HOPF BIFURCATIONS UNDER A SLOW SWEEP

WE CONSIDER THE NONLINEAR SYSTEM

$$(1)\quad \begin{cases} x' = -y + \mu x - x(x^2+y^2) \\ y' = x + \mu y - y(x^2+y^2) \end{cases}$$

WHERE THE BIFURCATION PARAMETER μ SATISFIES $\mu' = \varepsilon$ WITH $0 < \varepsilon \ll 1$.

NOW WE LET $x = \rho\cos\phi$, $y = \rho\sin\phi$ WITH $\rho = \rho(t)$ AND $\Phi = \Phi(t)$.

WE HAVE $\rho = (x^2+y^2)^{1/2}$ AND $\tan\phi = \dfrac{y}{x}$ SO THAT

$$\rho\rho' = xx' + yy' \quad \text{AND} \quad \phi' = \frac{xy' - yx'}{x^2+y^2}. \qquad (2)$$

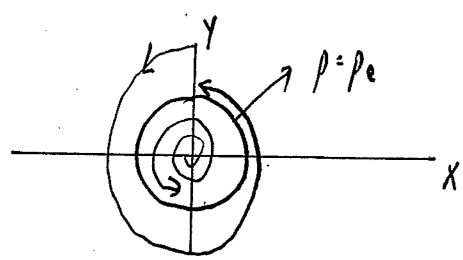
NOW USING (1) WE CALCULATE FROM (2) THAT

$$\rho\rho' = \mu(x^2+y^2) - (x^2+y^2)^2 \quad \text{AND} \quad \phi' = \frac{x[x+\mu y - y(x^2+y^2)] - y[-y+\mu x - x(x^2+y^2)]}{x^2+y^2}$$

THUS $\rho' = \mu\rho - \rho^3$ AND $\phi' = 1$.

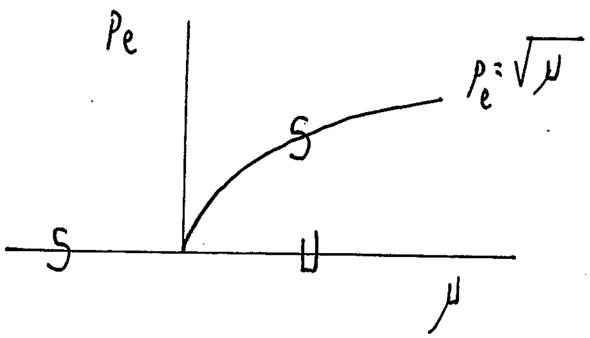
WE CONCLUDE THAT (1) BECOMES

$$(3)\quad \begin{cases} \rho' = \rho(\mu - \rho^2) \\ \phi' = 1 \\ \mu' = \varepsilon \end{cases}$$



EQUILIBRIUM VALUES $\rho = \rho_e$ CORRESPOND TO PERIODIC SOLUTIONS IN THE x VS t, y VS t PLANE. THEY ARE LIMIT CYCLES, "ISOLATED PERIODIC SOLUTIONS"

FOR μ FIXED WE OBTAIN THE BIFURCATION DIAGRAM



THIS YIELDS THE BIFURCATION DIAGRAM AND NOW WE WILL LET

$$\mu' = \varepsilon \quad \text{SO} \quad \mu = \mu_0 + \varepsilon t$$

WILL BE THE SLOW SWEEP.

PREDICTION OF TRANSITION TIME

WE LET $\mu = \mu(\varepsilon t)$ TO GET

$$p' = p\,[\mu(\varepsilon t) - p^2].$$

THEN $\tau = \varepsilon t$ AND $d/dt = \varepsilon\, d/d\tau$ SO THAT

$$\varepsilon \dot{p} = p\,[\mu(\tau) - p^2] \qquad \text{WITH} \quad \dot{\mu} = 1 \quad \text{AND} \quad \dot{} = d/d\tau$$

NOW SUPPOSE $\mu(0) = \mu_0 < 0$ AND ASK HOW LONG CAN THE SOLUTION REMAIN NEAR $p = 0$?

