Chapter 1

Nonlinear Oscillators

1.1 Introductory Theory

We consider a couple different situations where we add a nonlinearity of order ϵ to a simple oscillator. These problems are of the form

$$x'' + x + \epsilon h(x, x') = 0. \tag{1.1}$$

In subsequent sections, we will consider specific examples.

1.1.1 Method of Eliminating Secular Terms

We start by letting $x = x(T, \tau)$ where $T = t$ and $\tau = \epsilon t$. Our problem becomes

$$x_{TT} + 2\epsilon x_{T\tau} + \epsilon^2 x_{\tau\tau} + x + \epsilon h(x, x') = 0. \tag{1.2}$$

We then let

$$x = x_0 + \epsilon x_1 + \dots \tag{1.3}$$

so that equation 1.2 becomes to order 1

$$x_{0TT} + x_0 = 0 \tag{1.4}$$

so that

$$x_0 = A(\tau)\cos(\theta) = A(\tau)\cos(T + \tau\phi(\tau)) = A(\tau)\cos((1 + \epsilon\phi)t) \tag{1.5}$$

so that

$$x_{0T} = -A(\tau)\sin(T + \tau\phi(\tau)) \tag{1.6}$$

$$x_{0T\tau} = -A'\sin(T + \tau\phi) - A(\phi + \tau\phi')\cos(T + \tau\phi)\,. \tag{1.7}$$

Equating terms in equation 1.2 of order ϵ, we obtain that

$$x_{1TT} + x_1 = -2x_{0T\tau} - h(x_0, x_{0T}) \tag{1.8}$$

since

$$\begin{aligned} h(x,x') &= h\left(x_0 + \epsilon x_1, x_{0T} + \epsilon\left(x_{1T} + x_{0\tau}\right)\right) & (1.9)\\ &\simeq h\left(x_0, x_{0T}\right) + h_a\left(a,b\right)\big|_{(x_0,x_{0T})}\epsilon x_1 \\ &\qquad\qquad + h_b\left(a,b\right)\big|_{(x_0,x_{0T})}\epsilon\left(x_{1T}+x_{0\tau}\right) & (1.10)\\ &\simeq h\left(x_0,x_{0T}\right) + O\left(\epsilon\right) & (1.11)\end{aligned}$$

Equation 1.8 becomes

$$\begin{aligned} x_{1TT} + x_1 = -2\left(-A'\sin\left(\theta\right) - A\left(\phi + \tau\phi'\right)\cos\left(\theta\right)\right) \\ - h\left(A\cos\left(\theta\right), -A\sin\left(\theta\right)\right). \end{aligned} \tag{1.12}$$

Now we wish to eliminate the secular terms.

We can write

$$\begin{aligned}\sum_n a_n \sin\left(n\theta\right) + b_n\cos\left(n\theta\right) = -2\left(-A'\sin\left(\theta\right) - A\left(\phi+\tau\phi'\right)\cos\left(\theta\right)\right) \\ - h\left(A\cos\left(\theta\right), -A\sin\left(\theta\right)\right),\end{aligned} \tag{1.13}$$

where, in order to eliminate the secular terms, we must have $a_1 = b_1 = 0$. We have, as before, that

$$\begin{aligned} a_1 &= \frac{1}{\pi}\int_0^{2\pi}\left[-2\left(-A'\sin\left(\theta\right) - A\left(\phi+\tau\phi'\right)\cos\left(\theta\right)\right)\right. \\ &\qquad\qquad \left. - h\left(A\cos\left(\theta\right), -A\sin\left(\theta\right)\right)\right]\sin\left(\theta\right)\, d\theta & (1.14)\\ b_1 &= \frac{1}{\pi}\int_0^{2\pi}\left[-2\left(-A'\sin\left(\theta\right) - A\left(\phi+\tau\phi'\right)\cos\left(\theta\right)\right)\right. \\ &\qquad\qquad \left. - h\left(A\cos\left(\theta\right), -A\sin\left(\theta\right)\right)\right]\cos\left(\theta\right)\, d\theta. & (1.15)\end{aligned}$$

Upon requiring that $a_1 = b_1 = 0$, we then obtain that

$$A' = \frac{1}{2\pi}\int_0^{2\pi} h\left(A\cos\left(\theta\right), -A\sin\left(\theta\right)\right)\sin\left(\theta\right)\, d\theta \tag{1.16}$$

$$\phi + \tau\phi' = \frac{1}{2\pi A}\int_0^{2\pi} h\left(A\cos\left(\theta\right), -A\sin\left(\theta\right)\right)\cos\left(\theta\right)\, d\theta \tag{1.17}$$

1.1.2 Alternate Method

Our problem is once again

$$x'' + x + \epsilon h(x,x') = 0 \tag{1.18}$$

Let $\Theta(\epsilon)$ be the period of the solution. Then we can make the change of variables

$$t = \frac{\Theta(\epsilon)}{2\pi}\theta \tag{1.19}$$

so that the period is 2π in terms of the new time θ. We then obtain

$$\left(\frac{2\pi}{\Theta}\right)^2 \ddot{x} + x + \epsilon h\left(x, \left(\frac{2\pi}{\Theta}\right)\dot{x}; \epsilon\right) = 0 \tag{1.20}$$

$$\ddot{x} + x + \epsilon h\left(x, \left(\frac{2\pi}{\Theta}\right)\dot{x}; \epsilon\right) = \left(1 - \left(\frac{2\pi}{\Theta}\right)^2\right)\ddot{x}. \tag{1.21}$$

We now expand $\frac{2\pi}{\Theta} = 1 + \epsilon\phi$, so that we obtain

$$\ddot{x} + x + \epsilon h\left(x, (1+\epsilon\phi)\dot{x}; \epsilon\right) = -2\epsilon\phi\ddot{x} \tag{1.22}$$

so that $\ddot{x} + x \sim 0$ and hence

$$\ddot{x} + x + \epsilon h(x, (1+\epsilon\phi)\dot{x}; \epsilon) = 2\epsilon\phi x \tag{1.23}$$

$$\ddot{x} + x = \epsilon g(x, \dot{x}, \phi) \tag{1.24}$$

where we have defined

$$g(x, \dot{x}, \phi) = 2\phi x - h(x, (1+\epsilon\phi)\dot{x}; \epsilon). \tag{1.25}$$

Now by assumption our solution is 2π-periodic, and so we wish to find such a solution to this equation.

Now the solutions to the homogeneous equation are

$$y_1 = \sin(\theta), \qquad y_2 = \cos(\theta). \tag{1.26}$$

Using the method of variation of parameters, we write

$$x = v_1 y_1 + v_2 y_2 \tag{1.27}$$

$$x' = v_1' y_1 + v_2' y_2 + v_1 y_1' + v_2 y_2'. \tag{1.28}$$

Now if we specify that

$$v_1' y_1 + v_2' y_2 = 0 \tag{1.29}$$

we obtain that

$$x' = v_1 y_1' + v_2 y_2' \tag{1.30}$$

$$x'' = v_1' y_1' + v_2' y_2' + v_1 y_1'' + v_2 y_2'' \tag{1.31}$$

so that, from our inhomogeneous equation,

$$v_1' y_1' + v_2' y_2' = \epsilon g(x). \tag{1.32}$$

Using the conditions

$$v_1' y_1 + v_2' y_2 = 0. \tag{1.33}$$

$$v_1' y_1' + v_2' y_2' = \epsilon g(x), \tag{1.34}$$

together with

$$y_1 = \sin(\theta), \qquad\qquad y_2 = \cos(\theta), \tag{1.35}$$

we obtain that

$$v_1 = \int_0^\theta \epsilon g(x)\cos(s)\,ds + A, \tag{1.36}$$

$$v_2 = -\int_0^\theta \epsilon g(x)\sin(s)\,ds + B, \tag{1.37}$$

so that

$$x = \sin(\theta)\left(\int_0^\theta \epsilon g(x)\cos(s)\,ds + C_1\right) + \cos(\theta)\left(-\int_0^\theta \epsilon g(x)\sin(s)\,ds + C_2\right) \tag{1.38}$$

