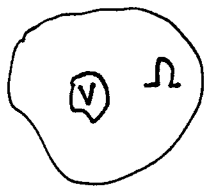
DERIVATION OF THE HEAT EQUATION

LET V BE AN ARBITRARILY SMALL DOMAIN INSIDE A GIVEN VOLUME Ω

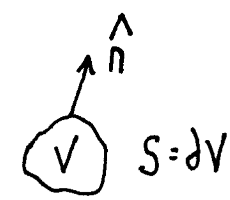


rate of change of heat energy = heat flowing across boundaries per unit time + heat energy generated internally per unit time

heat energy $= \int_V c\rho u \, d\underline{x}$ WITH c: specific heat, ρ: density, $u = u(\underline{x}, t)$ temperature

heat energy generated internally per unit time $= \int_V Q(\underline{x}, t, u) \, d\underline{x}$

LET $\underline{\phi}$: heat FLUX VECTOR ON ∂V. (boundary OF V).



THE NORMAL COMPONENT ON V IS $\underline{\phi} \cdot \underline{\hat{n}}$

IF $\underline{\phi} \cdot \hat{n} < 0$ heat Flux IS INWARD TO V

THEN
$$\frac{d}{dt} \int_V c\rho u \, d\underline{x} = - \int_S \underline{\phi} \cdot \underline{\hat{n}} \, dS + \int_V Q \, d\underline{x}$$

NOW USE THE DIVERGENCE THEOREM $\int_V \nabla \cdot \underline{A} \, d\underline{x} = \int_S \underline{A} \cdot \underline{\hat{n}} \, dS$.

THUS
$$\frac{d}{dt} \int_V c\rho u \, d\underline{x} + \int_V \nabla \cdot \underline{\phi} \, dx - \int_V Q \, d\underline{x} = 0 .$$

NOW IF c, ρ INDEPENDENT OF t WE HAVE

$$\int_V \left(c\rho u_t + \nabla \cdot \underline{\phi} - Q \right) d\underline{x} = 0 .$$

SINCE V IS ARBITRARY DOMAIN WE HAVE

$$c\rho\, u_t = -\nabla\cdot\underline{\phi} + Q(\underline{x},t,u).$$

FINALLY BY FOURIER'S LAW (OF FICK'S LAW OF DIFFUSION)

$$\underline{\phi} = -K\nabla u. \qquad K: \text{thermal conductivity.}$$

THIS LEADS TO $\quad u_t = D\nabla^2 u + \mathcal{Q}(\underline{x},t,u)$

WITH $\mathcal{Q}(\underline{x},t,u) = Q/\rho c. \qquad D = K/\rho c$ thermal diffusivity

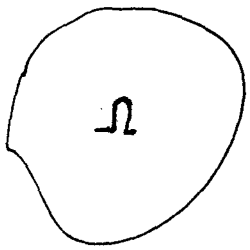
WE THEN HAVE THE BOUNDARY CONDITION THAT

$$\underline{\phi}\cdot\hat{n} = h(u-u_b) \qquad h>0 \quad h = \text{CONSTANT.}$$

THEN WE WRITE $\underline{\phi} = -K\nabla u$ SO THAT NEWTON'S LAW OF COOLING IS

$$\nabla u\cdot\hat{n} = \partial_n u = -b(u-u_b) \quad \text{ON } \partial\Omega \qquad b = h/K \text{ BIOT NUMBER,}$$

WITH u_b GIVEN, IS THE OUTSIDE TEMPERATURE.
THIS GIVES THE HEAT CONDUCTION PROBLEM



$$u_t = D\nabla^2 u + \mathcal{Q}(\underline{x},t,u) \quad \text{IN } \Omega$$

$$\partial_n u = -b(u-u_b) \quad \text{ON } \partial\Omega \quad \text{(BOUNDARY CONDITION)}$$

$$u(\underline{x},0) = u_0(\underline{x}) \quad \text{INITIAL TEMPERATURE.}$$

<u>REMARK</u> (i) $\nabla^2 u = u_{xx} + u_{yy} + u_{zz}$ WHEN $\underline{x} = (x,y,z)$.

OCCASIONALLY ONE WRITES $\nabla^2 = \Delta$.

(ii) WE CAN ONLY SOLVE LINEAR PROBLEMS WHERE

$$\mathcal{Q}(\underline{x},t,u) = b(\underline{x},t)\,u + F(\underline{x},t);$$

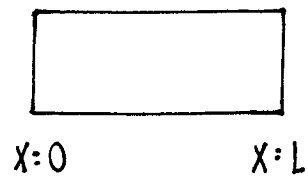
IF $b<0$ SINK OF HEAT PROPORTIONAL TO u.

$F<0$ CORRESPONDS TO SINK OF HEAT.

THE ONE-DIMENSIONAL PROBLEM WITH $\varphi(\underline{x}, t, u) = F(\underline{x})$

IS SIMPLY

$$U_t = D U_{xx} + F(x) \qquad 0 \leq x \leq L$$



$$U_x = -h_2 (U - U_2) \qquad \text{ON } x = L$$

$$U_x = h_1 (U - U_1) \qquad \text{ON } x = 0$$

$$U(x, 0) = U_0(x)$$

THE STEADY-STATE SOLUTION CORRESPONDING TO

$$U_s(x) = \lim_{t \to \infty} U(x, t)$$

SATISFIES

$$\left\{ \begin{array}{l} D U_s'' = -F(x) \qquad 0 < x < L \\ U_s' = -h_2 (U_s - U_2) \text{ ON } x = L; \quad U_s' = h_1 (U_s - U_1), \ x = 0 \end{array} \right.$$

<u>REMARK</u> (i) FOR RADIALLY SYMMETRIC DIFFUSION IN A <u>CIRCLE</u> THEN $\nabla^2 U = U_{rr} + \frac{1}{r} U_r$

(ii) FOR RADIALLY SYMMETRIC DIFFUSION IN A <u>SPHERE</u> THEN $\nabla^2 U = U_{rr} + \frac{2}{r} U_r$.

(iii) TO SOLVE $U_{rr} + \frac{(n-1)}{r} U_r = F_0$ (F_0 = CONSTANT), LET $U = Ar^2$

TO OBTAIN $2A + 2(n-1)A = F_0$ SO THAT $A = F_0 / 2n$

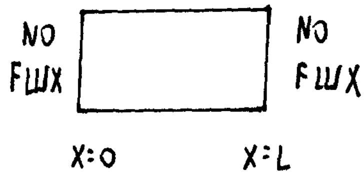
(iii) IN A CIRCLE $\nabla^2 U = U_{rr} + \frac{1}{r} U_r + \frac{1}{r^2} U_{\theta\theta}$.