WE USE THE WKB ANSATZ IN THE LINEARIZATION

$$\varepsilon \dot{p} = \mu(\tau)\, p$$

GIVEN BY $p = A\, e^{\Phi(\tau)/\varepsilon}$. WE GET

$$\varepsilon \left[\frac{\dot{\Phi}}{\varepsilon} A + \dot{A} \right] e^{\Phi/\varepsilon} = A \mu\, e^{\Phi/\varepsilon} \longrightarrow \dot{\Phi} + \varepsilon \frac{\dot{A}}{A} = \mu$$

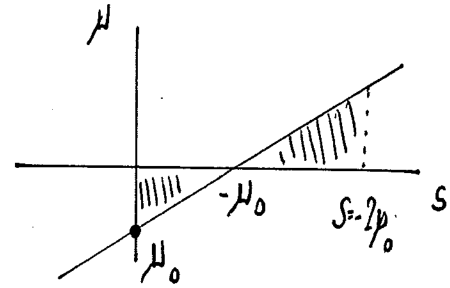
THUS FOR $\varepsilon \ll 1$ WE OBTAIN TO LEADING ORDER THAT

$$\dot{\Phi} = \mu .$$

FOR THE LINEARIZATION TO BE VALID, I.E. FOR THE SOLUTION TO REMAIN NEAR $p = 0$ WE MUST HAVE $\Phi < 0$ FOR THE RANGE OF τ.

NOW WE CALCULATE

$$\Phi(\tau) = \int_0^{\tau} \mu(s)\, ds \qquad \text{WITH} \quad \mu = \mu_0 + s$$



WHERE WE TAKE $\mu_0 < 0$.