From the initial conditions, we have that $C_1 = 0$, $C_2 = -A_0$, so that

$$x = \sin(\theta)\int_0^\theta \epsilon g(x)\cos(s)\,ds - \cos(\theta)\int_0^\theta \epsilon g(x)\sin(s)\,ds + A_0\cos(\theta). \tag{1.39}$$

Now in order for our solution to be 2π-periodic, we must have $x(0) = x(2\pi)$ and $x(\pi/2) = x(2\pi + \pi/2)$, so that

$$0 = \int_0^{2\pi} g(s)\sin(s)\,ds = P, \tag{1.40}$$

$$0 = \int_0^{2\pi} g(s)\cos(s)\,ds = Q. \tag{1.41}$$

Now we wish to show that for $\epsilon \ll 1$ there is a A, k_1 that satisfy this condition. First consider $\epsilon = 0$. Then we have from above

$$0 = \int_0^{2\pi} g(s)\sin(s)\,ds \tag{1.42}$$

$$0 = \int_0^{2\pi} (2\phi_0 x_0 - h(x_0, \dot{x}_0; 0))\sin(s)\,ds \tag{1.43}$$

$$0 = \int_0^{2\pi} (2\phi_0 A_0\cos(s) - h(x_0, \dot{x}_0; 0))\sin(s)\,ds \tag{1.44}$$

$$0 = \int_0^{2\pi} h(x_0, \dot{x}_0; 0)\sin(s)\,ds = P_0 \tag{1.45}$$

and

$$0 = \int_0^{2\pi} g(s)\cos(s)\,ds \tag{1.46}$$

$$0 = \int_0^{2\pi} (2\phi_0 x_0 - h(x_0, \dot{x}_0; 0))\cos(s)\,ds \tag{1.47}$$

$$0 = \int_0^{2\pi} (2\phi_0 A_0 \cos(s) - h(x_0, \dot{x}_0; 0))\cos(s)\,ds \tag{1.48}$$

$$0 = 2\pi\phi_0 A_0 - \int_0^{2\pi} h(x_0, \dot{x}_0; 0)\cos(s)\,ds. = 2\pi\phi_0 A_0 - Q_0 \tag{1.49}$$

We can solve for A_0 and phi_0 from equations 1.45 and 1.49. We first need to confirm, though, that for ϵ small the full solution will have A and ϕ approximately given by A_0 and ϕ_0.

As a result, we with to show that for ϵ sufficiently small, there are some functions $\phi = \phi(\epsilon)$ and $A = A(\epsilon)$ analytic in ϵ and which satisfy $\phi(0) = \phi_0$, $A(0) = A_0$. The implicit function theorem will guarantee these provided that

$$\lim_{\epsilon \to 0} \left(\det \begin{bmatrix} \frac{\partial P}{\partial \phi} & \frac{\partial P}{\partial A} \\ \frac{\partial Q}{\partial \phi} & \frac{\partial Q}{\partial A} \end{bmatrix} \right) \neq 0, \tag{1.50}$$

which means that we must have

$$\det \begin{bmatrix} 0 & \frac{\partial P_0}{\partial A_0} \\ 2\pi A_0 & 2\pi\phi_0 + \frac{\partial P_0}{\partial A_0} \end{bmatrix} \neq 0 \tag{1.51}$$

$$-2\pi A_0 \frac{\partial P_0}{\partial A_0} \neq 0. \tag{1.52}$$

As a result, we can approximate the solution for ϵ small by the solution to equations 1.45 and 1.49 provided that $A_0 \neq 0$ and $\frac{\partial P_0}{\partial A_0} \neq 0$.

Note that equations 1.45 and 1.49 give the equilibrium values of A and ϕ as found from equations 1.16 and 1.17. So the result of this section is the same as that in the previous section.

1.2 Constant-Amplitude Oscillations

The first case that we consider is when a small nonlinearity is only dependent on x. We can write such a problem as

$$x'' + x + \epsilon h(x) = 0 \tag{1.53}$$

Equation 1.16 tells us that

$$A' = \frac{1}{2\pi} \int_0^{2\pi} h\left(A\cos(\theta)\right)\sin(\theta)\; d\theta \tag{1.54}$$

$$A' = 0 \tag{1.55}$$

since h is a function of a cosine, which is orthogonal to sine. As a result the amplitude of oscillations is always constant. Equation 1.17 tells us that at equilibrium

$$\phi = \frac{1}{2\pi A} \int_0^{2\pi} h\left(A\cos\left(\theta\right)\right)\cos\left(\theta\right)\, d\theta. \tag{1.56}$$

Note that in the alternate method in §1.1.2, we can no longer apply the implicit function theorem since $\frac{\partial P_0}{\partial A_0} = 0$. This makes sense since we have equilibrium for any value of A, which is A_0 in the notation of §1.1.2. Instead, however, for A given, we can apply the inverse function theorem to equation 1.41 provided that

$$\lim_{\epsilon \to 0} \frac{\partial Q}{\partial \phi} = 2\pi A_0 \neq 0. \tag{1.57}$$

1.2.1 Example

Consider the example $h(x) = x^3$. Equation 1.12 becomes

$$x_{1TT} + x_1 = 2A' \sin\left(\theta\right) + 2A\left(\phi + \tau\phi'\right)\cos\left(\theta\right) - h\left(A\cos\left(\theta\right)\right) \tag{1.58}$$

Note that for problems where h depends only on x and not on any of its derivatives, we can make the assumption that A and ϕ are time-independent, which simplifies the above equation considerably. However for the sake of consistency with the more complex examples, we will keep those terms.

$$x_{1TT} + x_1 = 2A' \sin\left(\theta\right) + 2A\left(\phi + \tau\phi'\right)\cos\left(\theta\right) - A^3\cos^3\left(\theta\right) \tag{1.59}$$

$$= 2A' \sin\left(\theta\right) + 2A\left(\phi + \tau\phi'\right)\cos\left(\theta\right) - A^3\left(\frac{3\cos\theta + \cos 3\theta}{4}\right) \tag{1.60}$$

using an identity from Appendix A. To eliminate secular terms, we must have

$$A' = 0 \tag{1.61}$$

and

$$0 = 2A\left(\phi + \tau\phi'\right) - A^3\frac{3}{4} \tag{1.62}$$

so that at equilibrium ($\phi' = 0$),

$$\phi = \frac{3}{8}A^2. \tag{1.63}$$

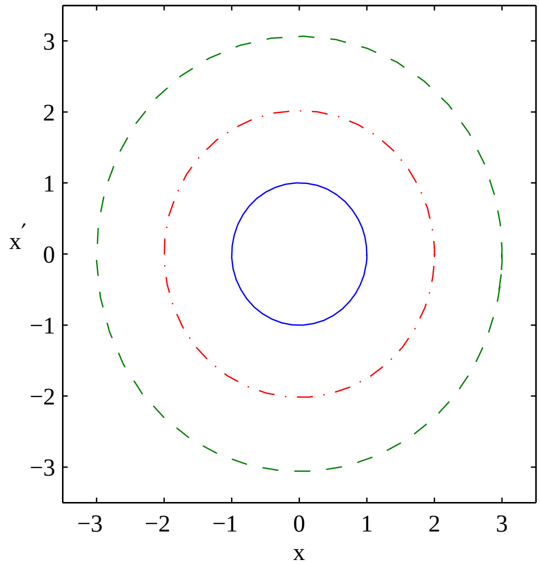

Figure 1.1: The solution to equation 1.53 with $h(x) = x^3$ and $\epsilon = 0.01$, using various initial conditions.

