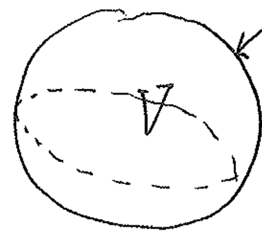
SOLID COMBUSTION THEORY

WE CONSIDER A POROUS PELLET SUBJECT TO GASEOUS DIFFUSION THROUGH THE PORES AND WE ASSUME THREE PHYSICAL PROCESSES:

(i) conductive transfer of heat

(ii) fickian diffusion of reactant

(iii) chemical reaction



THE GASEOUS REACTANT A WITH CONCENTRATION c FLOWS PAST A SOLID POROUS CATALYST PELLET V, AND THE RATE OF REACTION IS

$$A \xrightarrow{r} B \qquad r = K_0\, c\, \exp\left(-E/RT\right) \quad \frac{\text{moles}}{\text{cm}^3\text{-sec}}$$

LET ΔH (cal/mole) BE HEAT ABSORPTION

SO $\Delta H > 0$ (endothermic $\rightarrow$ absorbs heat)

$\Delta H < 0$ (exothermic)

R = gas constant $8.3 J/\text{mole-}K$

E = activation energy

T : temperature $K_0 > 0.$ $(1/\text{sec})$

NOW THE CONSERVATION OF REACTANT IS THAT

$$\frac{d}{dt}\int_V c\, d\underset{\sim}{x} = -\int_S \left(-D\,\nabla c\cdot\hat{n}\right) ds - \int_V r\, d\underset{\sim}{x}$$

USING THE DIVERGENCE THEOREM THIS YIELDS

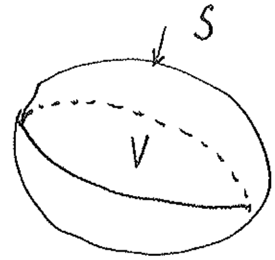
$$c_t = \nabla\cdot\left[D\nabla c\right] - r \qquad \text{IN} \quad V$$

SIMILARLY, WE OBTAIN

$$\frac{\partial}{\partial t}\left(\rho\, c_p\, T\right) = \nabla\cdot\left[\underset{\sim}{K}\nabla T\right] - r\,\Delta H$$

where ρ = density of pellet, c_p is specific heat, T = temperature, $\underset{\sim}{K}$ is thermal conductivity.

NOW ASSUMING THAT THESE PARAMETERS ARE CONSTANTS, WE OBTAIN



$$T_t = \kappa \Delta T - \frac{r \Delta H}{\rho c_p} \qquad \kappa = K/_{\rho c_p}$$

$$c_t = \nabla \cdot [D \nabla c] - r \qquad r = K_0 c \exp(-E/_{RT}).$$

WITH $T = T_0$ AND $c = c_0$ ON S.

NON-DIMENSIONALIZATION 1

WE LET $\tau = \omega t$, $T = \frac{E}{R} u$, $c = c_0 C$, $y = x/_L$.

THEN WE OBTAIN

$$\omega u_\tau = \frac{\kappa}{L^2} \Delta_y u - \left(\frac{r \Delta H}{\rho c_p}\right) \frac{R}{E}, \qquad \omega C_\tau = \frac{D}{L^2} \Delta_y C - K_0 C \exp(-1/_u).$$

HOOSE $\omega = \kappa/_{L^2}$ AND DEFINE

$L_e \equiv \kappa/_D$ ratio of thermal diffusivity to the diffusivity of Reactant.
(LEWIS NUMBER)

$$\mu \equiv \frac{K_0 L^2}{\kappa}, \qquad \lambda \equiv -\frac{\Delta H}{\rho c_p} \left(\frac{R}{E}\right) \frac{L^2 K_0}{\kappa} c_0$$

THEN,

(BVP)
$$\left\{ \begin{aligned} C_\tau &= \frac{1}{L_e} \Delta_y C - \mu C \exp(-1/_u) \\ u_\tau &= \Delta_y u + \lambda C \exp(-1/_u) \\ C &= 1 \text{ ON } S, \qquad u = T_0/_{(E/R)} \text{ ON } S. \end{aligned} \right.$$

ON-DIMENSIONALIZATION 2

let $y = x/_L$, $\tau = \omega t$, $c = c_0 C$, $T = T_0 + (\Delta T) u$.