(iv) SUPPOSE $U_t = \cdot [D \quad]$ THEN FOR THE STEADY-STATE SOLUTION WE REQUIRE $D_+ U_x |_{x_0^+} = D_- U_x |_{x_0^-}$ WHEN $D = \left\{ \begin{array}{ll} D_+, & x > x_0 \\ D_-, & x < x_0 \end{array} \right.$

INSULATING BOUNDARY CONDITIONS

CONSIDER

$$(*) \begin{cases} u_t = D u_{xx} & 0 < x < L, \quad t > 0 \\ u_x = 0 \ \text{ on } x = 0, L; & u(x,0) = f(x) \end{cases}$$



THE STEADY-STATE SOLUTION IS

$$u_s'' = 0, \quad 0 < x < L; \qquad u_s'(0) = u_s'(L) = 0.$$

THIS GIVES $\quad u_s = A \quad$ A = CONSTANT TO BE FOUND.

WE WILL SHOW THAT $\quad A = \frac{1}{L}\int_0^L f(x)\,dx = $ average of initial temperature.

TO SHOW THIS WE INTEGRATE

$$\int_0^L u_t \, dx = \int_0^L D u_{xx} \, dx = D u_x \Big|_0^L = 0.$$

HENCE $\quad \frac{d}{dt}\int_0^L u(x,t)\,dx = 0 \longrightarrow \int_0^L u(x,t)\,dx = C \quad$ CONSTANT

FOR ALL TIME. EVALUATE AT $t=0$ TO OBTAIN $\int_0^L u(x,t)\,dx = \int_0^L f(x)\,dx$

THEN LET $t \to \infty$ AND USE $\quad u(x,t) \to A \quad$ AS $\quad t \to \infty$

SO THAT $\quad \int_0^L A \, dx = AL = \int_0^L f(x)\,dx \longrightarrow A = \frac{1}{L}\int_0^L f(x)\,dx.$

IN THE MULTI-DIMENSIONAL CASE $\quad u_t = D\nabla^2 u \quad$ IN Ω

WITH $\quad \partial_n u = 0 \quad$ ON $\partial\Omega$ AND $\quad u(\underline{x},0) = f(\underline{x}) \quad$ YOU WILL SHOW THAT

$$\lim_{t \to \infty} u(\underline{x},t) = \frac{1}{\mathrm{vol}(\Omega)} \int_\Omega f(\underline{x})\,d\underline{x}.$$

ALTERNATIVELY FOR (*) IN 1-DIMENSION WE WRITE

$$U(x,t) = X(x)\,T(t) \qquad \frac{T'}{DT} = \frac{X''}{X} = -\lambda.$$

THUS
$$\phi'' + \lambda\phi = 0 \qquad \longrightarrow \qquad \lambda_0 = 0,\ \phi_0 = 0$$
$$\phi'(0) = \phi'(L) = 0 \qquad\qquad \lambda_n = \frac{n^2\pi^2}{L^2},\ \phi_n(x) = \cos\left(\frac{n\pi x}{L}\right)$$
$$n = 1, 2, 3, \ldots$$

THUS
$$U(x,t) = C_0 + \sum_{n=1}^{\infty} C_n e^{-\lambda_n D t} \cos\left(\frac{n\pi x}{L}\right)$$

THEN
$$f(x) = C_0 + \sum_{n=1}^{\infty} C_n \cos\left(\frac{n\pi x}{L}\right)$$

INTEGRATING FROM $x = 0, L$ $\longrightarrow$ $\int_0^L f(x)\,dx = C_0 L$

SO $C_0 = \frac{1}{L}\int_0^L f(x)\,dx.$

AS $t \to \infty$ THEN $\lim_{t\to\infty} U(x,t) = C_0 = \frac{1}{L}\int_0^L f(x)\,dx.$

STURM - LIOUVILLE PROBLEMS

THE SL PROBLEM HAS THE FORM

$$(p(x)\phi')' - q(x)\phi + \lambda w(x)\phi = 0 \qquad 0<x<1 \qquad w(x)>0, p(x)>0$$

$$\phi'(0) - h_1\phi(0) = 0, \quad \phi'(1) + h_2\phi(1) = 0. \qquad \text{ON } 0<x<1$$

WE WRITE THE EIGENVALUE PROBLEM AS

STURM-LIOUVILLE (*)

$$\left\{ \begin{array}{ll} \mathcal{L}\phi = \lambda w(x)\phi & \mathcal{L}\phi = -(p(x)\phi')' + q(x)\phi \\ \phi'(0) - h_1\phi(0) = 0, & \phi'(1) + h_2\phi(1) = 0 \end{array} \right. \qquad w(x)>0,\ p(x)>0 \text{ ON } 0<x<1$$

WE WILL DERIVE MANY PROPERTIES OF (*). TO DO SO WE FIRST DERIVE LAGRANGE'S IDENTITY:

$$(1) \quad \int_0^1 (v\mathcal{L}u - u\mathcal{L}v)\,dx = -p(x)u'v\Big|_0^1 + p(x)uv'\Big|_0^1.$$

<u>PROOF</u> WE WRITE $\int_0^1 v\mathcal{L}u\,dx = \int_0^1 [-v(pu')' + vqu]\,dx$

$$= -pu'v\Big|_0^1 + \int_0^1 ((pv')u + vqu\,dx)$$

$$= -pu'v\Big|_0^1 + pv'u\Big|_0^1 - \int_0^1 [(pv')'u - vqu]\,dx$$

THIS YIELDS THAT $\int_0^1 v\mathcal{L}u\,dx = -pu'v\Big|_0^1 + pv'u\Big|_0^1 + \int_0^1 u\mathcal{L}v\,dx$ □

NOW SUPPOSE THAT u, v SATISFY THE BOUNDARY CONDITIONS IN (*) SO THAT $u'(0) - h_1u(0) = 0$, $u'(1) + h_2u(1) = 0$, $v'(0) - h_1v(0) = v'(1) + h_2v(1) = 0$. THEN WE CAN ADD AND SUBTRACT IN (1):

$$\int_0^1 (v\mathcal{L}u - u\mathcal{L}v)\,dx = p(1)u(1)\big(v'(1) + h_2v(1)\big) - p(1)v(1)\big(u'(1) + h_2u(1)\big)$$

$$+ p(0)v(0)\big(u'(0) - h_1u(0)\big) - p(0)u(0)\big(v'(0) - h_1v(0)\big)$$

$$= 0$$

THEREFORE, WE OBTAIN

$$\int_0^1 v\,\mathcal{L}u\,dx = \int_0^1 u\,\mathcal{L}v\,dx \quad \text{WHENEVER } u, v \text{ SATISFY THE B.C.}$$

IF WE DEFINE $(a, b) \equiv \int_0^1 a\,b\,dx$ THEN WE CAN WRITE $(v, \mathcal{L}u) = (u, \mathcal{L}v)$

WE NOW DERIVE (OR STATE) MANY OF THE KEY PROPERTIES OF THE STURM-LIOUVILLE PROBLEM.