Checking this with the formula given in equation 1.17, we obtain

$$\begin{aligned} \phi + \tau\phi' &= \frac{1}{2\pi A}\int_0^{2\pi} h\left(A\cos\left(\theta\right)\right)\cos\left(\theta\right)\, d\theta && (1.64)\\ &= \frac{A^2}{2\pi}\int_0^{2\pi} \cos^4\left(\theta\right)\, d\theta && (1.65)\\ &= \frac{A^2}{2\pi}\int_0^{2\pi} \left(\frac{1}{8}\cos\left(4\theta\right) + \cos^2\left(\theta\right) - \frac{1}{8}\right) d\theta && (1.66)\\ &= \frac{A^2}{2\pi}\left(\pi - \frac{\pi}{4}\right) && (1.67)\\ &= \frac{3}{8}A^2. && (1.68) \end{aligned}$$

In figure 1.1, we can see that in the x-x' plane each solution remains a circle of constant radius, as expected.

Comparing with Analytical Solution

For this example, it's actually possible to find the analytical solution. Our problem is

$$x'' + x + \epsilon x^3 = 0. \tag{1.69}$$

By multiplying this by x' and integrating, we obtain that

$$\frac{x'^2}{2} + \frac{x^2}{2} + \epsilon\frac{x^4}{4} = C \tag{1.70}$$

and so we can write our problem as

$$\frac{x'^2}{2} + V(x) = E, \tag{1.71}$$

where, by the initial conditions, $E = V(A)$ and where we have defined

$$V(x) = \frac{x^2}{2} + \epsilon\frac{x^4}{4}. \tag{1.72}$$

Now $x'(t) = 0$ when

$$V(x) = V(A) \tag{1.73}$$

$$\frac{x^2}{2} + \epsilon\frac{x^4}{4} = \frac{A^2}{2} + \epsilon\frac{A^4}{4} \tag{1.74}$$

$$(x-A)(x+A) = -\frac{\epsilon}{2}(x-A)(x-A)\left(x^2+A^2\right) \tag{1.75}$$

so that either $x = \pm A$ or

$$1 = -\frac{\epsilon}{2}\left(x^2+A^2\right), \tag{1.76}$$

the later being impossible since the left-hand side is positive but the right-hand side is clearly negative.

As a result, we're in a potential well with turning points at $x = A$ and $x = -A$. Define Θ to be the period of the solution. Then by symmetry $x(\frac{\Theta}{2}) = -A$. We have that

$$\frac{x'^2}{2} + V(x) = V(A) \tag{1.77}$$

$$x' = \pm\sqrt{2\left(V(A) - V(x)\right)}, \tag{1.78}$$

where we take the negative sign as we go down from $x = A$ to $x = -A$, so that

$$\int_A^{-A} \frac{1}{\sqrt{V(A)-V(x)}}\,dx = -\int_0^{\Theta/2}\sqrt{2}\,dt \tag{1.79}$$

$$2\int_0^A \frac{1}{\sqrt{V(A)-V(x)}}\,dx = \frac{\Theta}{\sqrt{2}} \tag{1.80}$$

$$2\sqrt{2}\int_0^A \sqrt{2}\left(A^2-x^2\right)^{-1/2}\left(1+\frac{\epsilon}{2}\left(A^2+x^2\right)\right)^{-1/2} dx = \Theta. \tag{1.81}$$

Now if we let $x = A\sin(\theta)$, then this becomes

$$\Theta = 4\int_0^{\pi/2}\left(1+\frac{\epsilon A^2}{2}\left(1+\sin^2(\theta)\right)\right)^{-1/2} d\theta \tag{1.82}$$

$$\Theta \simeq 4\int_0^{\pi/2}\left(1-\frac{\epsilon A^2}{4}\left(1+\sin^2(\theta)\right)\right) d\theta \tag{1.83}$$

$$\Theta \simeq 2\pi\left(1-\epsilon\frac{3A^2}{8}\right) \tag{1.84}$$

$$\frac{\Theta}{2\pi} \simeq 1-\epsilon\frac{3A^2}{8} \tag{1.85}$$

$$\frac{2\pi}{\Theta} \simeq 1+\epsilon\frac{3A^2}{8} \tag{1.86}$$

$$\frac{2\pi}{\Theta} \simeq 1+\epsilon\phi, \tag{1.87}$$

which agrees with our previous result.

1.3 Limit Cycle

Here, we consider problems of the form

$$x'' + x + \epsilon h(x, x') = 0 \tag{1.88}$$

with $h(0,0) = 0$. This is exactly the problem considered in the introduction (§1.1). We now consider an example.

1.3.1 Example

Let

$$h(x, x') = (x^2 - 1)x'. \tag{1.89}$$

Then equations 1.16 and 1.17 tell us that we must have

$$A' = \frac{1}{2\pi}\int_0^{2\pi} h\left(A\cos(\theta), -A\sin(\theta)\right)\sin(\theta)\ d\theta, \tag{1.90}$$

$$\phi + \tau\phi' = \frac{1}{2\pi A}\int_0^{2\pi} h\left(A\cos(\theta), -A\sin(\theta)\right)\cos(\theta)\ d\theta. \tag{1.91}$$

As a result, we must have

$$A' = \frac{1}{2\pi}\int_0^{2\pi}\left(-A^3\cos^2(\theta)\sin(\theta) + A\sin(\theta)\right)\sin(\theta)\ d\theta \tag{1.92}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} -A^3\left(\frac{1+\cos(2\theta)}{2}\right)\left(\frac{1-\cos(2\theta)}{2}\right) + A\sin^2(\theta)\ d\theta \tag{1.93}$$

$$= -\frac{A^3}{8} + \frac{A}{2} \tag{1.94}$$

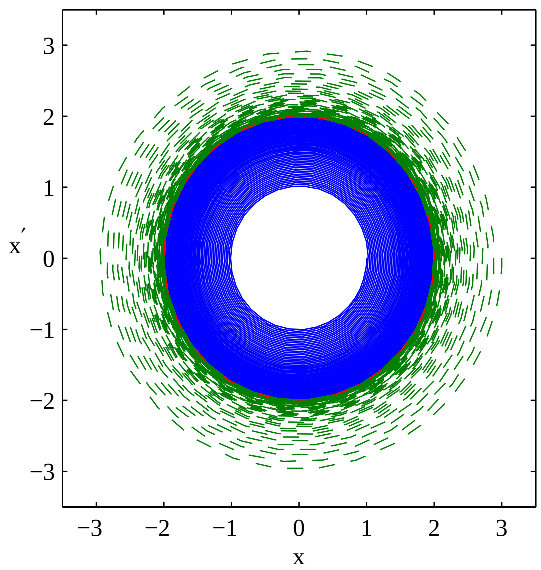

Figure 1.2: The solution to equation 1.88 with $h(x, x')$ given by equation 1.89 and $\epsilon = 0.01$, using various initial conditions.

and

$$\phi + \tau\phi' = \frac{1}{2\pi A}\int_0^{2\pi} \left(-A^3\cos^2(\theta)\sin(\theta) + A\sin(\theta)\right)\cos(\theta)\, d\theta \tag{1.95}$$

$$= \frac{1}{2\pi A}\int_0^{2\pi} -A^3\left(\frac{3\cos(\theta)+\cos(3\theta)}{4}\right)\sin(\theta) + A\sin(\theta)\cos(\theta)$$

$$= 0, \tag{1.96}$$

by using the identities in Appendix A. As a result,

$$\phi = \frac{C}{\tau} \tag{1.97}$$

where C is some constant, and so

$$\theta = T + C. \tag{1.98}$$

By setting $A' = 0$, we see that we get a steady state for $A = 2$ and this is stable. This is shown in figure 1.2, where all the solutions in the x-x' plane approach a circle of radius 2.