TO RECOVER THE OTHER NON-DIMENSIONALIZATION WE LET

$$u = RT_0/E + V.$$

THEN $$\exp\left(-\frac{1}{u}\right) = \exp\left(-\frac{E}{RT_0+VE}\right) = \exp\left(-\frac{E}{RT_0}\right)\exp\left(\frac{E}{RT_0}-\frac{E}{RT_0+VE}\right)$$

$$= \exp\left(-\frac{E}{RT_0}\right)\exp\left(\frac{VE^2}{RT_0(RT_0+VE)}\right) = \exp\left(\frac{V(E/RT_0)^2}{1+V(E/RT_0)}\right)$$

NOW WE LET $W = V(E/RT_0)^2$. THEN, THIS CORRESPONDS TO

$T = T_0[1+J_0RW/E]$

$$\begin{cases} C_\tau = \dfrac{1}{L_e}\Delta_y C - \tilde{\mu}\, C\exp\left(W/(1+BW)\right) \\ W_\tau = \Delta_y W + \hat{\lambda}\, C\exp\left(W/(1+BW)\right) \\ W=0 \text{ ON } S, \quad C=1 \text{ ON } S \end{cases}$$

l_e: ratio thermal diffusivity / diffusivity of Reactant.

WITH

$$B = RT_0/E \quad \text{(activation energy parameter)}.$$

$$\tilde{\mu} = \mu\exp\left(-E/RT_0\right)$$

$$\hat{\lambda} = \lambda\exp\left(-E/RT_0\right)\left(E/RT_0\right)^2.$$

NOTICE THAT $\lambda > 0$ IF THE REACTION IS EXOTHERMIC. (i.e. $\Delta H < 0$).

DIFFERENT APPROXIMATIONS

I) ASSUME THAT $Le \gg 1$, $\hat{\mu} \ll 1$ SO THAT $C_\tau \ll 1$. THIS YIELDS $C \sim 1$ ON SOME TIME INTERVAL. THEN,

$$W_\tau = \Delta_y W + \hat{\lambda}\exp\left(W/(1+BW)\right) \qquad W=0 \text{ ON } S.$$

IF IN ADDITION $B \ll 1 \longrightarrow W_\tau = \Delta_y W + \tilde{\lambda}\exp(W)$, $W=0$ ON S. BRATU EQUATION.

II) SUPPOSE WE NEGLECT DIFFUSION. THEN,

$$C_\tau = -\hat{\mu}\, C \exp\left(W/_{1+BW} \right) \qquad C(0)=1$$

$$W_\tau = \tilde{\lambda}\, C \exp\left(W/_{(1+BW)} \right) \qquad W(0)=0$$

THEN IT IS EASY TO SEE THAT $\dfrac{d}{d\tau}\left[C + \dfrac{\hat{\mu}}{\tilde{\lambda}} W \right] = 0$. THUS