PROPERTIES

(i) THE EIGENVALUES HAVE THE PROPERTIES

a) λ IS REAL

b) THERE ARE AN INFINITE NO. OF EIGENVALUES λ_j WITH

$$\lambda_1 < \lambda_2 < \lambda_3 < \dots \quad \text{AND} \quad \lambda_j \to +\infty \text{ AS } j \to \infty.$$

c) $\lambda_j > 0$ WHEN $h_1 \geq 0$ AND $h_2 \geq 0$, AND $q(x) \geq 0$, AND $h_1 h_2 > 0$

(ii) THE EIGENFUNCTIONS $y = \phi_j(x)$ FOR $j = 1, 2, 3, \dots$ HAVE THE PROPERTIES

a) $\phi_j(x)$ ARE REAL AND CAN BE NORMALIZED $\int_0^1 w\phi_j^2\,dx = 1$

b) $\int_0^1 \phi_j(x)\,\phi_k(x)\,w(x)\,dx = 0 \quad j \neq k$

(iii) EXPANSION PROPERTY

ANY FUNCTION $f(x)$ WITH $\int_0^1 (f(x))^2\,dx < \infty$ CAN BE EXPANDED AS

$$f(x) = \sum_{n=1}^{\infty} c_n\,\phi_n(x)$$

WHERE BY ORTHOGONALITY

$$c_n = \frac{\int_0^1 f(x)\,\phi_n(x)\,w(x)\,dx}{\int_0^1 (\phi_n(x))^2\,w(x)\,dx}$$

PROOF OF SOME OF THE PROPERTIES

(i) EIGENVALUES ARE REAL

LET $\mathcal{L}\phi = \lambda w \phi$ WITH $\phi'(0) - h_1 \phi(0) = 0$ AND $\phi'(1) + h_2 \phi(1) = 0$.

TAKE THE CONJUGATE $\mathcal{L}\bar{\phi} = \bar{\lambda} w \bar{\phi}$. NOW USE LAGRANGE'S IDENTITY

$$\int_0^1 (\phi \mathcal{L}\bar{\phi} - \bar{\phi}\mathcal{L}\phi)\,dx = 0 \longrightarrow (\phi, \mathcal{L}\bar{\phi}) - (\bar{\phi}, \mathcal{L}\phi) = 0.$$

THIS YIELDS THAT $0 = (\phi, \bar{\lambda}\bar{\phi}) - (\bar{\phi}, \lambda\phi) = (\bar{\lambda} - \lambda)(\bar{\phi}, \phi) = (\bar{\lambda} - \lambda)\int_0^1 \phi\bar{\phi}\,dx$

HENCE $(\bar{\lambda} - \lambda)\int_0^1 |\phi|^2\,dx = 0 \longrightarrow \lambda = \bar{\lambda} \longrightarrow \lambda$ IS REAL.

(ii) SHOW $\lambda_j > 0$ WHEN $h_1 \geq 0$ AND $h_2 \geq 0$, $q(x) \geq 0$, $h_1 h_2 > 0$

WE WRITE $\mathcal{L}\phi = \lambda w \phi$

MULTIPLY BY ϕ AND INTEGRATE $\int_0^1 \phi \mathcal{L} \phi\,dx = \lambda \int_0^1 w\phi^2\,dx$

NOW INTEGRATE BY PARTS: $-p(x)\phi'(x)\phi(x)\Big|_0^1 + \int_0^1 (p\phi'^2 + q\phi^2)\,dx$
$= \lambda \int_0^1 w\phi^2\,dx.$

NOW $\phi'(1) = -h_2\phi(1)$ AND $\phi'(0) = h_1\phi(0)$, WHICH YIELDS

$$p(1)h_2(\phi(1))^2 + p(0)h_1(\phi(0))^2 + \int_0^1 (p\phi'^2 + q\phi^2)\,dx = \lambda\int_0^1 w\phi^2\,dx.$$

NOW SINCE $q(x) \geq 0$ (BY ASSUMPTION) AND $p(x) > 0$, $w(x) > 0$ ON $0 < x < 1$ FOR STURM-LIOUVILLE THEN WE HAVE $\lambda > 0$.

REMARK IF $q \equiv 0$ FOR $0 \leq x \leq 1$ AND $h_1 = h_2 = 0$ THEN WE HAVE $(p(x)\phi')' + \lambda w(x)\phi = 0$ WITH $\phi'(0) = \phi'(1) = 0$.

THIS HAS AN EIGENVALUE $\lambda = 0$ AND EIGENFUNCTION $\phi = 1$.

(ii) EIGENFUNCTIONS CORRESPONDING TO DIFFERENT EIGENVALUES ARE ORTHOGONAL

WE WRITE $\mathcal{L}\phi_j = \lambda_j w \phi_j$ WITH ϕ_j AND ϕ_K

$\mathcal{L}\phi_K = \lambda_K w \phi_K$ SATISFYING THE BOUNDARY CONDITIONS.

THEN $(\phi_K, \mathcal{L}\phi_j) = (\phi_j, \mathcal{L}\phi_K)$ BY LAGRANGE'S IDENTITY.

THIS YIELDS $\lambda_j(\phi_K, w\phi_j) = \lambda_K(\phi_K, w\phi_j)$.

THEREFORE $(\lambda_j - \lambda_K)\int_0^1 w\phi_j\phi_K\,dx = 0.$

IF $\lambda_j \neq \lambda_K$ THEN $\int_0^1 w\phi_j\phi_K\,dx = 0 \rightarrow$ ORTHOGONALITY.

(iii) IT IS DIFFICULT TO PROVE THAT ANY $f(x)$ WITH $\int_0^1 (f(x))^2\,dx < \infty$ CAN BE EXPANDED AS $f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x)$.

TO DETERMINE THE COEFFICIENTS, MULTIPLY BY $w(x)\phi_m(x)$ AND GET

$$\int_0^1 f(x)\phi_m(x)w(x)\,dx = \sum_{n=1}^{\infty} c_n \int_0^1 \phi_m(x)\phi_n(x)w(x)\,dx$$

BY ORTHOGONALITY $c_m\int_0^1 (\phi_m(x))^2 w(x)\,dx = \int_0^1 f(x)\phi_m(x)w(x)\,dx,$

WHICH DETERMINES c_m.

EXAMPLE 1 FIND THE NORMALIZED EIGENFUNCTIONS FOR

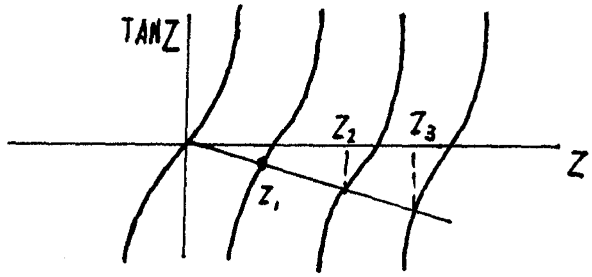
$$\phi'' + \lambda\phi = 0, \quad 0 < x < 1$$
$$\phi(0) = 0, \quad \phi'(1) + \phi(1) = 0$$

WE KNOW THAT $\lambda > 0$ SINCE $h_1 h_2 > 0$ (property ii).

WE OBTAIN $\phi = A\sin(\sqrt{\lambda}\,x)$. NOW $A\sqrt{\lambda}\cos(\sqrt{\lambda}) + A\sin(\sqrt{\lambda}) = 0$.

THUS λ SATISFIES $\tan[\sqrt{\lambda}] = -\sqrt{\lambda}$.

WITH $\lambda > 0$. PLOT $\tan z = -z$.



THERE ARE AN INFINITE # OF ROOTS WITH $z_n \approx \frac{(2n+1)}{2}\pi$ AS $n \to \infty$.