1.4 Forcing

Here we consider problems of the form

$$x'' + x + \epsilon h(x, x') = \epsilon \cos(\omega t) \tag{1.99}$$

$$x'' + x + \epsilon \left(h(x, x') - \cos(\omega t)\right) = 0 \tag{1.100}$$

where $\omega = 1 + \epsilon k_1$, so we're near resonance.

In this case, equation 1.16 becomes

$$A' = \frac{1}{2\pi} \int_0^{2\pi} \left[h\left(A\cos(\theta), -A\sin(\theta)\right) - \cos(\omega t)\right] \sin(\theta)\ d\theta \tag{1.101}$$

where

$$\cos(\omega t) = \cos(t + k_1 \epsilon t) \tag{1.102}$$

$$= \cos(t + \phi\tau + k_1\tau - \phi\tau) \tag{1.103}$$

$$= \cos(\theta + k_1\tau - \phi\tau) \tag{1.104}$$

$$= \cos(\theta)\cos(k_1\tau - \phi\tau) - \sin(\theta)\sin(k_1\tau - \phi\tau) \tag{1.105}$$

$$= \cos(\theta)\cos(\psi) + \sin(\theta)\sin(\psi) \tag{1.106}$$

where we have defined $\psi = \phi\tau - k_1\tau$, so that we obtain

$$A' = \frac{1}{2\pi} \int_0^{2\pi} \left(-\cos(\theta)\cos(\psi) - \sin(\theta)\sin(\psi)\right) \sin(\theta)\ d\theta + \frac{1}{2\pi} \int_0^{2\pi} h\left(A\cos(\theta), -A\sin(\theta)\right) \sin(\theta)\ d\theta \tag{1.107}$$

$$= -\frac{1}{2}\sin(\psi) + \frac{1}{2\pi} \int_0^{2\pi} h\left(A\cos(\theta), -A\sin(\theta)\right) \sin(\theta)\ d\theta. \tag{1.108}$$

Likewise, equation 1.17 becomes

$$\phi + \tau\phi' = \frac{1}{2\pi A} \int_0^{2\pi} \left[h\left(A\cos(\theta), -A\sin(\theta)\right) - \cos(\omega t)\right] \cos(\theta)\ d\theta \tag{1.109}$$

$$\phi + \tau\phi' = \frac{1}{2\pi A} \int_0^{2\pi} \left(-\cos(\theta)\cos(\psi) - \sin(\theta)\sin(\psi)\right) \cos(\theta)\ d\theta + \frac{1}{2\pi A} \int_0^{2\pi} h\left(A\cos(\theta), -A\sin(\theta)\right) \cos(\theta)\ d\theta \tag{1.110}$$

$$\phi + \tau\phi' = -\frac{1}{2A}\cos(\psi) + \frac{1}{2\pi A} \int_0^{2\pi} h\left(A\cos(\theta), -A\sin(\theta)\right) \cos(\theta)\ d\theta \tag{1.111}$$

$$\psi' + k_1 = -\frac{1}{2A}\cos(\psi) + \frac{1}{2\pi A} \int_0^{2\pi} h\left(A\cos(\theta), -A\sin(\theta)\right) \cos(\theta)\ d\theta \tag{1.112}$$

We can then solve this for the equilibrium values of A and ψ and then determine the stability of this equilibrium.

1.4.1 Example

Let

$$h(x,x') = \lambda x' + \kappa x^3. \tag{1.113}$$

Following the above analysis, we obtain that

$$2A' = -\sin(\psi) - \lambda A \tag{1.114}$$
$$2A\psi' = -\cos(\psi) + \frac{3}{4}\kappa A^3 - 2Ak_1. \tag{1.115}$$

Let (A_e, ψ_e) be an equilibrium solution to this. Then

$$0 = -\sin(\psi_e) - \lambda A_e \tag{1.116}$$
$$0 = -\cos(\psi_e) + \frac{3}{4}\kappa A_e^3 - 2A_e k_1, \tag{1.117}$$

which can be rewritten as

$$\lambda A_e = -\sin(\psi_e) \tag{1.118}$$
$$2A_e k_1 - \frac{3}{4}\kappa A_e^3 = -\cos(\psi_e), \tag{1.119}$$

so that

$$\lambda^2 A_e^2 + \left(2A_e k_1 - \frac{3}{4}\kappa A_e^3\right)^2 = 1 \tag{1.120}$$
$$\frac{3}{8}\kappa A_e^2 \pm \frac{1}{2}\sqrt{\frac{1}{A_e^2} - \lambda^2} = k_1. \tag{1.121}$$

As a result, for every $0 < A_e < 1/\lambda$, there are two values of k_1. Now we wish to determine the shape of the bifurcation diagram in the A-k_1 plane. First we look for bifurcation points in the A-k_1 plane. They will occur when

$$\frac{dk_1}{dA_e} = 0. \tag{1.122}$$

Differentiating the above equation, we obtain

$$\frac{dk_1}{dA_e} = \frac{3}{4}\kappa A_e \pm \frac{1}{4}\frac{1}{\sqrt{\frac{1}{A_e^2} - \lambda^2}}\frac{-2}{A_e^2} \tag{1.123}$$
$$0 = 3\kappa A_e^4 \pm \frac{-2}{\sqrt{\frac{1}{A_e^2} - \lambda^2}} \tag{1.124}$$
$$0 = \frac{4}{9\kappa^2 A_e^8 \lambda^2} - \frac{1}{A_e^2\lambda^2} + 1 \tag{1.125}$$
$$0 = b\xi^4 - \xi + 1 \tag{1.126}$$

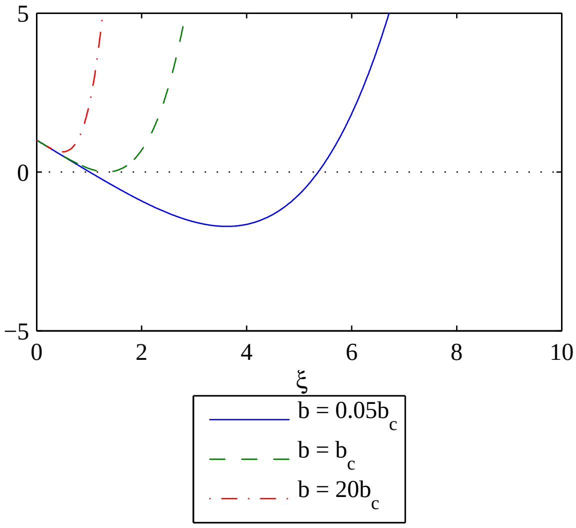

Figure 1.3: A plot of $h(\xi)$ for $\lambda = 1$ and for different values of b.

