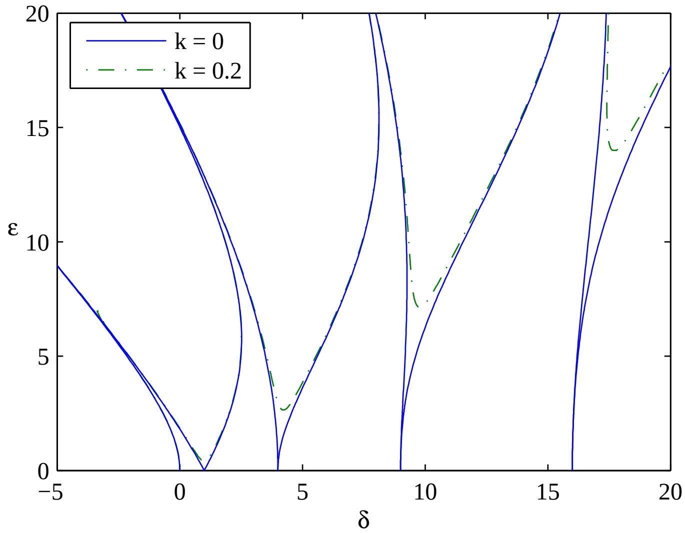

Figure 3.5: The border of the region of stability of the Mathieu equation, in the damped case.

Near $\delta = 1$

Suppose that k is of order ϵ. Then we can write $k = \epsilon k_1$ and expand near $\delta = 1$,

$$\delta = 1 + \epsilon\delta_1 + \dots \tag{3.273}$$
$$x = x_0 + \epsilon x_1 + \dots \tag{3.274}$$

Plugging this in and equating terms of equal order, we obtain

$$x_0'' + x_0 = 0 \tag{3.275}$$
$$x_1'' + x_1 = -k_1 x_0' - \cos(2t)\, x_0 - \delta x_0. \tag{3.276}$$

This tells us that

$$x_0 = a\cos(t) + b\sin(t) \tag{3.277}$$

so that

$$\begin{aligned} x_1'' + x_1 &= -k_1\left(-a\sin(t) + b\cos(t)\right) - \cos(2t)\left(a\cos(t) + b\sin(t)\right) \\ &\quad - \delta_1\left(a\cos(t) + b\sin(t)\right) \qquad (3.278) \\ &= k_1 a\sin(t) - k_1 b\cos(t) - \frac{a}{2}\left(\cos(t) + \cos(3t)\right) \\ &\quad - \frac{b}{2}\left(-\sin(t) + \sin(3t)\right) - \delta_1\left(a\cos(t) + b\sin(t)\right) \qquad (3.279) \end{aligned}$$

In order to eliminate secular terms, we must have

$$k_1 a + \frac{b}{2} - \delta_1 b = 0 \tag{3.280}$$

$$-k_1 b - \frac{a}{2} - \delta_1 a = 0 \tag{3.281}$$

which can be written as

$$\begin{bmatrix} k_1 & \frac{1}{2} - \delta_1 \\ -\frac{1}{2} - \delta_1 & -k_1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{3.282}$$

In order for this to have a nonzero solution, the determinant of the matrix must be zero, so we must have

$$0 = -k_1^2 + \left(\frac{1}{2} + \delta_1\right)\left(\frac{1}{2} - \delta_1\right) \tag{3.283}$$

$$0 = k_1^2 + \delta_1^2 - \frac{1}{4} \tag{3.284}$$

$$\delta_1 = \pm\sqrt{\frac{1}{4} - k_1^2} \tag{3.285}$$

so that

$$\delta = 1 + \epsilon\delta_1 + O\left(\epsilon^2\right) \tag{3.286}$$

$$= 1 \pm \sqrt{\frac{\epsilon^2}{4} - k^2} + O\left(\epsilon^2\right). \tag{3.287}$$