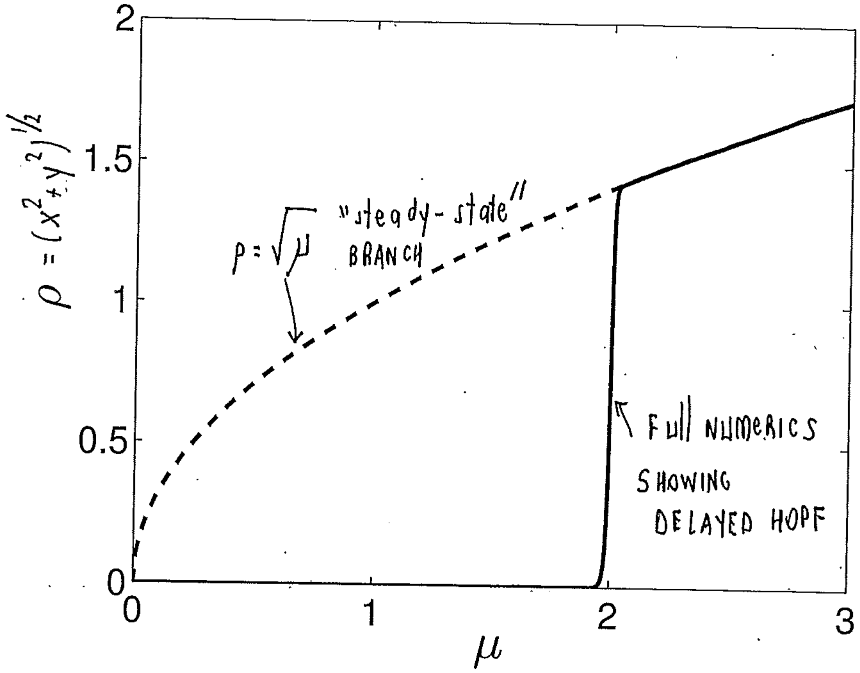
FROM THE PLOT WE OBSERVE THAT

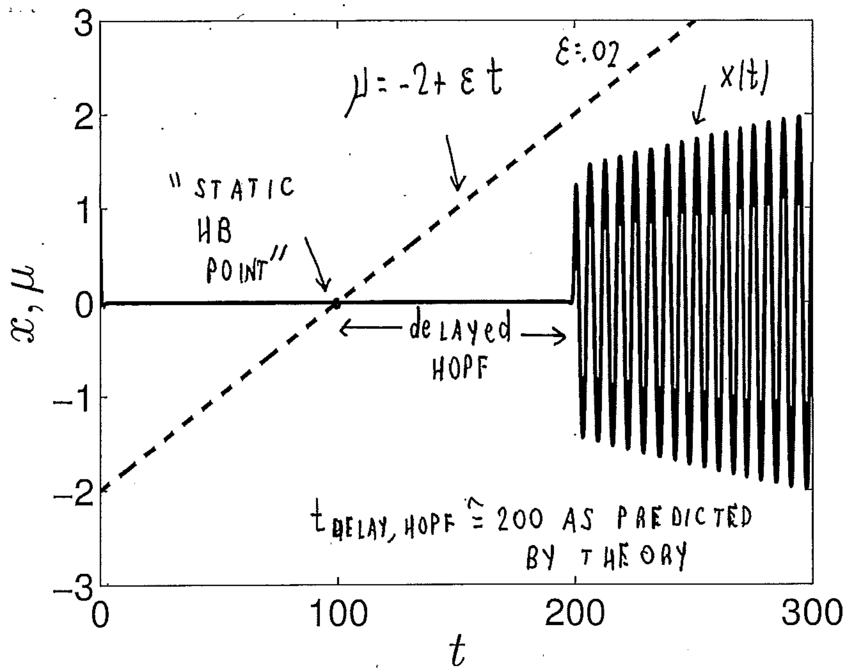
$$\Phi < 0 \quad \text{ONLY FOR} \quad 0 < \tau < -2\mu_0$$

AND THAT $\Phi = 0$ WHEN $\tau = -2\mu_0$, CALLED THE TRANSITION TIME. IN TERMS OF ORIGINAL t VARIABLE, TRANSITION OCCURS AT $t = -2\mu_0/\varepsilon$.

IF $\mu_0 = -2$ AND $\varepsilon = .02$ WE PREDICT $t_{TRANS} \cong 200$.

$\mu(\varepsilon t) = -2 + \varepsilon t$, $\varepsilon = .020$ $x(0) = y(0) = 1$





DELAYED - HOPF BIFURCATION ON A 2×2 NONLINEAR SYSTEM

WE CONSIDER THE NONLINEAR SYSTEM WITH SLOW SWEEP:

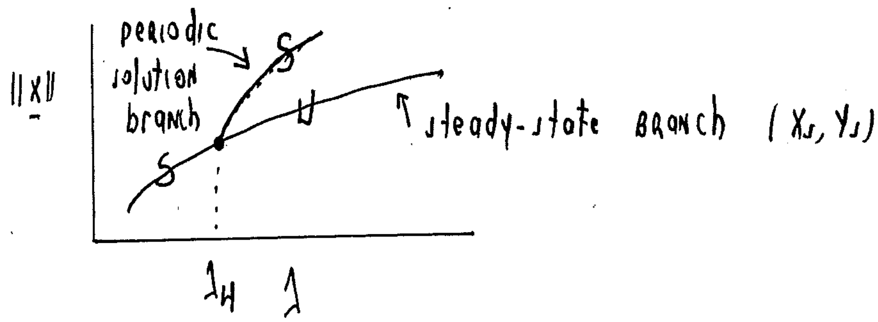
$$(1)\quad \begin{cases} x' = f(x,y,\lambda) \\ y' = g(x,y,\lambda) \\ \lambda' = \varepsilon \end{cases}$$

SO THAT $\lambda(\varepsilon t) = \lambda_i + \varepsilon t$.

WE ASSUME THAT $\exists$ STEADY-STATE SOLUTION WHEN λ IS A CONSTANT INDEPENDENT OF t WRITTEN AS

$$(x, y) = (x_s(\lambda), y_s(\lambda)).$$

WHERE $f(x_s, y_s, \lambda) = g(x_s, y_s, \lambda) = 0$. ASSUME THAT WHEN $\lambda = \lambda_H$ WE HAVE A SUPERCRITICAL HOPF BIFURCATION WITH LOCAL BIFURCATION DIAGRAM



THEREFORE, A NECESSARY CONDITION FOR THIS SCENARIO IN TERMS OF THE JACOBIAN

$$J(\lambda) \equiv \left.\begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}\right|_{\substack{x = x_s(\lambda) \\ y = y_s(\lambda)}}$$

IS THAT

(i) $\text{trace } J[\lambda_H] = 0$, $\det J[\lambda_H] > 0$. $\rightarrow$ $\exists$ HOPF AT $\lambda = \lambda_H$.

(ii) $\text{TRACE } J[\lambda] < 0$ WHEN $\lambda < \lambda_H$ AND $\det J[\lambda] > 0 \rightarrow$ stable FOR $\lambda < \lambda_H$

(iii) $\text{trace } J[\lambda] > 0$, $\det J(\lambda) > 0$ FOR $\lambda > \lambda_H \rightarrow$ unstable FOR $\lambda > \lambda_H$.