where we have defined

$$\xi = \frac{1}{A_e^2 \lambda^2}, \qquad b = \frac{4\lambda^6}{9\kappa^2}. \tag{1.127}$$

This is denoted $h(\xi)$ and is plotted for different values of b in figure 1.3. Now the number of zeros that

$$h(\xi) = b\xi^4 - \xi + 1 \tag{1.128}$$

has changes when $h(\xi) = 0$ and $h'(\xi) = 0$ simultaneously. This occurs only when

$$\xi = \xi_c = \frac{4}{3}, \qquad b = b_c = \frac{3^3}{4^4}, \tag{1.129}$$

which is when

$$\pm k = k_c(\lambda) = \left(\frac{4}{3}\right)^{5/2} \lambda^3. \tag{1.130}$$

As a result, $h(\xi)$ will have no solution for $|k| < k_c$, but will have two solutions for $|k| > k_c$. Note that for $h(\xi)$ to be zero, we must have $\xi > 1$ and so every zero is in the range $\xi > 1$ so that $A_e < 1/\lambda$. As a result, our bifurcation diagram in the A-k_1 plane will have no bifurcations for $|k| < k_c$, but will have two bifurcations for $|k| > k_c$. This is shown in figure 1.4.

We would like to be able to compare this expected amplitude of oscillation with that of a simulation by letting

$$k_1 = 3 + 5\sin(\delta t) \tag{1.131}$$

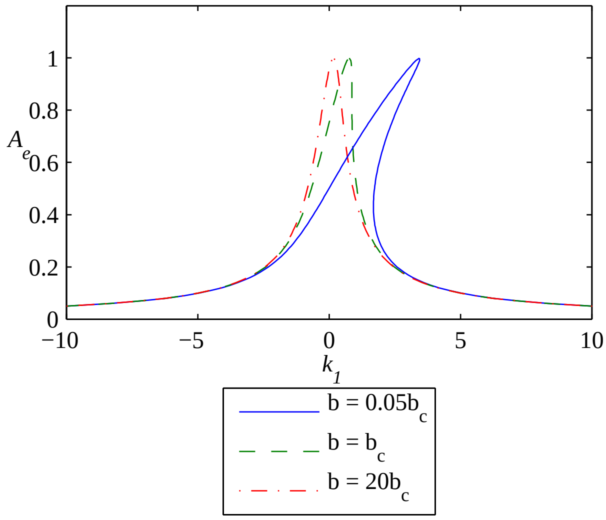

Figure 1.4: The location of the equilibrium solutions in the k_1-A plane for $\lambda = 1$ and various values of κ, here listed in terms of their corresponding b values.

for $\delta \ll \epsilon$. Unfortunately, this produces figure 1.5, where the solution differs wildly from the stable states. If we discretize the time-dependence of k_1 by letting

$$k_1 = \frac{\lfloor 10\left(3 + 5\sin\left(\delta t\right)\right)\rfloor}{10}, \tag{1.132}$$

we obtain figure 1.6, where the solution agrees very well with the stable states. We can see where the problem arises by looking at figure 1.7, a zoomed-in version of figure 1.6. Every time k_1 changes even very slightly, this causes the solution to oscillate wildly for a brief amount of time. As a result, the continuously changing k_1 used in figure 1.5 causes large deviations in the solution.

Stability

We have that

$$A' = -\frac{\lambda}{2}A - \frac{1}{2}\sin\left(\psi\right) = f\left(A, \psi\right) \tag{1.133}$$

$$\psi' = -k_1 + \frac{3\kappa A^2}{8} - \frac{1}{2A}\cos\left(\psi\right) = g\left(A, \psi\right). \tag{1.134}$$

The stability is then determined from the signs of the eigenvalues of

$$M = \begin{bmatrix} f_A & f_\psi \\ g_A & g_\psi \end{bmatrix} = \begin{bmatrix} -\frac{\lambda}{2} & -\frac{1}{2}\cos\left(\psi\right) \\ \frac{3\kappa A}{4} + \frac{\cos(\psi)}{2A^2} & \frac{\sin(\psi)}{2A} \end{bmatrix}. \tag{1.135}$$

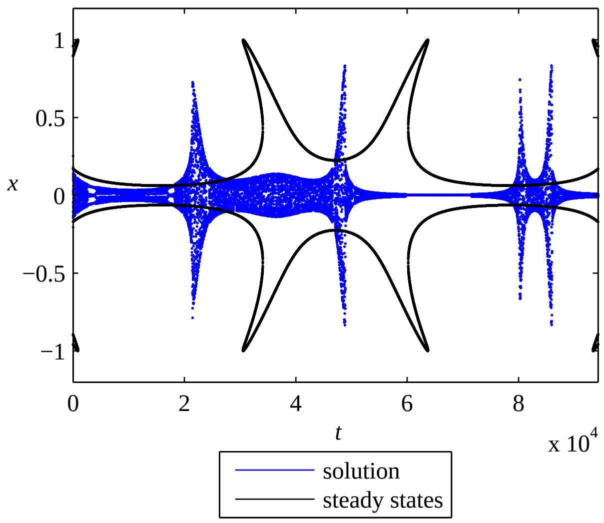

Figure 1.5: A simulation of equation 1.99 with $\lambda = 1$, $\kappa = \frac{2^6\sqrt{5}}{3^{5/2}}$, $\epsilon = 0.1$, $\delta = 0.0001$, $k_1 = 3 + 5\sin(\delta t)$.

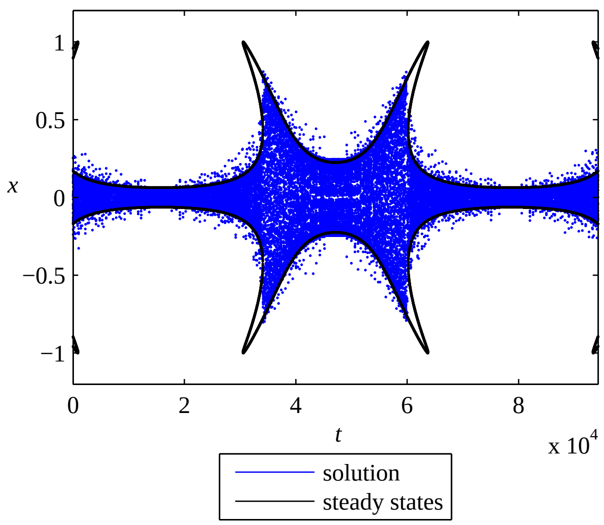

Figure 1.6: A simulation of equation 1.99 with $\lambda = 1$, $\kappa = \frac{2^6\sqrt{5}}{3^{5/2}}$, $\epsilon = 0.1$, $\delta = 0.0001$, $k_1 = \frac{\lfloor 10(3+5\sin(\delta t))\rfloor}{10}$.