Near $\delta = 4$

For larger values of δ, in order ϵ to still be small at the edge of stability, we must have k quite a bit smaller. As a result, near $\delta = 4$, we choose k to be of order ϵ^2. Then we can write $k = \epsilon^2 k_1$ and expand near $\delta = 4$,

$$\delta = 4 + \epsilon\delta_1 + \epsilon^2\delta_2 + \ldots \tag{3.288}$$

$$x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \ldots \tag{3.289}$$

We need to expand these to order ϵ^2 because it will turn out that $\delta_1 = 0$. Plugging this in and equating terms of equal order, we obtain

$$x_0'' + 4x_0 = 0 \tag{3.290}$$

$$x_1'' + 4x_1 = -\delta_1 x_0 - \cos(2t)\, x_0 \tag{3.291}$$

$$x_2'' + 4x_2 = -k_1 x_0' - \delta_1 x_1 - \delta_2 x_0 - \cos(2t)\, x_1. \tag{3.292}$$

This tells us that

$$x_0 = a\cos(2t) + b\sin(2t) \tag{3.293}$$

so that

$$
\begin{aligned}
x_1'' + 4x_1 &= -\delta_1 x_0 - \cos(2r)\, x_0 && (3.294)\\
&= -\delta_1 \left(a\cos(2t) + b\sin(2t)\right) \\
&\qquad\qquad - \cos(2t)\left(a\cos(2t) + b\sin(2t)\right) && (3.295)\\
&= -\delta_1 a\cos(2t) - \delta_1 b\sin(2t) \\
&\qquad\qquad - \frac{a}{2}\cos(4t) - \frac{a}{2} - \frac{b}{2}\sin(4t) - \frac{b}{2}\cdot 0 && (3.296)
\end{aligned}
$$

In order to eliminate secular terms, we must have

$$
\delta_1 a = 0, \qquad \delta_1 b = 0 \tag{3.297}
$$

so we must have $\delta_1 = 0$. As a result, we have

$$
x_1'' + 4x_1 = -\frac{a}{2} - \frac{a}{2}\cos(4t) - \frac{b}{2}\sin(4t)\,. \tag{3.298}
$$

Expanding x_1 in terms of sines and cosines and equating coefficients, we find that

$$
x_1 = \frac{a}{8} + \frac{a}{24}\cos(4t) + \frac{b}{24}\sin(4t)\,. \tag{3.299}
$$

As a result, we have that

$$
\begin{aligned}
x_2'' + 4x_2 &= -k_1 x_0' - \delta_1 x_1 - \delta_2 x_0 - \cos(2t)\, x_1 && (3.300)\\
&= -k_1\left(-2a\sin(2t) + 2b\cos(2t)\right) - 0 \\
&\quad - \delta_2\left(a\cos(2t) + b\sin(2t)\right) \\
&\quad - \cos(2t)\left(-\frac{a}{8} + \frac{a}{24}\cos(4t) + \frac{b}{24}\sin(4t)\right) && (3.301)\\
&= \left(2k_1 a - \delta_2 b - \frac{b}{48}\right)\sin(2t) \\
&\quad + \left(-2k_1 b - \delta_2 a + \frac{a}{8} - \frac{a}{48}\right)\cos(2t) \\
&\quad - \frac{a}{48}\cos(6t) - \frac{b}{48}\sin(6t)\,. && (3.302)
\end{aligned}
$$

In order to eliminate secular terms, we must have

$$
0 = 2k_1 a - \delta_2 b - \frac{b}{48} \tag{3.303}
$$

$$
0 = -2k_1 b - \delta_2 a + \frac{5b}{48} \tag{3.304}
$$

which can be written as

$$
\begin{bmatrix} 2k_1 & -\delta_2 - \frac{1}{48} \\ -\delta_2 + \frac{5}{48} & -2k_1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{3.305}
$$

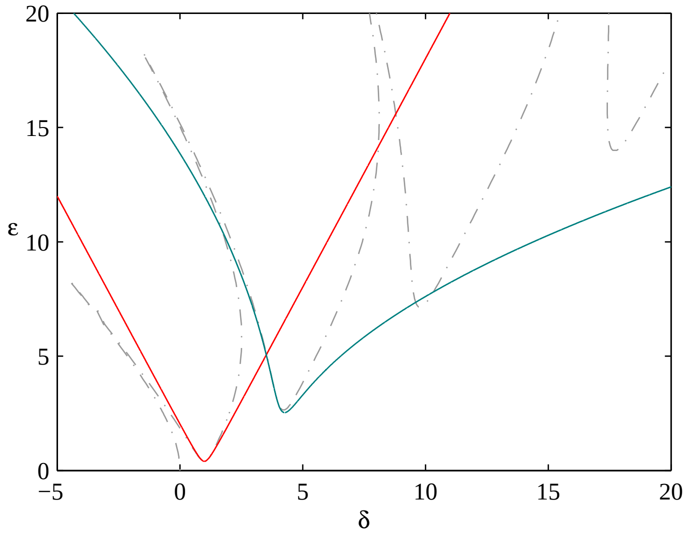

Figure 3.6: The approximation to the boundary between stability and instability of the Mathieu equation, in comparison with the numerical result from §3.3.3, with $k = 0.2$.