NEXT WE WRITE $\phi_n(x) = A_n \sin(\sqrt{\lambda_n}\,x)$. THEN TO NORMALIZE WE WRITE $\int_0^1 A_n^2 \sin^2(\sqrt{\lambda_n}\,x)\,dx = 1 = A_n^2 \int_0^1 \left(\frac{1-\cos(2\sqrt{\lambda_n}\,x)}{2}\right)dx$

THIS YIELDS
$$\frac{A_n^2}{2}\left[1 - \frac{1}{2\sqrt{\lambda_n}}\sin(2\sqrt{\lambda_n})\right] = 1$$

OR
$$\frac{A_n^2}{2}\left[1 - \frac{\sin(\sqrt{\lambda_n})\cos(\sqrt{\lambda_n})}{\sqrt{\lambda_n}}\right] = \frac{A_n^2}{2}\left[1 + \cos^2(\sqrt{\lambda_n})\right] = 1 \rightarrow A_n = \frac{\sqrt{2}}{[1+\cos^2(\sqrt{\lambda_n})]}$$

(HERE WE USED $\sin(\sqrt{\lambda_n}) = -\sqrt{\lambda_n}\cos(\sqrt{\lambda_n})$ FROM THE EIGENVALUE RELATION).

THIS YIELDS,
$$\phi_n(x) = \frac{\sqrt{2}}{[1+\cos^2(\sqrt{\lambda_n})]^{1/2}}\sin(\sqrt{\lambda_n}\,x)$$

NOW IF WE EXPAND $f(x) = x$ AS $f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x)$. WE HAVE
$$c_n = \int_0^1 x\,\phi_n(x)\,dx = A_n \int_0^1 x\sin(\sqrt{\lambda_n}\,x)\,dx = A_n\left[\frac{-x}{\sqrt{\lambda_n}}\cos(\sqrt{\lambda_n}\,x)\Big|_0^1 + \frac{1}{\sqrt{\lambda_n}}\int_0^1 \cos(\sqrt{\lambda_n}\,x)\,dx\right]$$

THIS YIELDS THAT
$$c_n = A_n\left[-\frac{1}{\sqrt{\lambda_n}}\cos(\sqrt{\lambda_n}) + \frac{1}{\lambda_n}\sin(\sqrt{\lambda_n})\right]$$

NOW REPLACE $\cos(\sqrt{\lambda_n}) = -\frac{1}{\sqrt{\lambda_n}} \sin(\sqrt{\lambda_n})$, WHICH GIVES $C_n = \frac{2}{\lambda_n} \sin(\sqrt{\lambda_n}) A_n$.

THE FINAL RESULT IS $f(x) = x = \sum_{n=1}^{\infty} \frac{2}{\lambda_n} \sin(\sqrt{\lambda_n}) A_n^2 \sin(\sqrt{\lambda_n}\, x)$

FINALLY $$f(x) = x = 4 \sum_{n=1}^{\infty} \frac{\sin(\sqrt{\lambda_n})}{\lambda_n (1 + \cos^2(\sqrt{\lambda_n}))} \sin(\sqrt{\lambda_n}\, x).$$

USE THIS EXPANSION TO SOLVE THE HEAT CONDUCTION PROBLEM GIVEN BY

$$\begin{cases} u_t = \alpha^2 u_{xx}, \quad 0 < x < 1 \\ u(0,t) = 1, \quad u_x(1,t) + u(1,t) = 0 \\ u(x,0) = f(x) \end{cases}$$

WE FIRST CALCULATE THE STEADY-STATE SOLUTION $u_s(x)$ WHICH SATISFIES

$$u_s'' = 0$$
$$u_s(0) = 1, \quad u_s'(1) + u_s(1) = 0.$$

WE GET $u_s(x) = Ax + B$ THEN $u_s(0) = 1$ GIVES $u_s(x) = Ax + 1$.

THIS YIELDS $A + (A+1)$ OR $A = -1/2$. $\longrightarrow u_s(x) = -x/2 + 1$.

FINALLY, WE WRITE $u = u_s + v$. THIS YIELDS THAT

$$(*) \begin{cases} v_t = \alpha^2 v_{xx} \\ v(0,t) = 0, \quad v_x(1,t) + v(1,t) = 0 \\ v(x,0) = f(x) - u_s(x) \end{cases}$$

WE WRITE $v = XT$ SO THAT $\frac{T'}{\alpha^2 T} = \frac{X''}{X} = -\lambda$

$T' = -\alpha^2 \lambda T$

$X'' + \lambda X = 0$

$T = e^{-\alpha^2 \lambda t}$

THIS YIELDS THE EIGENVALUE PROBLEM

$$\phi'' + \lambda\phi = 0, \qquad 0 < x < 1$$

$$\phi(0) = 0, \qquad \phi'(1) + \phi(1) = 0$$

THE NORMALIZED EIGENFUNCTIONS ARE $\phi_n(x) = \dfrac{\sqrt{2}}{[1 + \cos^2(\sqrt{\lambda_n})]^{1/2}} \sin(\sqrt{\lambda_n}\, x)$

WITH $\tan(\sqrt{\lambda_n}) = -\sqrt{\lambda_n}$. WE EXPAND

$$v(x,t) = \sum_{n=1}^{\infty} c_n e^{-\alpha^2 \lambda_n t} \phi_n(x)$$

NOW $\quad v(x,0) = f(x) - u_s(x) = \sum_{n=1}^{\infty} c_n \phi_n(x).$

BY ORTHOGONALITY $\quad c_n = \int_0^1 (f(x) - u_s(x))\, \phi_n(x)\, dx.$

AND THEN $\quad u(x,t) = u_s(x) + \sum_{n=1}^{\infty} c_n e^{-\alpha^2 \lambda_n t} \phi_n(x).$

FOR $t \to \infty$, $\quad u(x,t) \sim u_s(x) + c_1 e^{-\alpha^2 \lambda_1 t} \phi_1(x) + \cdots$

EXAMPLE FIND THE NORMALIZED EIGENFUNCTION FOR

$$\begin{cases} (x^2\phi')' + \lambda\phi = 0, & 1 < x < 2 \\ \phi(1) = 0, \ \phi(2) = 0 \end{cases}$$

WE EXPAND OUT TO OBTAIN $x^2\phi'' + 2x\phi' + \lambda\phi = 0$.

WE LET $\phi = x^r$ TO OBTAIN $r(r-1) + 2r + \lambda = 0$.

THIS YIELDS $r = \dfrac{-1 \pm \sqrt{1-4\lambda}}{2}$.

FOR AN EIGENVALUE WE NEED $1 - 4\lambda < 0$ OR $\lambda > 1/4$.

THIS YIELDS $r = -\dfrac{1}{2} \pm \dfrac{i}{2}\sqrt{4\lambda-1}$.

OUR SOLUTION IS

$$\phi(x) = C_1 x^{-1/2} \sin\left(\frac{\sqrt{4\lambda-1}}{2}\log x\right) + C_2 x^{-1/2}\cos\left(\frac{\sqrt{4\lambda-1}}{2}\log x\right)$$

NOW $\phi(1) = 0$ GIVES $C_2 = 0$.