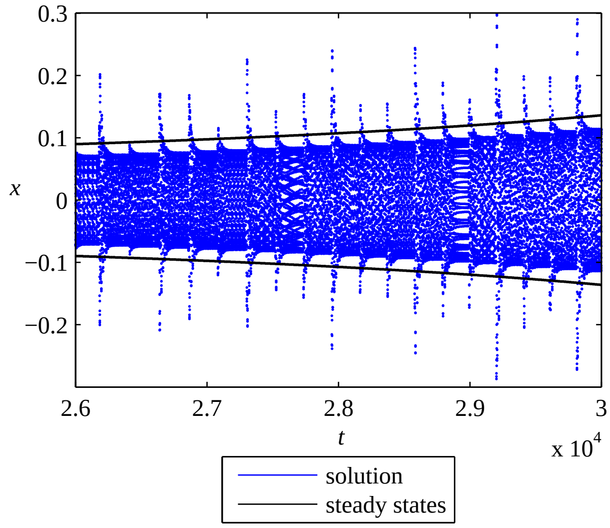

Figure 1.7: A zoomed-in version of figure 1.6, which has $\lambda = 1$, $\kappa = \frac{2^6\sqrt{5}}{3^{5/2}}$, $\epsilon = 0.1$, $\delta = 0.0001$, $k_1 = \frac{\lfloor 10(3+5\sin(\delta t))\rfloor}{10}$.

We determine the signs of the eigenvalues by finding the sign of the trace and determinant.

$$\mathrm{tr}(M) = -\frac{\lambda}{2} + \frac{\sin(\psi)}{2A} \tag{1.136}$$

Now we know that at the equilibrium points,

$$\lambda A = -\sin(\psi) \tag{1.137}$$

$$2Ak_1 - \frac{3}{4}\kappa A^3 = -\cos(\psi) \tag{1.138}$$

and so

$$\mathrm{tr}(M) = -\frac{\lambda}{2} + \frac{-\lambda A}{2A} = -\lambda < 0. \tag{1.139}$$

Now consider the determinant.

$$\det(M) = \left(-\frac{\lambda}{2}\right)\left(\frac{\sin(\psi)}{2A}\right) - \left(\frac{3\kappa A}{4} + \frac{\cos(\psi)}{2A^2}\right)\left(-\frac{1}{2}\cos(\psi)\right) \tag{1.140}$$

$$= \frac{\lambda^2}{4} + \frac{3\kappa A}{8}\cos(\psi) + \frac{\cos^2(\psi)}{4A^2} \tag{1.141}$$

$$= \frac{\lambda^2}{4} + \frac{3\kappa A}{8}\cos(\psi) + \frac{1}{4A^2} - \frac{\lambda^2}{4} \tag{1.142}$$

$$= \frac{3}{8}\kappa A^2\left(-2k_1 + \frac{3}{4}\kappa A^2 + \frac{2}{3\kappa A^4}\right) \tag{1.143}$$

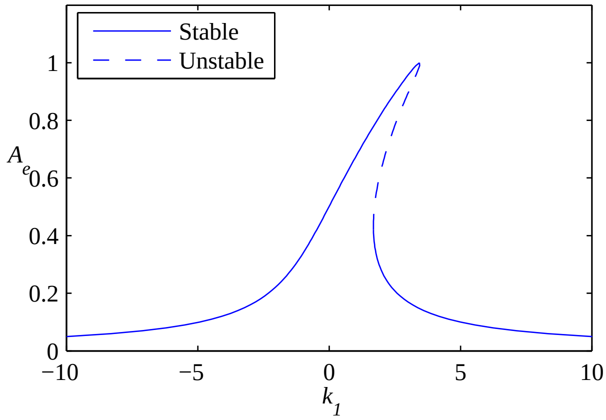

Figure 1.8: The bifurcation diagram for equation 1.99 with h given by equation 1.113. This is shown for $\lambda = 1$, $\kappa = \frac{2}{3}\frac{\lambda^3}{\sqrt{b}}$ where $b = 0.05b_c$.

This will be positive (and so the solution will be stable) if and only if

$$-2k_1 + \frac{3}{4}\kappa A^2 + \frac{2}{3\kappa A^4} > 0 \tag{1.144}$$

$$\left(2k_1 - \frac{3}{4}\kappa A^2\right)\frac{3\kappa}{2}A^4 < 1. \tag{1.145}$$

Note that we get equality at the bifurcation points. This can be seen by performing implicit differentiation on equation 1.120 and setting $\frac{dk_1}{dA_e} = 0$. As a result, the stability of the equilibrium solution only changes sign at the bifurcation points. Note that for k_1 near zero, we obtain approximately

$$-\frac{9}{8}\kappa^2 A^6 < 0 < 1 \tag{1.146}$$

so the equilibrium is stable. As a result, we must obtain bifurcation diagrams as shown in figure 1.8.

1.5 Entrainment

We now consider the forced van der Pol oscillator:

$$x'' + x - \epsilon\left(1 - x^2\right)x' = \epsilon F \cos(\omega t) \tag{1.147}$$

where we assume that ω is close to, but not exactly at, the resonant frequency of the problem. We write

$$\omega = 1 + \epsilon k_1. \tag{1.148}$$

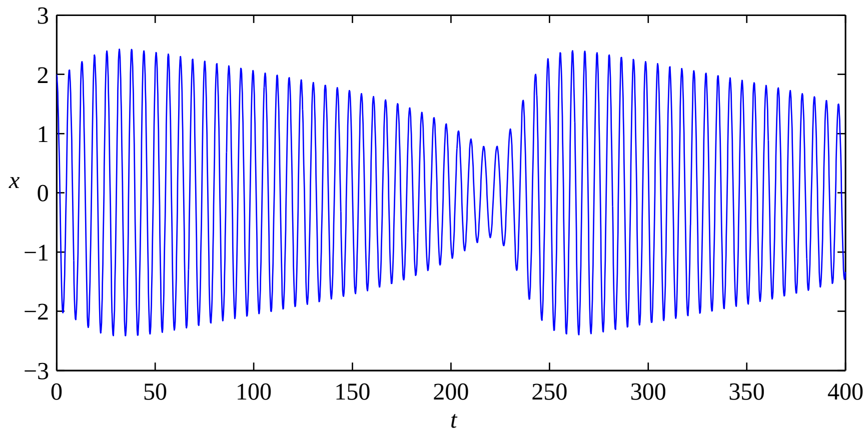

Figure 1.9: The solution to equation 1.147 with $\epsilon = 0.1$, $k_1 = 0.4$, $F = 1.2$ and with initial conditions $x(0) = 2$, $x'(0) = 0$.

This problem exhibits *entrainment*, where the unforced oscillations are quenched by the forcing function. For example, consider the simulations shown in figures 1.9 and 1.10. In figure 1.9, the solution travels in an out of phase with respect to the forcing function. In comparison, in figure 1.10, where the value of F has only been slightly increased, the solution has been entrained by the forcing function, so the solution goes toward a stable limit cycle with constant amplitude and phase.

We now wish to determine the region in the parameter-space where entrainment can occur. We will do this by looking for stable solutions of constant amplitude and phase. Performing the same analysis as before, we obtain

$$A' = \frac{A}{8}\left(4 - A^2\right) - \frac{F}{2}\sin\left(\phi\tau - k_1\tau\right) \tag{1.149}$$

$$\phi + \tau\phi' = \frac{-F}{2A}\cos\left(\phi\tau - k_1\tau\right). \tag{1.150}$$