In order to have a nonzero solution to this, we must have that the determinant of the matrix is zero. As a result,

$$0 = -4k_1^2 - \left(\delta_2 + \frac{1}{48}\right)\left(\delta_2 - \frac{5}{48}\right) \tag{3.306}$$

$$\delta_2 = \frac{\frac{1}{12} \pm \sqrt{\frac{1}{144} - 4\left(-\frac{5}{48^2} + 4k_1^2\right)}}{2}. \tag{3.307}$$

where

$$\delta = 1 + \epsilon^2\delta_2 + O\left(\epsilon^3\right). \tag{3.308}$$

These approximations are compared to the result from §3.3.3 in figure 3.6.

3.3.5 Hill's Equation

Consider Hill's equation, which is a generalised version of the Mathieu equation

$$x'' + (\delta + \epsilon b(t))\, x = 0 \tag{3.309}$$

where b is periodic with period π. Let us assume that

$$\int_0^\pi b(t)\, dt = 0 \tag{3.310}$$

and that we can expand $b(t)$ as

$$b(t) = \sum_{n=1}^{\infty} c_n \cos(2nt) + d_n \sin(2nt). \tag{3.311}$$

We wish to determine an expansion for the solution where ϵ is small. Now we know this occurs near $\delta = m^2$ for positive integers m, so we expand

$$\delta = m^2 + \epsilon\delta_1 + \dots \tag{3.312}$$

$$x = x_0 + \epsilon x_1 + \dots \tag{3.313}$$

Then we obtain that

$$x_0'' + m^2 x_0 = 0 \tag{3.314}$$

$$x_1'' + m^2 x_1 = -\delta_1 x_0 - b(t) x_0 \tag{3.315}$$

so that

$$x_0 = a\cos(mt) + b\sin(mt) \tag{3.316}$$

and

$$\begin{aligned} x_1'' + m^2 x_1 &= -\delta_1 x_0 - b(t)x_0 && (3.317)\\ &= -\delta_1 (a\cos(mt) + b\sin(mt)) - (a\cos(mt) + b\sin(mt)) \\ &\qquad \times \sum_{n=1}^{\infty} (c_n \cos(2nt) + d_n \sin(2nt)) && (3.318)\\ &= -\delta_1 a\cos(mt) - \delta_1 b\sin(mt) \\ &\quad + \sum_{n=1}^{\infty} \Big[-\frac{ac_n}{2} (\cos((2n+m)t) + \cos((2n-m)t)) \\ &\qquad - \frac{ad_n}{2} (\sin((2n+m)t) + \sin((2n-m)t)) \\ &\qquad - \frac{bc_n}{2} (\sin((2n+m)t) - \sin((2n-m)t)) \\ &\qquad - \frac{bd_n}{2} (-\cos((2n+m)t) + \cos((2n-m)t)) \Big] && (3.319) \end{aligned}$$

To eliminate secular terms, if $m = 0$, we must have $\delta_1 a = \delta_1 b = 0$, and so $\delta_1 = 0$. As a result for $m = 0$, we must expand everything to second order. We will return to this later. For $m \neq 0$, we must have

$$0 = -\delta_1 a - \frac{ac_m}{2} - \frac{bd_m}{2} \tag{3.320}$$

$$0 = -\delta_1 b - \frac{ad_m}{2} + \frac{bc_m}{2} \tag{3.321}$$

which we can rewrite as

$$\begin{bmatrix} -\delta_1 - \frac{c_m}{2} & -\frac{d_m}{2} \\ -\frac{d_m}{2} & -\delta_1 + \frac{c_m}{2} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{3.322}$$