WE WOULD THEN LIKE TO ANALYZE ANY DELAYED BIFURCATION BEHAVIOR AS $\lambda(\varepsilon t)$ CROSSES ABOVE λ_H.

WE NOW PUT $\tau = \varepsilon t$ IN (1) TO OBTAIN

$$\left.\begin{aligned} \varepsilon \dot{x} &= f[x, y, \lambda(\tau)] \\ \varepsilon \dot{y} &= g[x, y, \lambda(\tau)] \\ \text{WITH } \lambda(\tau) &= \lambda_I + \tau . \end{aligned}\right\} \qquad (2)$$

TWO-TERM REGULAR EXPANSION

WE PUT $x = x_s + \varepsilon x_1 + \cdots$, $y = y_s + \varepsilon y_1 + \cdots$

AND SUBSTITUTE TO GET IN TERMS OF THE JACOBIAN $J = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}_{(x,y)=(x_s,\, y_s)}$ THAT

$$J \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} \dot{x}_s \\ \dot{y}_s \end{pmatrix} \quad \text{OR EQUIVALENTLY} \quad \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = J^{-1} \begin{pmatrix} \dot{x}_s \\ \dot{y}_s \end{pmatrix}$$

WE CAN CONTINUE IN THIS WAY TO GET AN IMPROVED APPROXIMATION

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_s(\lambda) \\ y_s(\lambda) \end{pmatrix} + \varepsilon J^{-1} \begin{pmatrix} \dot{x}_s \\ \dot{y}_s \end{pmatrix} + O(\varepsilon^2) \quad \Big\} \qquad (3)$$

HERE $\begin{pmatrix} \dot{x}_s \\ \dot{y}_s \end{pmatrix} = \begin{pmatrix} x_{s\lambda} \\ y_{s\lambda} \end{pmatrix}$ SINCE $\dot{\lambda} = 1$ IS A LINEAR SWEEP.

NOW WITH THE CORRECTION (3) WE INTRODUCE A QUANTITY THAT CAN BE EXPONENTIALLY SMALL IN ε. IN PARTICULAR, WE PUT

$$\begin{pmatrix} x \\ y \end{pmatrix} = \left(\begin{pmatrix} x_s \\ y_s \end{pmatrix} + \varepsilon J^{-1} \begin{pmatrix} x_{s\lambda} \\ y_{s\lambda} \end{pmatrix} + \cdots \right) + \begin{pmatrix} U_0 \\ V_0 \end{pmatrix} \exp\left[\Phi(\tau)/\varepsilon \right]$$

THEN WE LINEARIZE (2) AND OBTAIN TO WITHIN $O(\varepsilon)$ ERROR OF THE JACOBIAN THAT

$$\begin{aligned} f_x^0 U_0 + f_y^0 V_0 &= \dot{\Phi} U_0 \\ g_x^0 U_0 + g_y^0 V_0 &= \dot{\Phi} V_0 \end{aligned} \qquad \text{WHERE } f_x^0 = \left.\frac{\partial f}{\partial x}\right|_{(x_s, y_s)} \text{ ETC..}$$

THIS YIELDS THAT

$$\begin{pmatrix} f_x^0 - \dot{\Phi} & f_y^0 \\ g_x^0 & g_y^0 - \dot{\Phi} \end{pmatrix} \begin{pmatrix} U_0 \\ V_0 \end{pmatrix} = \underline{0}$$

NOW FOR THIS TO HAVE A NONTRIVIAL SOLUTION WE NEED THAT $\det \begin{pmatrix} f_x^0 - \dot{\Phi} & f_y^0 \\ g_x^0 & g_y^0 - \dot{\Phi} \end{pmatrix} = 0$

THIS YIELDS THAT

$$\left.\begin{array}{l}(\Phi')^2 - \operatorname{trace} J(\lambda)\, \Phi' + \det J(\lambda) = 0 \\ \text{WITH} \quad \lambda = \lambda(\tau) \quad \text{AND} \quad \Phi' = d\Phi/d\tau .\end{array}\right\} \quad (4)$$

NOW WLOG WE CAN SET $\Phi(0) = 0$ (AS OTHERWISE IT CAN BE ABSORBED IN U_0, V_0).