$$C + \frac{\hat{\mu}}{\tilde{\lambda}} W = 1.$$

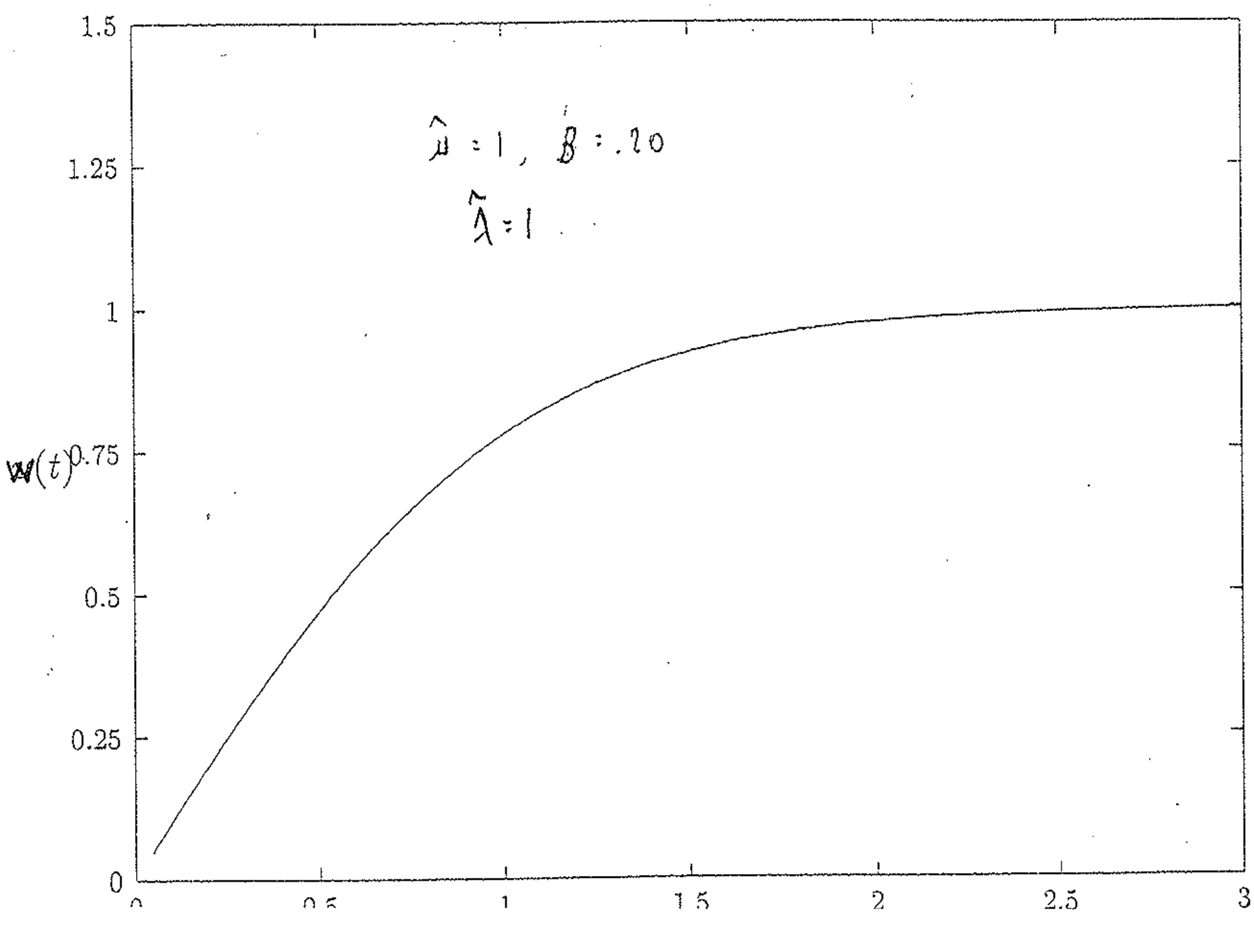
THIS YIELDS THE SIMPLE ODE

$$W_\tau = \tilde{\lambda}\left[1 - \frac{\hat{\mu}}{\tilde{\lambda}} W \right] \exp\left(W/_{1+BW} \right)$$

$$W(0) = 0.$$

THUS $W \longrightarrow \hat{\lambda}/_{\tilde{\mu}}$ AS $\tau \longrightarrow \infty$ AND $C \longrightarrow 0$. THUS THE TEMPERATURE REMAINS BOUNDED AND THE CONCENTRATION OF REACTANT VANISHES.

<u>NOTE</u>: IF WE SET $C=1$ AND $B=0$ WE OBTAIN FINITE TIME BLOW UP FOR $W_\tau = \hat{\lambda} e^W$.



CYLINDRICAL DOMAIN

WE NOW CONSIDER $u_{rr} + \frac{1}{r} u_r + \lambda e^{u} = 0$, $u(1) = 0$, $u'(0) = 0$, $0 < r < 1$.

WE LET $u = -2 \log v$ $\quad u_r = \frac{-2 v_r}{v}$ $\quad u_{rr} = -\frac{2 v_{rr}}{v} + \frac{2 v_r^2}{v^2}$.

THIS YIELDS,
$$-2 v v_{rr} + 2 v_r^2 - \frac{2}{r} v_r v + \lambda = 0.$$

WE WRITE THIS AS
$$v v_{rr} - v_r^2 + \frac{1}{r} v v_r = \frac{\lambda}{2}$$

NOW WE TRY $v = A(1 + \alpha r^2)$. SUBSTITUTING WE OBTAIN
$$2\alpha A^2 (1 + \alpha r^2) - 4 A^2 \alpha^2 r^2 + A(1 + \alpha r^2) 2 A \alpha = \frac{\lambda}{2}.$$

WE GROUP TERMS: $\quad 2\alpha A^2 + 2 A^2 \alpha = \frac{\lambda}{2}$; $\quad -2 A^2 \alpha^2 + 2 A^2 \alpha^2 = 0$.

THEREFORE, WE HAVE $\quad 8 A^2 \alpha = \lambda$, $\quad u = -2 \log \left[A(1 + \alpha r^2) \right]$.