$\phi(2) = 0$ GIVES $\sin\left(\dfrac{\sqrt{4\lambda-1}}{2}\log 2\right) = 0$ OR $\dfrac{\sqrt{4\lambda-1}}{2}\log 2 = n\pi$

$n = 1,2,3$

THIS YIELDS $\lambda_n = \dfrac{1}{4} + \dfrac{n^2\pi^2}{(\log 2)^2}$ $n = 1,2,3,\ldots$

AND $\phi_n(x) = C x^{-1/2} \sin\left(\dfrac{n\pi \log x}{\log 2}\right)$

THEN WE NORMALIZE WITH $\int_1^2 \phi_n^2(x)\,dx = 1$. THIS YIELDS THAT

$$C^2 \int_1^2 \frac{1}{x}\sin^2\left(\frac{n\pi\log x}{\log 2}\right)dx = 1.$$

LET $y = \log x / \log 2$ SO THAT $dy = \dfrac{1}{x\log 2}\,dx$

$$\longrightarrow \quad C^2 \log 2 \int_0^1 \sin^2(n\pi y)\,dy = 1 \rightarrow C^2(\log 2)/2 = 1.$$

THIS YIELDS THAT $C = (2/\log 2)^{1/2}$.

FINALLY, WE OBTAIN THAT

$$\phi_n(x) = \left(\frac{2}{\log 2}\right)^{1/2} x^{-1/2} \sin\left(\frac{n\pi \log x}{\log 2}\right) \qquad n = 1, 2, 3, \ldots$$

$$\lambda_n = \frac{1}{4} + \frac{n^2\pi^2}{(\log 2)^2} .$$

NOW IF WE EXPAND $f(x)$ IN TERMS OF THIS SERIES WE OBTAIN

$$f(x) = \sum_{n=1}^{\infty} C_n \phi_n(x) \qquad C_n = \int_1^2 f(x)\phi_n(x)\,dx .$$

<u>EXAMPLE</u> SOLVE THE HEAT CONDUCTION PROBLEM

$$\left\{ \begin{array}{ll} u_t = D(x^2 u_x)_x - u & 1 \le x \le 2,\ t > 0 \\ u(1,t) = u(2,t) = 0, & u(x,0) = f(x) \end{array} \right.$$

SEPARATING VARIABLES WE OBTAIN $u(x,t) = X(x)T(t)$

THEN $$XT' = DT(x^2X')' - XT \quad \rightarrow \frac{1}{D}\left(\frac{T'}{T} + 1\right) = \frac{(x^2X')'}{X} = -\lambda .$$

THIS LEADS TO THE EIGENVALUE PROBLEM

$$\left. \begin{array}{ll} (x^2\phi')' + \lambda\phi = 0 & 1 < x < 2 \\ \phi(1) = \phi(2) = 0 & \end{array} \right\} \qquad \phi_k(x) = \left(\frac{2}{\log 2}\right)^{1/2} x^{-1/2} \sin\left(\frac{n\pi \log x}{\log 2}\right)$$

$$\lambda_k = \frac{1}{4} + \frac{k^2\pi^2}{(\log 2)^2}, \quad k = 1, 2, \ldots$$

WE THEN OBTAIN $T_k' = -(1 + D\lambda_k)T$

THIS YIELDS THAT $T_k(t) = e^{-t} e^{-D\lambda_k t}$

$$u(x,t) = \sum_{n=1}^{\infty} C_n e^{-(1+D\lambda_n)t} \phi_n(x) \qquad \text{WITH} \quad u(x,0) = \sum_{n=1}^{\infty} C_n \phi_n(x)$$

THUS $f(x) = \sum_{n=1}^{\infty} C_n \phi_n(x)$ SO THAT $C_n = \int_1^2 f(x)\phi_n(x)\,dx$.

(i) $$\phi'' + x\phi' + \lambda\phi = 0$$
$$\phi(0) = \phi(1) = 0$$

MULTIPLY BY $e^{x^2/2}$ SO THAT $$(e^{x^2/2}\phi)' + \lambda e^{x^2/2}\phi = 0$$
$$\phi(0) = \phi(1) = 0.$$

THE WEIGHT FUNCTION IS $w = e^{x^2/2}$ SO THAT $\int_0^1 \phi_n \phi_m e^{x^2/2}\,dx = 0$ IF $n \neq m$.

(ii) $$\phi'' + \frac{2}{x}\phi' + \lambda\phi = 0$$
$$\phi(1) = \phi(2) = 0$$

IN STURM-LIOUVILLE FORM $(x^2\phi')' + \phi\lambda x^2 = 0$

THE WEIGHT FUNCTION IS $w = x^2$ AND $\int_0^1 \phi_n \phi_m x^2\,dx = 0$, $n \neq m$

WE CAN SOLVE FOR ϕ BY WRITING $\phi(x) = v(x)/x$

SO THAT $v'' + \lambda v = 0$ AND $v = A\cos(\sqrt{\lambda}\,x) + B\sin(\sqrt{\lambda}\,x)$.

THIS YIELDS $$\phi = \frac{1}{x}\left[A\cos(\sqrt{\lambda}\,x) + B\sin(\sqrt{\lambda}\,x)\right]$$

IT IS MORE CONVENIENT TO WRITE

$$\phi = \frac{1}{x}\left[A\cos(\sqrt{\lambda}(x-1)) + B\sin(\sqrt{\lambda}(x-1))\right].$$

NOW $\phi(1) = 0 \to A = 0$. $\phi(2) = 0 \longrightarrow \sin(\sqrt{\lambda}) = 0$

HENCE $\sqrt{\lambda} = n\pi$ OR $\lambda = n^2\pi^2$

AND $$\phi_n(x) = \frac{1}{x}\sin(n\pi(x-1))$$

WHEN $p(x)$ VANISHES AT ONE OF THE ENDPOINTS, OR WHEN EITHER ENDPOINT IS ∞ WE HAVE A <u>SINGULAR</u> STURM-LIOUVILLE PROBLEM.

(iii) $\phi'' + \frac{1}{x}\phi' + \lambda\phi = 0, \quad 0 < x < 1; \quad \phi(0)$ FINITE, $\phi(1) = 0$.

THIS IS BESSEL'S EQUATION WITH $\phi = A J_0(\sqrt{\lambda}x) + B Y_0(\sqrt{\lambda}x)$

AND IN STURM-LIOUVILLE FORM $(x\phi')' + \lambda x \phi = 0$

SO THAT $\int_0^1 x\,\phi_n\phi_m\,dx = 0$ FOR $n \neq m$.

(iv) $\phi'' - 2x\phi' + \lambda\phi = 0 \quad -\infty < x < \infty$; HERMITE'S EQUATION

IN STURM-LIOUVILLE FORM $(e^{-x^2}\phi')' + \lambda e^{-x^2}\phi = 0$

SO THAT $\int_{-\infty}^{\infty} e^{-x^2}\phi_n\phi_m\,dx = 0$ FOR $n \neq m$.