WE WOULD LIKE TO FIND A τ_0 FOR WHICH

$$\operatorname{RE}[\Phi(\tau_0)] = 0 \quad \text{WITH} \quad \operatorname{RE}[\Phi(\tau)] > 0 \text{ FOR } \tau > \tau_0 \text{ AND } \operatorname{RE}[\Phi(\tau)] < 0$$

FOR $\tau < \tau_0$. THEN SINCE $\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} U \\ V \end{pmatrix} e^{\Phi/\varepsilon}$ IT FOLLOWS THAT THE LINEARIZATION FAILS FOR $\tau \geq \tau_0$ AND THE SOLUTION JUMPS AWAY FROM $\begin{pmatrix} X_s \\ Y_s \end{pmatrix}$, AND BEGINS THE OSCILLATION.

NOW FROM (4) WE HAVE WITH $\lambda = \lambda(\tau)$ THAT

$$\left.\Phi'(\tau) = \frac{\operatorname{trace} J(\lambda)}{2} \pm \frac{1}{2}\sqrt{(\operatorname{trace} J(\lambda))^2 - 4 \det J(\lambda)}\right\} \quad (5)$$

DEFINE $\quad D(\lambda) \equiv (\operatorname{trace} J(\lambda))^2 - 4 \det J(\lambda)$.

SINCE λ_H IS THE HOPF BIFURCATION POINT FOR THE "STATIC" PROBLEM WE HAVE $\det J(\lambda_H) > 0$ AND $\operatorname{trace} J(\lambda_H) = 0$ SO THAT

$$D(\lambda_H) < 0,$$

SO THAT $D(\lambda) < 0$ FOR $|\lambda - \lambda_H|$ SMALL (AT LEAST).

WE SUPPOSE THAT $D(\lambda) = 0$ HAS TWO ROOTS λ_- AND λ_+ WITH

$$\lambda_- < \lambda_H < \lambda_+ .$$

ASSUME THAT THE STARTING POINT $\lambda(0) = \lambda_I$ OF THE SLOW RAMP SATISFIES

$$\lambda_- < \lambda_i < \lambda_H < \lambda_+$$

AND $\lambda = \lambda_I + \tau$. THEN ON $\lambda_i < \lambda < \lambda_+$ WE HAVE $D(\lambda) < 0$.

THEREFORE FROM (5) WE HAVE

$$\Phi' = \frac{\text{trace } J(\lambda)}{2} \pm \frac{i}{2}\sqrt{|D(\lambda)|}$$

HENCE, WE GET

$$RE[\Phi'] = \frac{\text{trace } J(\lambda)}{2}$$

AS SUCH WE HAVE

$$RE[\Phi(T)] = \frac{1}{2}\int_0^{T} \text{trace } J[\lambda(s)]\, ds. \quad (6)$$

WITH A LINEAR RAMP $\lambda = \lambda_i + S$, SUCH THAT $d\lambda/ds = 1$ WE HAVE

$$\left. RE[\Phi(T)] = \frac{1}{2}\int_{\lambda_i}^{\lambda(T)} (\text{trace } J(\lambda))\, d\lambda \right\} \quad (7)$$

NOW SUPPOSE THAT $\exists\, \lambda_F$ SATISFYING $\lambda_H < \lambda_F < \lambda_+$ (SO THAT $D(\lambda_F) < 0$)

FOR WHICH

$$\frac{1}{2}\int_{\lambda_i}^{\lambda_F} (\text{trace } J(\lambda))\, d\lambda = 0. \quad (8)$$

SINCE trace $J(\lambda) < 0$ FOR $\lambda < \lambda_H$ AND trace $J(\lambda) > 0$ FOR $\lambda > \lambda_H$ WE MUST HAVE $\lambda_F > \lambda_H$, i.e. A DELAYED BIFURCATION. EQUATION (8) IS AN <u>EQUAL-AREA-RULE</u>. THE TIME FOR THE ONSET OF OSCILLATIONS IS NOT SIMPLY $(\lambda_H - \lambda_I)/\varepsilon$ BUT INSTEAD IS THE DELAYED VALUE $t_{HOPF,DEL}$ DEFINED BY

$$t_{HOPF,DEL} \equiv \frac{(\lambda_F - \lambda_I)}{\varepsilon}. \quad (9).$$

EXAMPLE 1

CONSIDER
$$x' = -y + \lambda x - x(x^2+y^2)$$
$$y' = x + \lambda y - y(x^2+y^2)$$
WITH $\lambda = \lambda_I + \tau$ AND $\tau = \varepsilon t$.