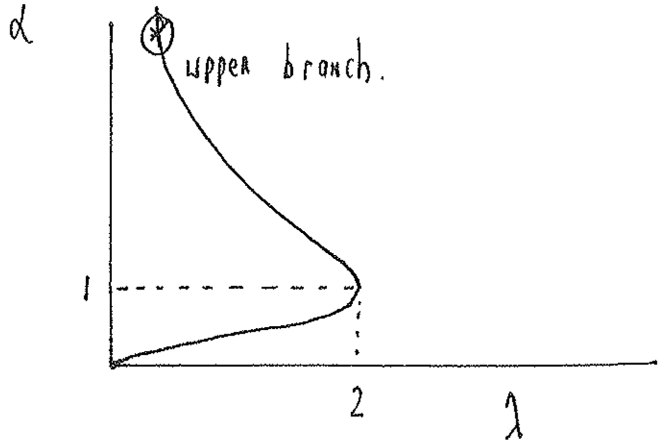
NOTICE THAT $u(1) = 0 \longrightarrow A(1 + \alpha) = 1$ AND $u'(0) = 0$.

THIS YIELDS THAT
$$\lambda = \frac{8\alpha}{(1 + \alpha)^2}, \quad 0 \le \alpha < \infty. \qquad u = -2 \log \left(\frac{1 + \alpha r^2}{1 + \alpha} \right)$$

NOW DEFINE $u(0) = u_0$ SO THAT $u_0 = -2 \log A = +2 \log (1 + \alpha)$. THIS YIELDS THAT $\alpha = -1 + e^{u_0/2}$ OR $1 + \alpha = e^{u_0/2}$.

NOW WE CALCULATE
$$\lambda_\alpha = 8(1+\alpha)^{-2} - 16\alpha(1+\alpha)^{-3} = 8(1+\alpha)^{-3} \left[(1+\alpha) - 2\alpha \right]$$

THUS WE HAVE A FOLD POINT AT $\alpha_c = 1$, where $\lambda_c = 2$. THE CORRESPONDING TEMPERATURE AT THE CENTER u_0 IS $u_0 = 2 \log 2$.



NOW AS $\alpha \to \infty$ we have $\lambda \sim 8/\alpha$ AND SO $U_0 \sim 2\log\alpha = 2\log\left(\frac{8}{\lambda}\right)$.

THIS YIELDS THAT

$$U_0 \sim -2\log\lambda \quad \text{AS} \quad \lambda \to 0 \quad \text{ON upper branch (point } \circledast)$$

NOW IF WE ASSUME THAT $\exists$ A THIN LAYER NEAR $r = 0$ where $U \sim -2\log\lambda + W$, THEN WE OBTAIN FROM $U_{rr} + \frac{1}{r}U_r + \lambda e^U = 0$ THAT WITH $\rho = r/\sigma$, $\sigma \ll 1$, $U_r = U_\rho \frac{1}{\sigma}$,

$$\frac{1}{\sigma^2}\left[W_{\rho\rho} + \frac{1}{\rho}W_\rho\right] + \lambda e^{-2\log\lambda}\, e^W = 0.$$

THIS YIELDS THAT $\sigma = \sqrt{\lambda} \ll 1$ AND

$$(\text{INNER}) \left\{ \begin{array}{ll} W_{\rho\rho} + \frac{1}{\rho}W_\rho + e^W = 0 & 0 < \rho < \infty \\ W'(0) = 0 & \end{array} \right.$$

N THE OUTER REGION WE HAVE $\lambda e^U \ll 1$ AS $\lambda \to 0$ SO THAT

$$(\text{outer}) \left\{ \begin{array}{ll} U_{rr} + \frac{1}{r}U_r = 0 & \text{FOR} \quad \sigma \ll r \leq 1 \\ U(1) = 0 & \\ U \text{ IS singular AS } r \to 0 & \end{array} \right.$$

MATCHING THE TWO SOLUTIONS together determines the Asymptotic solution FOR $\lambda \to 0$ ON the upper branch

REMARK WE NOW CONSIDER THE EFFECT OF A PARTIALLY INSULATED BOUNDARY. THEN, WE MUST SOLVE

$$(*)\quad \begin{cases} u_{rr} + \frac{1}{r} u_r + \lambda e^{u} = 0 & \text{IN } 0 < r < 1 \\ u'(0) = 0 & \\ u'(1) + b\,u(1) = 0 & \text{WITH } b > 0 \end{cases}$$