USING FROBENIUS SERIES THERE ARE POLYNOMIAL SOLUTIONS TO THIS EQUATION WHEN $\lambda = 2n$, FOR $n = 0, 1, 2, ..$

THEN $\phi_n(x) = H_n(x)$ HERMITE POLYNOMIALS

$$\lambda = 2n, \quad n = 0, 1, 2, ..$$

$H_0(x) = 1, \quad H_1(x) = 2x, \quad H_2(x) = 4x^2 - 2, \; --$

(v) $\phi'' - \frac{2x}{1-x^2}\phi' + \frac{\lambda}{1-x^2}\phi = 0$ IN $-1 < x < 1$, $\phi(\pm 1)$ FINITE.

THIS IS LEGENDRE'S EQUATION AND IN SL FORM WE GET

$[(1-x^2)\phi']' + \lambda\phi = 0$ SO $\int_{-1}^{1} \phi_n(x)\phi_m(x)\,dx = 0$
$n \neq m$

THE SOLUTIONS ARE $\phi = A P_n(x) + B Q_n(x)$ WHEN $\lambda = n(n+1)$

NOW $P_n(x)$ ARE LEGENDRE POLYNOMIALS OF DEGREE n.

INHOMOGENEOUS PROBLEMS: EIGENFUNCTION EXPANSIONS

SUPPOSE THAT WE HAVE A HEAT EQUATION ON $0 \le x \le 1$, $t > 0$

$$w(x)\, u_t = (p(x)\, u_x)_x - q(x)\, u + G(x,t) \qquad p(x) > 0 \text{ ON } 0 < x < 1$$

$$w(x) > 0 \text{ ON } 0 < x < 1$$

$$u_x(0,t) = h_1\, u(0,t) \qquad h_1 \geq 0, \quad h_2 \geq 0$$

$$u_x(1,t) = -h_2\, u(1,t)$$

$$u(x,0) = f(x)$$

WE FIRST CONSIDER THE EIGENVALUE PROBLEM FOR THE HOMOGENEOUS PROBLEM

$$(p(x)\,\phi')' - q(x)\,\phi + \lambda\, w(x)\,\phi = 0, \qquad 0 < x < 1$$

$$\phi'(0) - h_1\,\phi(0) = 0, \quad \phi'(1) + h_2\,\phi(1) = 0$$

THE NORMALIZED EIGENFUNCTIONS ARE $\phi_n(x)$ WITH EIGENVALUES λ_n FOR $n = 1, 2, 3, \ldots$. THE NORMALIZATION CONDITION is $\int_0^1 w(x)\,(\phi_n(x))^2\, dx = 1$.

WE LOOK FOR A SOLUTION IN THE FORM

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t)\,\phi_n(x)$$

WHERE $b_n(t)$ FOR $n = 1, 2, 3, \ldots$ ARE TO BE FOUND.

WE SUBSTITUTE INTO THE PDE TO OBTAIN

$$\sum_{n=1}^{\infty} w(x)\, b_n'(t)\,\phi_n(x) = \sum_{n=1}^{\infty} b_n(t)\, \mathcal{L}\phi_n + \frac{G(x,t)}{w(x)}\, w(x)$$

WHERE $\mathcal{L}\phi_n = (p\,\phi_n')' - q\,\phi_n = -\lambda_n\, w\,\phi_n$. NOW EXPAND

$$\frac{G(x,t)}{w(x)} = \sum_{n=1}^{\infty} c_n\,\phi_n(x) \qquad c_n = \int_0^1 G(x,t)\,\phi_n(x)\, dx.$$

THIS YIELDS THAT

$$\sum_{n=1}^{\infty} w(x)\, b_n' \phi_n = \sum_{n=1}^{\infty} -\lambda_n b_n w \phi_n + \sum_{n=1}^{\infty} c_n \phi_n w$$

EQUIVALENTLY, WE OBTAIN THAT

$$\sum_{n=1}^{\infty} (b_n' + \lambda_n b_n - c_n)\, \phi_n w = 0$$

BY ORTHOGONALITY OF THE EIGENFUNCTIONS ϕ_n, $n = 1, 2, \ldots$

WE OBTAIN THAT

$$(*) \quad b_n' + \lambda_n b_n = c_n .$$

NOW $\quad u(x,0) = f(x) = \sum_{n=1}^{\infty} b_n(0)\, \phi_n(x).$

THIS YIELDS THAT $\quad b_n(0) = \int_0^1 f(x)\, \phi_n(x)\, w(x)\, dx$

BY ORTHOGONALITY AND $\int_0^1 w \phi_n^2\, dx = 1.$

WE SOLVE $(*)$ BY WRITING

$$(b_n e^{\lambda_n t})' = c_n(t) e^{\lambda_n t}$$

WE OBTAIN $\quad b_n e^{\lambda_n t} = b_n(0) + \int_0^t e^{\lambda_n \tau} c_n(\tau)\, d\tau.$

THIS YIELDS THAT $\quad b_n(t) = b_n(0) e^{-\lambda_n t} + e^{-\lambda_n t} \int_0^t e^{\lambda_n \tau} c_n(\tau)\, d\tau$

THE EIGENFUNCTION FUNCTION EXPANSION

SOLUTION IS $\quad u(x,t) = \sum_{n=1}^{\infty} b_n(t)\, \phi_n(x)$

WITH $b_n(t)$ AS GIVEN ABOVE.

IMPORTANT REMARK

IN ORDER TO OBTAIN AN EIGENFUNCTION EXPANSION WITH GOOD CONVERGENCE PROPERTIES IT IS CRITICAL TO MAKE A CHANGE OF VARIABLES TO OBTAIN HOMOGENEOUS BOUNDARY CONDITIONS.

EXAMPLE TO SOLVE $u_t = u_{xx} \quad 0 < x < L, \; 0 < t$

$$u(0,t) = T_0 \qquad u_x(L,t) = -h(u(L,t) - e^{-t})$$

WITH h, T_0 CONSTANTS.

WE FIRST WRITE THE SOLUTION TO

$$W_{xx} = 0 \qquad w(0,t) = T_0, \quad W_x(L,t) = -h(w(L,t) - e^{-t})$$

THEN $W = Ax + B$. $\quad w(0,t) = T_0 \longrightarrow B = T_0$

$$w_x = -h(w - e^{-t}) \longrightarrow A = -h[AL + B - e^{-t}].$$

THIS YIELDS $A = -\dfrac{h(T_0 - e^{-t})}{1 + hL} = +\dfrac{h(e^{-t} - T_0)}{1 + hL}.$