As a result, we must have

$$\delta_1^2 = \frac{1}{4}\left(c_m^2 + d_m^2\right) \tag{3.323}$$

and so

$$\delta = m^2 \pm \frac{\epsilon}{2}\sqrt{c_m^2 + d_m^2}. \tag{3.324}$$

Case m = 0

Recall that we determined that in the $m = 0$ case, we must expand everything to second order. As a result, we expand

$$\delta = \epsilon\delta_1 + \epsilon^2\delta_2 + \dots \tag{3.325}$$

$$x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \dots \tag{3.326}$$

From before, plugging in $m = 0$, we have $x_0 = a$ and $\delta_1 = 0$, so that

$$x_1'' = -\delta_1 a - \sum_{n=1}^{\infty} c_n a \cos(2nt) + d_n a \sin(2nt) \tag{3.327}$$

$$x_1'' = -\sum_{n=1}^{\infty} c_n a \cos(2nt) + d_n a \sin(2nt) \tag{3.328}$$

$$x_1 = \sum_{n=1}^{\infty} \frac{c_n a}{4n^2}\cos(2nt) + \frac{d_n a}{4n^2}\sin(2nt). \tag{3.329}$$

The second-order equation gives us

$$x_2'' = -\delta_1 x_2 - \delta_2 x_0 - b(t)x_1 \tag{3.330}$$

$$= -\delta_2 a - \left(\sum_{i=1}^{\infty} c_i \cos(2it) + d_i \sin(2it)\right) \times \left(\sum_{j=1}^{\infty} \frac{c_j a}{4j^2}\cos(2jt) + \frac{d_j a}{4j^2}\sin(2jt)\right) \tag{3.331}$$

In order to eliminate the secular-like terms, we must have

$$0 = -\delta_2 a - \sum_{i=1}^{\infty} \frac{c_i^2 a}{8i^2} + \frac{d_i^2 a}{8i^2} \tag{3.332}$$

$$\delta_2 = -\frac{1}{8}\sum_{i=1}^{\infty} \frac{c_i^2 + d_i^2}{i^2} \tag{3.333}$$

so that

$$\delta = -\epsilon^2 \frac{1}{8}\sum_{i=1}^{\infty} \frac{c_i^2 + d_i^2}{i^2}. \tag{3.334}$$

3.4 Applications of Mathieu's Equation

3.4.1 Pendulum with Oscillating Pivot

Suppose we have a mass m attached at the end of a massless pendulum of length L. Suppose the pivot point P oscillates in the vertical direction according to some function $p(t)$. Then the angle θ from the vertical to the pendulum obeys

$$\theta'' + \left(\frac{g + p''(t)}{L}\right) \sin(\theta) = 0. \tag{3.335}$$

We choose to measure the angle θ such that when the pendulum is vertical, pointed upward (at what is usually the unstable stationary solution), $\theta = \pi$. When the pendulum is near the top, $\theta \approx \pi$. Let $x = \theta - \pi$ so that $|x| \ll 1$. Then our model is approximately

$$x'' + \left(\frac{g + p''(t)}{L}\right)(-x) = 0. \tag{3.336}$$

Let $p(t) = A\cos(\omega t)$ to obtain

$$x'' + \left(\frac{-g + A\omega^2 \cos(\omega t)}{L}\right) x = 0. \tag{3.337}$$

Now let $2\tau = \omega t$ so that

$$\ddot{x} + \left(-\frac{4g}{\omega^2 L} + \frac{4A}{L}\cos(2\tau)\right) x = 0. \tag{3.338}$$

We can finally let

$$\delta = -\frac{4g}{\omega^2 L}, \qquad \epsilon = \frac{4A}{L} \tag{3.339}$$

to obtain

$$\ddot{x} + (\delta + \epsilon \cos(2\tau))\, x = 0 \tag{3.340}$$

where ϵ will be small if the amplitude of oscillations of the pivot is small compared to the length of the pendulum.

We wish to determine an ϵ and δ, and hence an A and ω, such that the solution to the above equation (Mathieu's equation) is stable for x small. Notice that the usual problem ($A = 0$ so $\epsilon = 0$) is unstable; near $x = 0$ the solution grows exponentially in time.

3.4.2 Variable Length Pendulum

Consider now a pendulum with an oscillatory length. This time, the pendulum is pointed downward.