NOW LINEARIZING AROUND $(x_s, y_s) = (0,0)$ WE GET $J = \begin{pmatrix} \lambda & -1 \\ 1 & \lambda \end{pmatrix}$

SO THAT TRACE $J = 2\lambda$ AND $\det J = \lambda^2+1$ FOR WHICH $D(\lambda) = (\mathrm{tr}\,J)^2 - 4\det J = -4 < 0$

FOR ALL $\lambda > 0$. THE CRITERION FOR THE JUMP IS THAT
$$\frac{1}{2}\int_{\lambda_I}^{\lambda_F} \mathrm{trace}\, J(\lambda)\, d\lambda = 0, \quad \text{WITH } \lambda_I < 0 \text{ AND } \lambda_F > 0$$

SINCE $\mathrm{trace}\, J(\lambda) = 2\lambda$, WE CONCLUDE THAT $\int_{\lambda_I}^{\lambda_F} \lambda\, d\lambda = 0 \longrightarrow \lambda_F = -\lambda_I > 0.$

THUS THE DELAYED HOPF POINT IS
$$t_{\text{HOPF, DELAY}} = \frac{(\lambda_F - \lambda_I)}{\varepsilon} = \frac{-2\lambda_I}{\varepsilon}. \quad (10)$$

EXAMPLE 2 (GLYCOLOSIS OSCILLATIONS)

WE CONSIDER
$$x' = -x + ay + x^2 y = f(x,y)$$
$$y' = b - ay - x^2 y = g(x,y)$$

WE WILL THINK OF b AS FIXED AND "a" AS THE BIFURCATION PARAMETER, WHICH WILL BE ALLOWED TO DEPEND SLOWLY ON TIME (SEE BELOW).

NOW THE STEADY-STATE SATISFIES
$$\begin{matrix} -x_s + a y_s + x_s^2 y_s = 0 \\ b - a y_s - x_s^2 y_s = 0 \end{matrix} \longrightarrow x_s = b, \quad y_s = \frac{b}{a+b^2}.$$

NOW WE CALCULATE
$$J = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} = \begin{pmatrix} 2xy-1 & a+x^2 \\ -2xy & -(a+x^2) \end{pmatrix}\Bigg|_{(x,y)=(x_s,y_s)} = \begin{pmatrix} \frac{2b^2}{a+b^2} - 1 & a+b^2 \\ -\frac{2b^2}{a^2+b^2} & -(a+b^2) \end{pmatrix}$$

THIS YIELDS THAT

$$J = \begin{pmatrix} \dfrac{b^2-a}{b^2+a} & a+b^2 \\ \dfrac{-2b^2}{b^2+a} & -(a+b^2) \end{pmatrix}$$

THUS $\det J = a + b^2$, $\quad \operatorname{tr} J = \dfrac{(b^2-a)-(b^2+a)^2}{b^2+a}$. THUS $\det J > 0 \quad \forall a, b > 0$.

WE HAVE $\operatorname{tr} J > 0$ (UNSTABLE) WHEN $b^2 - a > (b^2+a)^2$. SETTING $c = b^2$

THEN $c - a > c^2 + 2ac + a^2 \longrightarrow c^2 + (2a-1)c + (a^2+a) = 0$. SINCE $(2a-1)^2 - 4a^2 - 4a = -8a+1$

WE HAVE THAT $C_+ = C_-$ WHEN $a = 1/8$ WHICH YIELDS FROM SETTING $\operatorname{tr} J = 0$ THAT $b = \sqrt{\frac{3}{8}}$.