WE THEN WRITE $u(x,t) = W + v(x,t)$

AND WE WILL BE GUARANTEED HOMOGENEOUS BOUNDARY CONDITIONS FOR $v(x,t)$. SINCE $W_t = -xhe^{-t}/(1+hL)$

WE OBTAIN THAT

$$v_t = v_{xx} + \frac{h}{(1+hL)}(e^{-t}x)$$

$$v(0,t) = 0, \qquad v_x(L,t) = -hv(L,t)$$

$$v(x,0) = u(x,0) - W|_{x=0} = u(x,0) - T_0.$$

WE CAN THEN WRITE $v(x,t) = \sum_{n=1}^{\infty} b_n(t)\,\phi_n(x)$ $\qquad \phi_n'' + \lambda_n \phi_n = 0$

$$\phi_n(0) = 0, \quad \phi_n'(L) = -h\phi_n(L)$$

EXAMPLE CONSIDER HEAT FLOW IN A SPHERE WITH PERIODIC TEMPERATURE VARIATIONS ON THE BOUNDARY OF THE SPHERE, MODELED BY

$$u_t = D\left(u_{rr} + \frac{2}{r} u_r\right) \qquad 0 < r < a,\ t > 0$$

$$u(a,t) = \sin(\omega t), \quad u \text{ BOUNDED AS } r \to 0$$

$$u(r,0) = 1 - r/a$$

WE LET $u(r,t) = \sin(\omega t) + v(r,t)$ WHICH YIELDS

$$\begin{cases} v_t = D\left(v_{rr} + \frac{2}{r} v_r\right) - \omega\cos(\omega t) \\ v(a,t) = 0, \quad v \text{ BOUNDED AS } r \to 0 \\ v(r,0) = 1 - r/a \end{cases}$$

WE SEPARATE VARIABLES TO OBTAIN $\dfrac{T'}{DT} = \dfrac{Y'' + 2/r\, Y'}{Y} = -\lambda$.

THIS YIELDS THE EIGENVALUE PROBLEM

$$\phi'' + \frac{2}{r}\phi' + \lambda\phi = 0 \quad \longrightarrow \quad (r^2\phi')' + \lambda r^2 \phi = 0$$

SO THAT THE WEIGHT FUNCTION IS $w = r^2$.

$$\text{HENCE} \qquad \phi = A\,\frac{\sin(\sqrt{\lambda}\, r)}{r} + B\,\frac{\cos(\sqrt{\lambda}\, r)}{r}$$

$B = 0$ FOR BOUNDEDNESS AT $r = 0$. NOW $\phi(a) = 0$ SO $\sqrt{\lambda}\, a = n\pi$.

THIS YIELDS $\lambda_n = n^2\pi^2/a^2$.

WE LABEL $\phi_n(r) = \sin(n\pi r/a)/r$

$$\text{THEN} \qquad v(r,t) = \sum_{n=1}^{\infty} b_n(t)\,\phi_n(r)$$

$$\text{WITH} \qquad b_n(0) = \frac{\int_0^a v(r,0)\, r^2 \phi_n(r)\, dr}{\int_0^a r^2 \phi_n^2(r)\, dr} = \frac{\int_0^a (1 - r/a)\, r^2 \phi_n(r)\, dr}{\int_0^a r^2 \phi_n^2(r)\, dr}.$$

NOW WE SUBSTITUTE INTO THE PDE TO OBTAIN:

$$\sum_{n=1}^{\infty} b_n' \phi_n = \sum_{n=1}^{\infty} D b_n \left(\phi_n'' + \frac{2}{r} \phi_n' \right) - \omega \cos(\omega t)$$

BUT $\phi_n'' + \frac{2}{r} \phi_n' = -\lambda_n \phi_n$ SO THAT

$$\sum_{n=1}^{\infty} \left(b_n' + D \lambda_n b_n \right) \phi_n = -\omega \cos(\omega t)$$

WE EXPAND $1 = \sum_{n=1}^{\infty} c_n \phi_n(r)$ $\quad c_n = \dfrac{\int_0^a r^2 \phi_n(r)\, dr}{\int_0^a r^2 (\phi_n(r))^2\, dr} = \dfrac{2}{n\pi} \left[1 - (-1)^n \right].$

THIS YIELDS THAT $\sum_{n=1}^{\infty} \left(b_n' + D \lambda_n b_n \right) \phi_n = -\sum_{n=1}^{\infty} \omega \cos(\omega t)\, c_n \phi_n .$

WE THEN OBTAIN THAT

$$\begin{cases} b_n' + \lambda_n D\, b_n = -\omega c_n \cos(\omega t) \\ b_n(0) \text{ given} \end{cases}$$

NOW WE WRITE $\left(b_n e^{\lambda_n D t} \right)' = -\omega c_n \cos(\omega t)\, e^{\lambda_n D t}.$

THIS YIELDS $b_n e^{\lambda_n D t} = b_n(0) - \omega c_n \int_0^t \cos(\omega \tau)\, e^{\lambda_n D \tau}\, d\tau .$

BUT $\int_0^t e^{\gamma \tau} \cos(\omega \tau)\, d\tau = \dfrac{\gamma}{\gamma^2 + \omega^2} \left[e^{\gamma t} \cos(\omega t) - 1 + \dfrac{\omega}{\gamma} \sin(\omega t)\, e^{\gamma t} \right].$

WE OBTAIN

$$b_n(t) = b_n(0)\, e^{-\lambda_n D t} - \omega c_n \left[\frac{\lambda_n D}{\lambda_n^2 D^2 + \omega^2} \left(\cos \omega t - e^{-\lambda_n D t} \right) + \frac{\omega}{\lambda_n D} \sin(\omega t) \right]$$

THEN FINALLY $u(r,t) = \sin(\omega t) + \sum_{n=1}^{\infty} b_n(t)\, \dfrac{\sin(n\pi r/a)}{r} .$

NOW FOR LARGE t WITH $t>>1$ WE USE $\lambda_n>0$ TO OBTAIN

$$b_n(t) \approx -w\, C_n \left[\frac{\lambda_n D}{\lambda_n^2 D^2 + w^2} \cos(wt) - \frac{w}{\lambda_n D} \sin(wt) \right] \qquad t>>1$$

WE OBTAIN $\quad b_n(t) \approx w\, C_n\, A_n \sin(wt - \gamma_n)$

WITH $\quad \tan \gamma_n = \dfrac{\lambda_n^2 D^2/w}{\lambda_n^2 D^2 + w^2}$, $\quad A_n = \sqrt{\left(\dfrac{\lambda_n D}{\lambda_n^2 D^2 + w^2}\right)^2 + \dfrac{w^2}{\lambda_n^2 D^2}}$

HENCE FOR $t>>1$ WE HAVE

$$u(0,t) = \sin(wt) + \sum_{n=1}^{\infty} \left(\frac{n\pi}{a}\right) w\, C_n\, A_n \sin(wt - \gamma_n)$$

WITH $\quad C_n = \dfrac{2}{n\pi}\left(1-(-1)^n\right)$. NOTICE $\quad A_n = O\left(1/n\right)$ FOR $n>>1$

WE CONCLUDE THAT THERE IS A PHASE LAG FOR THE TEMPERATURE AT THE CENTER OF THE SPHERE.

EXAMPLE FIND THE SOLUTION TO THE FORCED WAVE EQUATION

$$U_{tt} = c^2 U_{xx} + F(x,t) \quad , \quad 0 < x < L, \ t > 0$$

$$U(0,t) = U(L,t) = 0$$

$$U(x,0) = g(x) \, , \quad U_t(x,0) = 0$$

WHERE $F(x,t) = \delta(x - x_0) \sin\left(\frac{N\pi c t}{L}\right)$ N a positive integer.