Derivation of Model

Suppose that there is some force F on the mass along the pendulum. Then the forces on the mass at the end of the pendulum are given by

$$mx'' = -F\sin(\theta) \tag{3.341}$$
$$my'' = F\cos(\theta) - mg, \tag{3.342}$$

where

$$x = L\sin(\theta) \tag{3.343}$$
$$y = -L\cos(\theta). \tag{3.344}$$

By letting $z = x + iy = -iLe^{i\theta}$, we obtain

$$z'' = \left(2L'\theta' + L\theta'' - iL'' + iL\theta'^2\right)e^{i\theta} \tag{3.345}$$

so that

$$m\left(2L'\theta' + L\theta'' + iL\theta'^2 - iL''\right) = iF - imge^{i\theta}. \tag{3.346}$$

By equating real parts, we then obtain

$$2L'\theta' + L\theta'' + g\sin(\theta) = 0. \tag{3.347}$$

Letting $\phi = L\theta$, this becomes

$$\phi'' - \phi\frac{L''}{L} + g\sin\left(\frac{\phi}{L}\right) = 0. \tag{3.348}$$

For $\theta \ll 1$, this is approximately

$$\phi'' + \left(\frac{g - L''}{L}\right)\phi = 0. \tag{3.349}$$

Transformation to Mathieu's Equation

Let

$$L = L_0\left(1 + \Delta\cos(\omega t)\right) \tag{3.350}$$

for $\Delta \ll 1$. Then we obtain

$$0 = \phi'' + \left(\frac{g - L''}{L}\right)\phi \tag{3.351}$$
$$0 = \phi'' + \left(\frac{g}{L_0\left(1 + \Delta\cos(\omega t)\right)} - \frac{-L_0\Delta\omega^2\cos(\omega t)}{L_0\left(1 + \Delta\cos(\omega t)\right)}\right)\phi \tag{3.352}$$
$$0 = \phi'' + \left(\frac{g}{L_0}\left(1 - \Delta\cos(\omega t)\right) + \Delta\omega^2\cos(\omega t)\right)\phi \tag{3.353}$$
$$0 = \phi'' + \left(\frac{g}{L_0} + \Delta\left(\omega^2 - \frac{g}{L_0}\right)\cos(\omega t)\right)\phi. \tag{3.354}$$

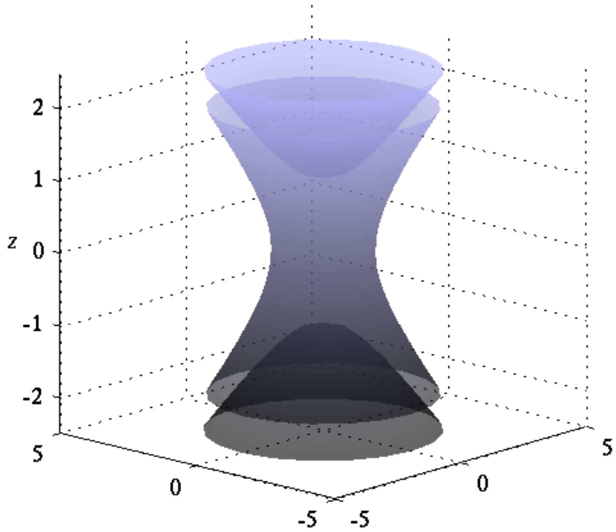

Figure 3.7: The physical ion trap, for $z_0 = 1$, $r_0 = \sqrt{2}$.

Letting $\Omega^2 = g/L_0$, this becomes

$$\phi'' + \left(\Omega^2 + \Delta\left(\omega^2 - \Delta^2\right)\cos\left(\omega t\right)\right)\phi = 0. \tag{3.355}$$

Letting

$$\tau = \frac{\omega}{2}t, \qquad \delta = \frac{4\Omega^2}{\omega^2}, \qquad \epsilon = 4\Delta\left(1 - \frac{\Omega^2}{\omega^2}\right) \tag{3.356}$$

this becomes Mathieu's equation:

$$\ddot{\phi} + \left(\delta + \epsilon\cos\left(2\tau\right)\right)\phi = 0. \tag{3.357}$$

3.4.3 Ion Traps

As in the honours thesis by Fischer [8], we consider an ion trap as shown in figure 3.7. The side walls are described by

$$r^2 = 2z^2 + r_0^2 \tag{3.358}$$

where r_0 is the radius at the narrowest point. The end caps are described by

$$z^2 = \frac{r^2}{2} + z_0^2 \tag{3.359}$$

where $2z_0$ is the shortest distance between the two end caps.