Now we let $\psi = \phi\tau - k_1\tau$ to obtain

$$A' = \frac{A}{8}\left(4 - A^2\right) - \frac{F}{2}\sin\left(\psi\right) \tag{1.151}$$

$$-\psi' = k_1 + \frac{F}{2A}\cos\left(\psi\right). \tag{1.152}$$

At an equilibrium solution, we must then have

$$\frac{A_e}{8}\left(4 - A_e^2\right) = -\frac{F}{2}\sin\left(\psi_e\right) \tag{1.153}$$

$$-k_1 A_e = \frac{F}{2}\cos\left(\psi_e\right), \tag{1.154}$$

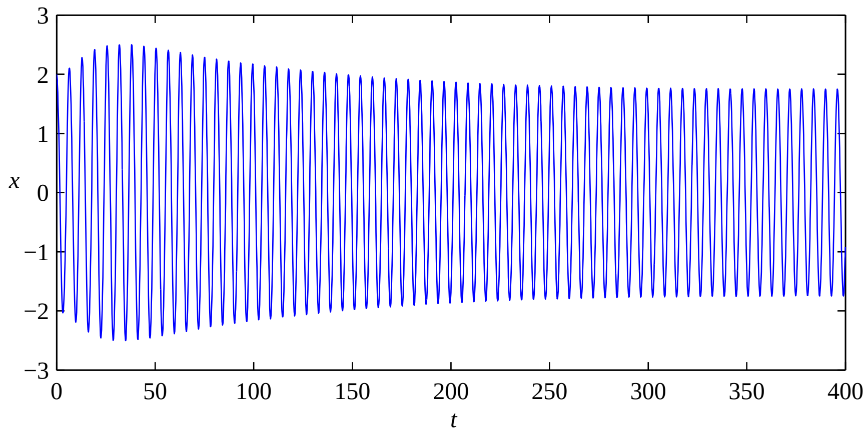

Figure 1.10: The solution to equation 1.147 with $\epsilon = 0.1$, $k_1 = 0.4$, $F = 1.5$ and with initial conditions $x(0) = 2$, $x'(0) = 0$.

so that

$$F^2 = A_e^2 \left(1 - \frac{A_e^2}{4}\right)^2 + 4k_1^2 A_e^2 \tag{1.155}$$

and so we define

$$g(A_e) = \frac{u^3}{16} - \frac{u^2}{2} + \left(4k_1^2 + 1\right) u - F^2 = 0 \tag{1.156}$$

where we have defined $u = A_e^2$.

Now we look at the stability of the equilibrium solution. If it is stable, we have entrainment; if it is not, we do not. Let $A = A_e + A_\Delta$ and $\psi = \psi_e + \psi_\Delta$. Then from equation 1.151, linearising in A_Δ and ψ_Δ, we obtain

$$A'_\Delta = \frac{A_e + A_\Delta}{2} - \frac{(A_e + A_\Delta)^3}{8} - \frac{F}{2} \sin(\psi_e + \psi_\Delta) \tag{1.157}$$

$$= \frac{A_e}{2} + \frac{A_\Delta}{2} - \frac{A_e^3}{8} - \frac{3A_e^2 A_\Delta}{8} - \frac{F}{2} \left(\sin(\psi_e)\cos(\psi_\Delta) + \cos(\psi_e)\sin(\psi_\Delta)\right) \tag{1.158}$$

$$= \frac{A_e}{2} + \frac{A_\Delta}{2} - \frac{A_e^3}{8} - \frac{3A_e^2 A_\Delta}{8} - \frac{F}{2}\sin(\psi_e) - \frac{F}{2}\cos(\psi_e)\psi_\Delta \tag{1.159}$$

$$= \frac{A_\Delta}{2} - \frac{3A_e^2 A_\Delta}{8} - \frac{F}{2}\cos(\psi_e)\psi_\Delta \tag{1.160}$$

$$= \frac{A_\Delta}{2} - \frac{3A_3^2 A_\Delta}{8} + k_1 A_e \psi_\Delta. \tag{1.161}$$

Likewise from equation 1.152 we obtain

$$-\psi'_\Delta = k_1 + \frac{F}{2\left(A_e + A_\Delta\right)} \cos\left(\psi_e + \psi_\Delta\right) \tag{1.162}$$

$$= k_1 + \frac{F}{2}\left(\frac{1}{A_e} - \frac{A_\Delta}{A_e^2}\right)\left(\cos\left(\psi_e\right)\cos\left(\psi_\Delta\right) - \sin\left(\psi_e\right)\sin\left(\psi_\Delta\right)\right) \tag{1.163}$$

$$= k_1 + \frac{F}{2}\left(\frac{1}{A_e} - \frac{A_\Delta}{A_e^2}\right)\left(\cos\left(\psi_e\right) - \sin\left(\psi_e\right)\psi_\Delta\right) \tag{1.164}$$

$$= -\frac{F}{2A_e^2}\cos\left(\psi_e\right) A_\Delta - \frac{F}{2A_e}\sin\left(\psi_e\right)\psi_\Delta \tag{1.165}$$

$$= \frac{k_1 A_e}{A_e^2} A_\Delta + \frac{1}{A_e}\left(-\frac{A_e}{2} + \frac{A_e^3}{8}\right)\psi_\Delta \tag{1.166}$$

$$= \frac{k_1}{A_e} A_\Delta + \left(-\frac{1}{2} + \frac{A_e^2}{8}\right)\psi_\Delta \tag{1.167}$$

We can put these together and write

$$\begin{bmatrix} A'_\Delta \\ \psi'_\Delta \end{bmatrix} = \begin{bmatrix} \frac{1}{2} - \frac{3}{8}A_e^2 & k_1 A_e \\ -\frac{k_1}{A_e} & \frac{1}{2} - \frac{1}{8}A_e^2 \end{bmatrix} \begin{bmatrix} A_\Delta \\ \psi_\Delta \end{bmatrix}. \tag{1.168}$$

As a result, the stability of the equilibrium solution is determined by the signs of the eigenvalues of the matrix

$$M = \begin{bmatrix} \frac{1}{2} - \frac{3}{8}A_e^2 & -k_1 A_e \\ \frac{k_1}{A_e} & \frac{1}{2} - \frac{1}{8}A_e^2 \end{bmatrix}. \tag{1.169}$$

We will calculate this by considering the sign of its determinant and trace:

$$\operatorname{tr}\left(M\right) = 1 - \frac{A_e^2}{2}, \tag{1.170}$$

$$\det\left(M\right) = \left(\frac{1}{2} - \frac{3}{8}A_e^2\right)\left(\frac{1}{2} - \frac{1}{8}A_e^2\right) + k_1^2. \tag{1.171}$$

In order for the eigenvalues of M to both be negative, we must have $\operatorname{tr}(M) < 0$ and $\det(M) > 0$. For the trace to be negative, we need

$$u = A_e^2 > 2. \tag{1.172}$$

By equation 1.155, we get equality when

$$F^2 = \frac{1}{2} + 8k_1^2. \tag{1.173}$$

Now in the k_1-F plane, $\operatorname{tr}(M)$ can also change sign at the turning points of equation 1.155, which occur when $g'(u) = 0$. Inserting this condition into $g(u) = 0$ from equation 1.155, we obtain that the turning points occur at

$$\frac{F^4}{16} - \frac{F^2}{27}\left(1 + 36k_1^2\right) + \frac{16}{27}k_1^2\left(1 + 4k_1^2\right)^2 = 0 \tag{1.174}$$

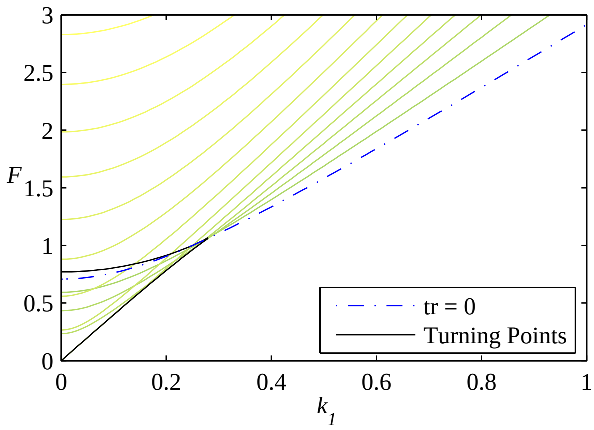

Figure 1.11: The lines corresponding to the turning points and to $\text{tr}(M) = 0$. The yellow/ green lines are lines of constant u indicating the region in which $\text{tr}(M) < 0$.

in the k_1-F plane. The lines in the k_1-F plane corresponding to the turning points and to $\text{tr}(M) = 0$ are shown in figure 1.11, together with the region in which $\text{tr}(M) < 0$.