WE CAN SOLVE THIS PROBLEM EXPLICITLY TO OBTAIN

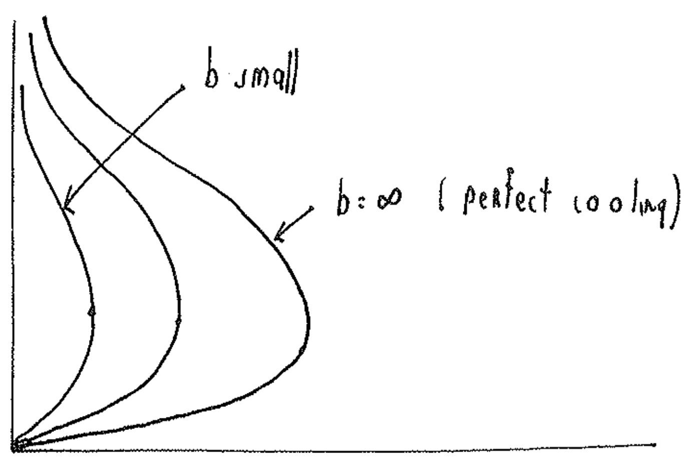
$$u = 2 \log\left(\frac{1+\alpha}{1+\alpha r^2} \right) + \frac{4\alpha}{b(1+\alpha)}$$

$$\lambda = \frac{8\alpha}{(1+\alpha)^2} \exp\left(-\frac{4\alpha}{b(1+\alpha)} \right)$$

NOW IF we set $\lambda' = 0$, we get $\alpha_0^2 - 1 + 4\alpha_0/b = 0$ SO THAT

$$\alpha_0 = -\frac{2}{b} + \left(1 + \frac{4}{b^2} \right)^{1/2}$$

α



NOTE $\quad u(0) = 2\log(1+\alpha) + \dfrac{4\alpha}{b(1+\alpha)}$

SCALE INVARIANCE

$$u_{rr} + \frac{(N-1)}{r} u_r + \lambda e^{u} = 0 \quad \text{IN} \quad 0 < r < 1$$

$$u(1) = 0, \quad u'(0) = 0.$$

LET $\quad u = \alpha + w(\rho), \quad \rho = \gamma r$

$$u(1) = 0 \longrightarrow \alpha + w(\gamma) = 0 \longrightarrow \text{LET} \quad \alpha = -w(\gamma).$$

THE BVP IS $\quad \gamma^2 \left(w'' + \frac{(N-1)}{\rho} w' \right) + \lambda e^{\alpha} e^{w} = 0$

SO

$$(*) \begin{cases} w'' + \frac{(N-1)}{\rho} w' + e^{w} = 0 \\ w(0) = 0, \quad w'(0) = 0 \end{cases} \qquad \begin{aligned} \lambda &= e^{-\alpha} \gamma^2 = e^{w(\gamma)} \gamma^2 \\ \alpha &= -w(\gamma) = u(0). \end{aligned}$$

THUS $\quad u(0)$ VS λ IS GIVEN PARAMETRICALLY BY

$$\lambda = \gamma^2 e^{w(\gamma)}$$

$$u(0) = -w(\gamma)$$

NOW PUT $\quad w = -2 \log \rho + V(\Lambda) \qquad \Lambda = \log \rho.$ WE WANT BEHAVIOR OF $w(\rho)$ AS $\rho \to \infty$ FOR $(*)$

$$w' = -2/\rho + V' \, 1/\rho \qquad w'' = 2/\rho^2 - 1/\rho^2 \, V' + 1/\rho^2 \, V''.$$

$$\frac{1}{\rho^2} V'' - \frac{1}{\rho^2} V' + \frac{2}{\rho^2} + \frac{(N-1)}{\rho} \left[\frac{V'}{\rho} - \frac{2}{\rho} \right] + \frac{1}{\rho^2} e^{V} = 0.$$

THUS

$$V'' - V' + 2 + (N-1) V' - 2(N-1) + e^{V} = 0$$

$$V'' + V'(N-2) + e^{V} + (4-2N) = 0.$$

NOW IF $N = 3$:

$$v'' + v' + e^v - 2 = 0.$$

SO $\quad v \sim \ln 2$ AS $\Lambda \to \infty$. $\qquad g(v) = e^v - 2.$

$$v = \ln 2 + \phi \qquad \phi \ll 1$$

$$\phi'' + \phi' + g'(\ln 2)\,\phi = 0$$

$$\phi'' + \phi' + 2\phi = 0 \qquad \lambda^2 + \lambda + 2 = 0 \qquad \lambda = \frac{-1 \pm \sqrt{1-8}}{2}$$

$$\phi = A e^{-\Lambda/2} \cos\left(\frac{\sqrt{7}}{2}\Lambda + \varphi\right) \qquad \Lambda = \log \rho$$