WE HAVE $$b_{\pm}^2 = -\frac{(2a-1)}{2} \pm \sqrt{\frac{(1-8a)}{4}}, \quad \text{FOR} \quad 0 < a < 1/8.$$

OUR PLOT IS THE FOLLOWING IN THE PARAMETER SPACE b VERSUS a:

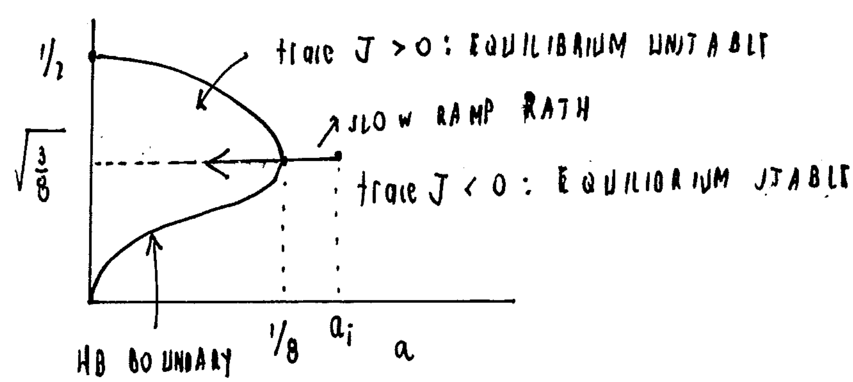


FIG.1

THIS YIELDS A QUALITATIVE PICTURE AS

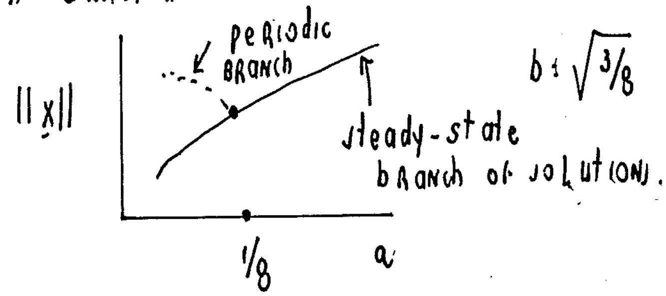


FIG.2

NOW WE LET $a = a_i - \varepsilon t$, $\varepsilon \ll 1$ WITH $a_i > 1/8$, AS THE SLOW RAMP PATH AS SEEN IN THE BIFURCATION PLOT FIG.1. WE BEGIN IN THE STABLE REGIME AND THEN SLOWLY RAMP INTO THE UNSTABLE REGION WHERE TRACE $J > 0$. WE WILL FIX $b = \sqrt{3/8}$.

NOW PROVIDED THAT

$$D(a) \equiv (\text{trace } J(a))^2 - 4 \det(J) < 0$$

ALONG THE SLOW SWEEP, WE HAVE THAT THE TRANSITION POINT TO A HOPF BIFURCATION SHOULD OCCUR AT $a = a_f$ GIVEN BY THE EQUAL-AREA RULE

$$\int_{a_f}^{a_i} \text{trace } J(a)\, da = 0. \qquad (11)$$

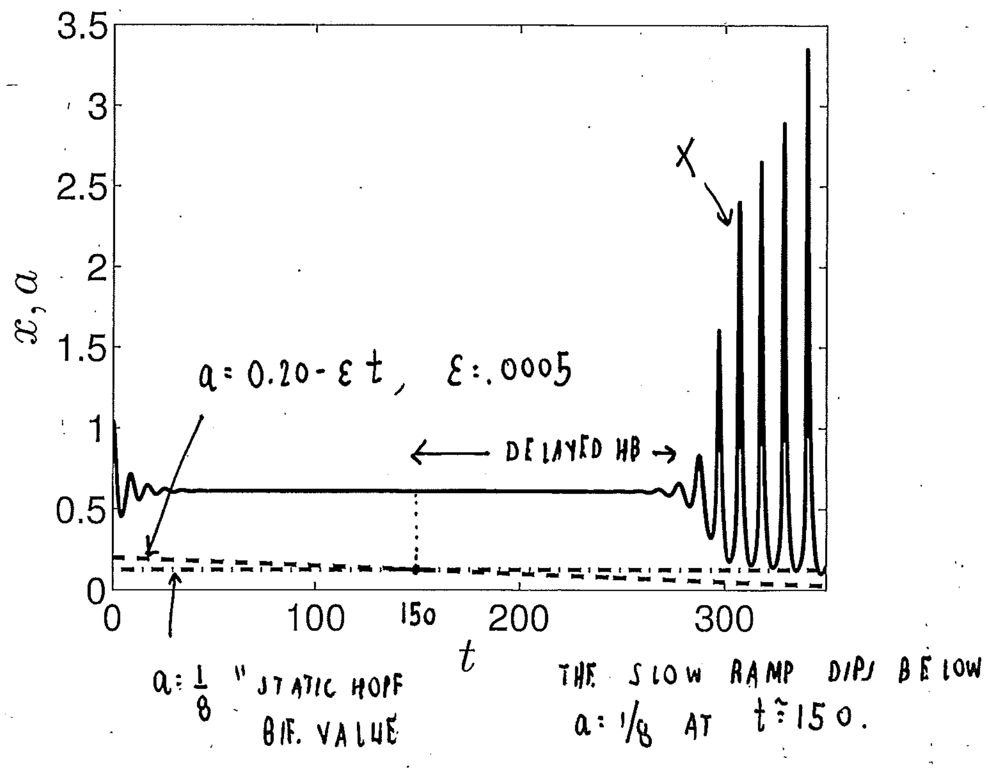
WITH $b^2 = 3/8$ WE GET THAT a_f SATISFIES