THIS CORRESPONDS TO PLUCKING THE STRING LOCALLY AT SOME $x = x_0$ WITH $0 < x_0 < L$.

REMARK RECALL THAT $\int_0^1 \delta(x - x_0) g(x) \, dx = g(x_0)$ WHEN $0 < x_0 < 1$.

ALSO $\delta(x - x_0) = \lim_{\sigma \to 0} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x - x_0)^2/2\sigma^2}$ IN generalized FUNCTION sense:

$$\delta(x - x_0) = \begin{cases} 0 & \text{IF } x \neq x_0 \\ \infty & \text{IF } x = x_0 \end{cases} \qquad 1 = \int_a^b \delta(x - x_0)\, dx \text{ FOR } a < x_0 < b.$$

FOR THE HOMOGENEOUS PROBLEM THE EIGENFUNCTIONS ARE $\sin(n\pi x/L)$ AND THE NORMALIZED EIGENFUNCTIONS ARE

$$\phi_n(x) = \left(\frac{2}{L}\right)^{1/2} \sin\left(\frac{n\pi x}{L}\right) \quad \text{WITH} \quad \int_0^L (\phi_n(x))^2 \, dx = 1$$

WE LOOK FOR A SOLUTION OF THE FORM

$$(*) \qquad U(x,t) = \sum_{n=1}^{\infty} b_n(t) \, \phi_n(x).$$

THEN $U(x,0) = g(x) = \sum_{n=1}^{\infty} b_n(0) \phi_n(x) \longrightarrow b_n(0) = \int_0^L g(x) \phi_n(x) \, dx$

$U_t(x,0) = 0 = \sum_{n=1}^{\infty} b_n'(0) \phi_n(x) \longrightarrow b_n'(0) = 0$

WE THEN SUBSTITUTE (*) INTO THE PDE TO OBTAIN

$$\sum_{n=1}^{\infty} (b_n'' + w_n^2 b_n)\,\phi_n(x) = \sum_{n=1}^{\infty} d_n(t)\,\phi_n(x),$$

WITH $w_n = cn\pi/L$ AND $d_n(t) = \int_0^L \phi_n(x)\,F(x,t)\,dx = \int_0^L \phi_n(x)\,F(x,t)\,dx.$

NOW WE CALCULATE

$$d_n(t) = \int_0^L \left(\frac{2}{L}\right)^{1/2} \sin\left(\frac{n\pi x}{L}\right) \delta(x-x_0) \sin\left(\frac{N\pi c t}{L}\right) dx$$

THIS YIELDS $d_n(t) = \left(\frac{2}{L}\right)^{1/2} \sin\left(\frac{n\pi x_0}{L}\right) \sin(w_N t)$, $w_N = N\pi c/L$.

BY ORTHOGONALITY $u(x,0) = g(x) = \sum_{n=1}^{\infty} b_n(0)\,\phi_n(x)$.

THIS YIELDS THAT $b_n(0) = \int_0^L g(x)\,\phi_n(x)\,dx$

THE PROBLEM FOR $b_n(t)$ BECOMES:

$$\begin{cases} b_n'' + w_n^2 b_n = \left(\frac{2}{L}\right)^{1/2} \sin\left(\frac{n\pi x_0}{L}\right) \sin(w_N t), & w_n = n\pi c/L \\ b_n(0) = \int_0^L g(x)\,\phi_n(x)\,dx, & b_n'(0) = 0 \end{cases}$$

<u>CASE 1</u> $(n \neq N)$ NON-RESONANT CASE

THE PARTICULAR SOLUTION HAS THE FORM

$b_n(t) = A_n \sin(w_N t)$ SO THAT $A_n(w_n^2 - w_N^2) = \left(\frac{2}{L}\right)^{1/2} \sin\left(\frac{n\pi x_0}{L}\right)$

THIS YIELDS $A_n = \left(\frac{2}{L}\right)^{1/2} \frac{1}{w_n^2 - w_N^2} \sin\left(\frac{n\pi x_0}{L}\right)$

THE GENERAL SOLUTION IS

$$b_n(t) = \alpha_n \sin(w_n t) + B_n \cos(w_n t) + A_n \sin(w_N t)$$

BUT $b_n'(0) = 0 \longrightarrow \alpha_n w_n + A_n w_N = 0 \longrightarrow \alpha_n = -\frac{w_N}{w_n} A_n.$

$b_n(0) = B_n.$

THIS YIELDS THAT FOR $n \neq N$.

$$b_n(t) = A_n\left[\sin(\omega_N t) - \frac{\omega_N}{\omega_n}\sin(\omega_n t)\right] + b_n(0)\cos(\omega_n t).$$

CASE 2 $(n = N)$ RESONANT CASE.

WE WRITE THE PARTICULAR SOLUTION AS $b_{Np} = \mathrm{IM}(\tilde{b}_N)$.

THIS YIELDS $\quad \tilde{b}_N'' + \omega_N^2 \tilde{b}_N = \left(\frac{2}{L}\right)^{1/2} \sin\left(\frac{N\pi x_0}{L}\right) e^{i\omega_N t}$.

WE SUBSTITUTE $\quad \tilde{b}_N = B_N t e^{i\omega_N t}$

TO OBTAIN THAT $\quad B_N = \frac{-i}{2\omega_N}\left(\frac{2}{L}\right)^{1/2}\sin\left(\frac{N\pi x_0}{L}\right)$

THIS YIELDS THAT $\quad b_{Np}(t) = \mathrm{IM}(\tilde{b}_N) = -\frac{1}{2\omega_N}\left(\frac{2}{L}\right)^{1/2}\sin\left(\frac{N\pi x_0}{L}\right) t\cos(\omega_N t)$

FINALLY WE WRITE

$$b_N(t) = \alpha_N \sin(\omega_N t) + B_N \cos(\omega_N t) - \frac{t}{2\omega_N}\left(\frac{2}{L}\right)^{1/2}\sin\left(\frac{N\pi x_0}{L}\right)\cos(\omega_N t)$$

$$b_N'(0) = 0 \longrightarrow \quad \alpha_N \omega_N - \frac{1}{2\omega_N}\left(\frac{2}{L}\right)^{1/2}\sin\left(\frac{N\pi x_0}{L}\right) = 0$$

$$\alpha_N = \frac{1}{2\omega_N^2}\left(\frac{2}{L}\right)^{1/2}\sin\left(\frac{N\pi x_0}{L}\right)$$

THIS YIELDS THAT

$$b_N(t) = b_N(0)\cos(\omega_N t) + \frac{1}{2\omega_N^2}\left(\frac{2}{L}\right)^{1/2}\sin\left(\frac{N\pi x_0}{L}\right)\sin(\omega_N t) - \frac{t}{2\omega_N}\left(\frac{2}{L}\right)^{1/2}\sin\left(\frac{N\pi x_0}{L}\right)\cos(\omega_N t)$$

THE FINAL SOLUTION IS

$$u(x,t) = b_N(t)\sin\left(\frac{N\pi x}{L}\right) + \sum_{n=1, n\neq N}^{\infty} b_n(t)\left(\frac{2}{L}\right)^{1/2}\sin\left(\frac{n\pi x}{L}\right)$$