SO $$\phi \sim \frac{A}{\sqrt{\rho}} \cos\left(\frac{\sqrt{7}}{2}\log \rho + \varphi_0\right)$$

$$v \sim \ln 2 + \frac{A}{\sqrt{\rho}} \cos\left(\frac{\sqrt{7}}{2}\log \rho + \varphi_0\right) \quad \text{AS} \quad \rho \to \infty.$$

$$w \sim -2\log \rho + \ln 2 + \frac{A}{\sqrt{\rho}} \cos\left(\frac{\sqrt{7}}{2}\log \rho + \varphi_0\right)$$

AND $$\lambda \sim 2 + \frac{A}{\sqrt{\gamma}} \cos\left(\frac{\sqrt{7}}{2}\log \gamma + \varphi_0\right)$$

$$u(0) \sim 2\log \gamma - \log 2$$

NOW IF $N = 2$ $\qquad V'' + e^V = 0.$ NO FIXED POINTS!

HOWEVER, THIS SUGGESTS AN INTEGRAL EQUATION.

$$W'' + \frac{1}{\rho} W' + e^W = 0$$

$$W(0) = 0, \quad W'(0) = 0.$$

LET $\qquad W = -2 \log S. \qquad W_\rho = -\frac{2S'}{S}, \quad W_{\rho\rho} = -2\left[\frac{S''}{S} + \frac{S'^2}{S^2}\right]$

THIS GIVES $\qquad -\frac{2S''}{S} - \frac{2S'^2}{S^2} - \frac{2S'}{S\rho} + \frac{1}{S^2} = 0.$

SO $\qquad SS'' - S'^2 + \frac{S'S}{\rho} = \frac{1}{2}.$

$$S'(0) = 0, \quad S(0) = 1.$$

TRY $\qquad S = 1 + a\rho^2. \qquad$ THEN $\qquad S' = 2a\rho, \quad S'' = 2a$

$$(1 + a\rho^2)(2a) - (4a^2\rho^2) + 2a(1 + a\rho^2) = \frac{1}{2}.$$

$$-4a^2\rho^2 + 4a^2\rho^2 + 4a = \frac{1}{2} \quad \rightarrow a = 1/8$$

THUS $\qquad S = 1 + \rho^2/8.$

$$W = -2\log\left(1 + \rho^2/8\right).$$

NOW USE THIS TO OBTAIN $U = U(r)$.

RECALL $\lambda = \gamma^2 e^{W(\gamma)}$.

BUT $W(\gamma) = -2\log\left(1 + \frac{\gamma^2}{8}\right)$

SO $$\lambda = \frac{\gamma^2}{(1+\gamma^2/8)^2}.$$

BUT $W(\gamma) = -U(0) = -\alpha$

SO $U(0) = 2\log\left(1 + \gamma^2/8\right)$

$$U(r) = \alpha + W[\gamma r] \qquad \alpha = -W(\gamma)$$

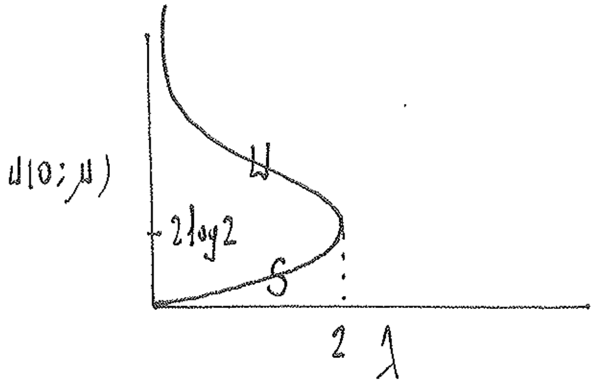
$$= +2\log\left(1 + \frac{\gamma^2}{8}\right) - 2\log\left(1 + \frac{\gamma^2 r^2}{8}\right)$$

SO $$U(r) = -2\log\left(\frac{1+\gamma^2 r^2/8}{1+\gamma^2/8}\right)$$

WITH $$\lambda = \frac{\gamma^2}{(1+\gamma^2/8)^2}.$$

NOW LET $\mu \equiv \gamma^2/8$. THEN

$$\left\{\begin{array}{l} U(r) = -2\log\left(\dfrac{1+\mu r^2}{1+\mu}\right), \qquad \lambda = \dfrac{8\mu}{(1+\mu)^2}. \\ \\ U(0) = 2\log(1+\mu) \end{array}\right.$$