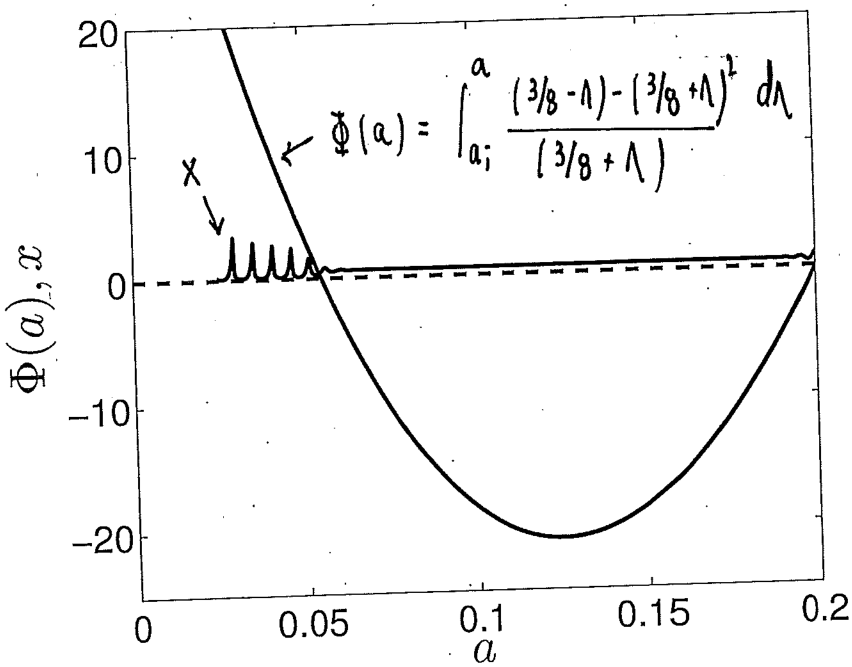
$$\int_{a_f}^{a_i} \frac{(3/8 - a) - (3/8 + a)^2}{(3/8 + a)}\, da = 0. \longrightarrow t_{\text{DELAY, HOPF}} = \frac{a_i - a_f}{\varepsilon}.$$

IN THE PLOT BELOW WE CHECK THIS PREDICTION FOR $a_i = 0.20$, $\varepsilon = .0005$ $X(0) = Y(0) = 1$ FOR WHICH $a_f = .05524 < \frac{1}{8}$. WE DEFINE

$$\Phi(a) = \int_{a_i}^{a} \frac{(3/8 - \lambda) - (3/8 + \lambda)^2}{(3/8 + \lambda)}\, d\lambda$$

AND PLOT $\Phi(a)$ VS a ALONG THE SLOW SWEEP NUMERICALLY, IN OUR EXAMPLE WE HAVE VERIFIED THAT $D(a) = (\text{trace } J(a))^2 - 4 \det J(a) < 0$ ON THE PATH $a_f < a < a_i$.





NOTICE THAT OSCILLATION IS INITIATED WHEN $\Phi(a)$ CROSSES ABOVE ZERO.

WE CALCULATE $\Phi(a) = -(a_i - a) + \frac{3}{4} \ln\left(\frac{\frac{3}{8} + a_i}{\frac{3}{8} + a} \right) - \frac{1}{2}\left[\left(\frac{3}{8} + a_i\right)^2 - \left(\frac{3}{8} + a\right)^2 \right].$

SETTING $\Phi(a) = 0$ USING NEWTON'S METHOD WE GET $a = a_f \mathrel{\hat{=}} 0.05524$