Now if we apply a potential difference A between the side walls and the end caps, taking the end caps to be ground, we obtain a potential of

$$V(z,r) = A\frac{r^2 - 2\left(z^2 - z_0^2\right)}{r_0^2 + 2z_0^2} \tag{3.360}$$

and hence an electric field of

$$E = -\nabla V = \frac{A}{r_0^2 + 2z_0^2}\left(-2r\hat{\mathbf{r}} + 4z\hat{\mathbf{z}}\right). \tag{3.361}$$

As a result, in the z-direction, we have

$$mz'' = \frac{4QA}{d_0^2}z \tag{3.362}$$

where prime denotes differentiation with respect to t and we have let $d_0^2 = r_0^2 + 2z_0^2$. If

$$A = U_0 - V_0\cos(\omega t), \tag{3.363}$$

as in the thesis of King [15], our problem then becomes

$$z'' - \frac{4Q}{md_0^2}\left(U_0 - V_0\cos(\omega t)\right)z. \tag{3.364}$$

Following King [15], we can then make the substitutions

$$\tau = \frac{\omega}{2}t, \qquad \delta = \frac{-16QU_0}{md_0^2\omega^2}, \qquad \epsilon = \frac{16QV_0}{md_0^2\omega^2} \tag{3.365}$$

so that our equation once more takes the familiar form of Mathieu's equation:

$$\ddot{z} + \left(\delta + \epsilon\cos(2\tau)\right)z = 0. \tag{3.366}$$

Stability for $\mathbf{U_0 = 0}$

In the case that $U_0 = 0$, our equation becomes

$$mz'' = -\frac{4QV_0}{d_0^2}\cos(\omega t)\,z. \tag{3.367}$$

We follow King [15]. We assume that the solution is composed of two parts: one which has large amplitude and small acceleration, the other which has small amplitude but large acceleration (something small but quickly oscillating). We approximate $z = z_M + z_\mu$ so that we can approximate our equation by

$$mz_\mu'' = -\frac{4QV_0}{d_0^2}\cos(\omega t)\,z_M \tag{3.368}$$

so that

$$z_\mu \sim \frac{4QV_0}{md_0^2\omega^2} \cos(\omega t)\, z_M. \tag{3.369}$$

As a result, we obtain

$$mz'' = -\frac{4QV_0}{d_0^2} \cos(\omega t)\, z \tag{3.370}$$

$$z_M'' + z_\mu'' = -\frac{4QV_0}{md_0^2} \cos(\omega t)\,(z_M + z_\mu) \tag{3.371}$$

$$z_M'' - \frac{4QV_0}{md_0^2} \cos(\omega t)\, z_M = -\frac{4QV_0}{md_0^2} \cos(\omega t)\, z_M - \frac{16Q^2V_0^2}{m^2 d_0^4 \omega^2} \cos^2(\omega t)\, z_M \tag{3.372}$$

Averaging over one period, this becomes

$$z_M'' = -\frac{8Q^2V_0^2}{m^2 d_0^4 \omega^2} z_M, \tag{3.373}$$

which is a harmonic oscillator with frequency

$$\frac{2\sqrt{2}QV_0}{md_0^2\omega}. \tag{3.374}$$

As a result, for $U_0 = 0$, the ion trap acts like a harmonic oscillator, trapping the ion at its centre.

See King [15] and Brewer et al. [4] for further reference.

A physical analogy to the trap is shown in figure 3.8. If one constantly rotates the base at the correct frequency, the ball will be not roll down the base [25, 27].

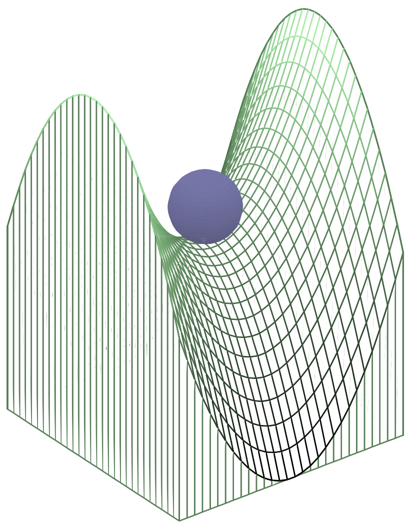

Figure 3.8: A physical analogy to the ion trap.