LOWER: $0<\mu<1 \rightarrow d\lambda/d\mu > 0$

UPPER $\mu>1 \rightarrow d\lambda/d\mu = 0.$

SET $d\lambda/d\mu = 0 \rightarrow \dfrac{8}{(1+\mu)^2} - \dfrac{16\mu}{(1+\mu)^3} = 0 \rightarrow 8(1+\mu) = 16\mu \rightarrow \mu = 1$

$$u(0;1) = 2\log 2. \qquad \lambda_{MAX} = \frac{8}{(2)^2} = 2.$$

<u>LEMMA</u> THE LOWER BRANCH IS LINEARLY STABLE, WHILE THE UPPER BRANCH IS UNSTABLE.

<u>PROOF</u> WE HAVE $u_e(r;\mu) = -2\log\left(1+\mu r^2\right) + 2\log(1+\mu)$.

WE CALCULATE
$$\frac{du_e}{d\mu} = -\frac{2r^2}{1+\mu r^2} + \frac{2}{1+\mu} = \frac{-2r^2 - 2\mu r^2 + 2 + 2\mu r^2}{(1+\mu)(1+\mu r^2)}$$
$$\frac{du}{d\mu} = \frac{2(1-r^2)}{(1+\mu)(1+\mu r^2)} > 0 \text{ ON } 0<r<1.$$

WE HAVE $u_{e\mu} \equiv \dfrac{\partial u_e(r;\mu)}{\partial\mu} > 0$ ON $0<r<1$.

NOW $\Delta_r u_e + \lambda e^{u_e} = 0$

WE CALCULATE
$$\begin{cases} \Delta u_{e\mu} + \lambda e^{u_e} u_{e\mu} = -\lambda_\mu e^{u_e} & \text{IN } B \\ u_{e\mu} = 0 & \text{ON } \partial B. \end{cases}$$

NOW $u_t = \Delta u + \lambda e^{u}$ IN Ω.

WE LET $u = u_e + e^{-\sigma t}\phi$.

THEN

$$\Delta\phi + \lambda e^{u_e}\phi = -\sigma\phi$$

SO

$$\Delta\phi + \sigma\phi + \lambda e^{u_e}\phi = 0 \quad \text{IN} \quad B$$
$$\phi = 0 \quad \text{ON} \quad \partial B.$$

WE WANT TO PROVE THAT $\sigma_1 > 0$ IFF $\lambda_\mu > 0$. FOR THEN SINCE $\sigma_1 < \sigma_2 \dots$

i.e. $\sigma_j < \sigma_{j+1} \dots \quad \forall j$

WE HAVE ALL $\sigma_j > 0 \rightarrow$ LOWER BRANCH IS STABLE.

NOW G- SECOND IDENTITY,

$$\int_B (\phi_1 \Delta u_{e\mu} - u_{e\mu}\Delta\phi_1)\, d\underline{x} = 0.$$

$$\int_B (\phi_1[-\lambda e^{u_e} u_{e\mu} - \lambda_\mu e^{u_e}] - u_{e\mu}[-\lambda e^{u_e}\phi_1 - \sigma_1\phi_1])\, dx = 0$$

$$-\lambda_\mu \int_B \phi_1 e^{u_e}\, d\underline{x} + \sigma_1 \int_B u_{e\mu}\phi_1\, dx = 0.$$

THUS

$$\sigma_1 \int_B \underbrace{u_{e\mu}}_{>0} \phi_1\, d\underline{x} = \lambda_\mu \int \phi_1 e^{u_e}\, d\underline{x}$$

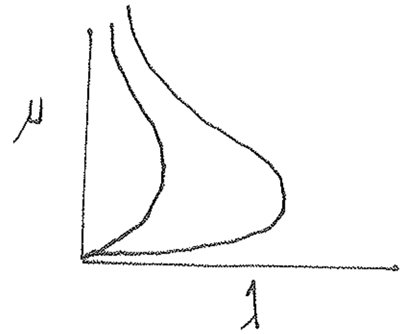
THUS IF $\lambda_\mu > 0 \rightarrow \sigma_1 > 0$ stable; if $\lambda_\mu < 0$, $\sigma_1 < 0$ UNSTABLE.