Now the determinant of M is zero when

$$0 = \left(\frac{1}{2} - \frac{3}{8}A_e^2\right)\left(\frac{1}{2} - \frac{1}{8}A_e^2\right) + k_1^2. \tag{1.175}$$

Solving this for u, plugging the result into equation 1.155 and multiplying the two answers together, we obtain that

$$\frac{F^4}{16} - \frac{F^2}{27}\left(1 + 36k_1^2\right) + \frac{16}{27}k_1^2\left(1 + 4k_1^2\right)^2 = 0, \tag{1.176}$$

which can be solved for F in terms of k_1. Note that this is the same as the equation for the turning points. These are shown in figure 1.12, together with the region in which $\det(M) > 0$.

Putting this information together, we then obtain that the region of entrainment is the region as shown in figure 1.13. The lines $\text{tr}(M) = 0$ and $\det(M) = 0$ which form the border for this region intersect at some point P. Since at this point, $\det(M) = 0$, we must have

$$\frac{F^4}{16} - \frac{F^2}{27}\left(1 + 36k_1^2\right) + \frac{16}{27}k_1^2\left(1 + 4k_1^2\right)^2 = 0. \tag{1.177}$$

Since we also have $\text{tr}(M) = 0$, we must have

$$F^2 = \frac{1}{2} + 8k_1^2. \tag{1.178}$$

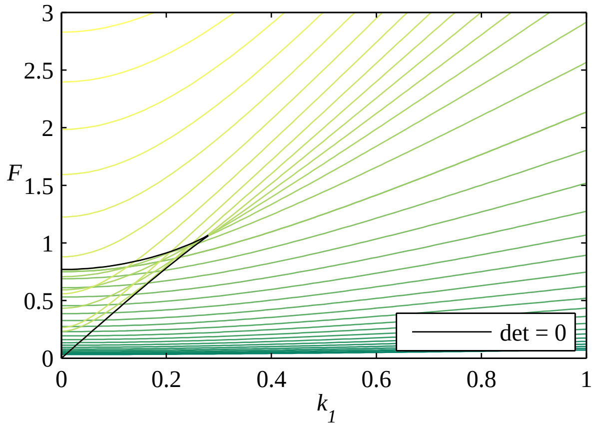

Figure 1.12: The lines corresponding to $\det(M) = 0$. The yellow/ green lines are lines of constant u indicating the region in which $\det(M) > 0$.

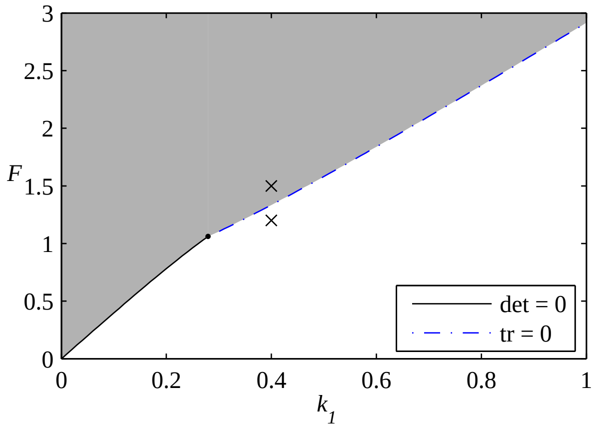

Figure 1.13: The region of entrainment for equation 1.147. The point of intersection between the two lines is referred to in the text as the point P. The two other points are the parameter values used in figures 1.9 and 1.10.

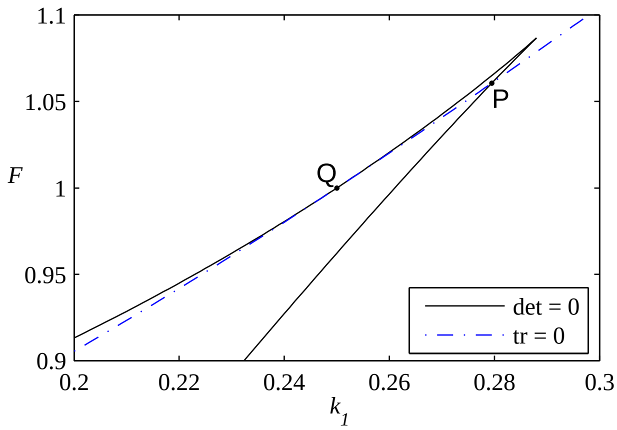

Figure 1.14: A zoomed-in version of figure 1.11, showing the intersections between the lines $\mathrm{tr}(M) = 0$ and $\det(M) = 0$.

Putting these two equations together, and letting $\xi = k_1^2$, we obtain

$$\frac{\left(\frac{1}{2} + 8\xi\right)}{16} - \frac{\left(\frac{1}{2} + 8\xi\right)(1 + 36\xi)}{27} + \frac{16}{27}\xi\,(1 + 4\xi)^2 = 0. \tag{1.179}$$

Solving this for ξ, we obtain that $\xi = 5/64, 1/16$. These correspond to $k_1 = \sqrt{5}/8$, $F = 3/\sqrt{8}$ and $k_1 = 1/4$, $F = 1$. The intersection point which corresponds to P, the intersection point along the border of the region of entrainment, is $k_1 = \sqrt{5}/8$, $F = 3/\sqrt{8}$. We label the other intersection point, $k_1 = 1/4$, $F = 1$, as Q. These two intersections are shown in the zoomed-in figure 1.14.

Appendix A

Trigonometric Identities

$$\cos(A)\cos(B) = \frac{1}{2}\left(\cos(A-B) + \cos(A+B)\right) \quad \text{(A.1)}$$

$$\sin(A)\cos(B) = \frac{1}{2}\left(\sin(A-B) + \sin(A+B)\right) \quad \text{(A.2)}$$

$$\sin(A)\sin(B) = \frac{1}{2}\left(\cos(A-B) - \cos(A+B)\right) \quad \text{(A.3)}$$

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$\sin\theta\cos\theta = \frac{\sin 2\theta}{2}$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$\sin^3\theta = \frac{3\sin\theta-\sin 3\theta}{4}$$

$$\sin^2\theta\cos\theta = \frac{\cos\theta-\cos 3\theta}{4}$$

$$\sin\theta\cos^2\theta = \frac{\sin\theta+\sin 3\theta}{4}$$

$$\cos^3\theta = \frac{3\cos\theta+\cos 3\theta}{4}$$

$$\sin^4\theta = \frac{3-4\cos 2\theta+\cos 4\theta}{8}$$

$$\sin^3\theta\cos\theta = \frac{2\sin 2\theta-\sin 4\theta}{8}$$

$$\sin^2\theta\cos^2\theta = \frac{1-\cos 4\theta}{8}$$

$$\sin\theta\cos^3\theta = \frac{2\sin 2\theta+\sin 4\theta}{8}$$

$$\cos^4\theta = \frac{3+4\cos 2\theta+\cos 4\theta}{8}$$