REMARK FOR $u'' + \frac{1}{r} u' + \lambda e^{u} = 0$ IN $0 < r < 1$

$$u'(0) = 0, \quad u'(1) + b\, u(1) = 0$$

WE HAVE

$$u = -2 \log \left(\frac{1 + \mu r^2}{1 + \mu} \right) + \frac{4\mu}{b(1+\mu)}$$



$$\lambda = \frac{8\mu}{(1+\mu)^2} \exp\left(-\frac{4\mu}{b(1+\mu)} \right)$$

$$u(0) = 2 \log (1+\mu) + \frac{4\mu}{b(1+\mu)}$$

PERTURBATION THEORY

CONSIDER $\Delta u + \lambda e^{u} = 0$ IN $0 \le r \le 1, \; 0 \le \varphi \le 2\pi$

$$u(1, \varphi) = \varepsilon\, h(\varphi)$$

HOW DOES THE FIRST FOLD POINT DEPEND ON $h(\varphi)$?

WE EXPAND FOR $\varepsilon \to 0$

$$u = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \ldots \qquad \lambda = \lambda_0(\mu) + \varepsilon \lambda_1(\mu) + \varepsilon^2 \lambda_2(\mu) + \ldots$$

WE SUBSTITUTE TO OBTAIN $e^{u_0 + \varepsilon u_1 + \varepsilon^2 u_2} [\lambda_0 + \varepsilon \lambda_1 + \varepsilon^2 \lambda_2]$

$$= e^{u_0} [1 + (\varepsilon u_1 + \varepsilon^2 u_2) + \frac{\varepsilon^2}{2} u_1^2][\lambda_0 + \varepsilon \lambda_1 + \varepsilon^2 \lambda_2]$$

$$= e^{u_0} [\lambda_0 + \varepsilon(\lambda_0 u_1 + \lambda_1) + \varepsilon^2 [\lambda_2 + \lambda_1 u_1 + \frac{\lambda_0 u_1^2}{2}]$$

SO

$$\begin{cases} \Delta u_0 + \lambda_0 e^{u_0} = 0 \\ u_0 = 0 \text{ ON } r = 1 \end{cases}$$

$$\begin{cases} \Delta u_1 + \lambda_0 e^{u_0} u_1 = -\lambda_1 e^{u_0} \\ u_1 = H(\varphi) \text{ ON } r = 1 \end{cases}$$

$$\Delta u_2 + \lambda_0 e^{u_0} = -\lambda_1 u_1 e^{u_0} - \lambda_0 \frac{u_1^2}{2} e^{u_0} - \lambda_2 e^{u_0}$$

$$u_2 = 0 \text{ ON } r = 1.$$

WE HAVE $\quad u_0 = -2\log\left(\dfrac{1+\mu r^2}{1+\mu}\right) \qquad \lambda = \dfrac{8\mu}{(1+\mu)^2}$.

NOW LET $\mu = \mu_0 + \varepsilon \mu_1 + \cdots$ BE WHERE FOLD POINT OCCURS:

$$\frac{d\lambda}{d\mu} = 0 \to \lambda_0'(\mu) + \varepsilon \lambda_1'(\mu) + \varepsilon^2 \lambda_2'(\mu) = 0.$$

PUT $\mu = 1 + \varepsilon \mu_1 + \cdots$ AND USE $\lambda_0'(1) = 0$:

$$\varepsilon \lambda_0''(1)\mu_1 + \varepsilon \lambda_1'(1) = 0 \to \mu_1 = -\frac{\lambda_1'(1)}{\lambda_0''(1)}$$

NOW AT $\mu_c = 1 + \varepsilon \mu_1 + \cdots$

$$\lambda_c(\mu) = \lambda_0(1+\varepsilon\mu_1) + \varepsilon\lambda_1(1+\varepsilon\mu_1) + \varepsilon^2\lambda_2(1+\varepsilon\mu_1) + \cdot$$

$$= \lambda_0(1) + \varepsilon^2 \frac{\lambda_0''(1)}{2}\mu_1^2 + \varepsilon\lambda_1(1) + \varepsilon^2\lambda_1'(1)\mu_1 + \varepsilon^2\lambda_2(1).$$

THUS YIELDS $\quad \lambda_c = \lambda_0(1) + \varepsilon\lambda_1(1) + \varepsilon^2\Big[\underbrace{\frac{\lambda_0''(1)}{2}\mu_1^2 + \lambda_2(1) + \lambda_1'(1)\mu_1}_{\lambda_2(1) - [\lambda_1'(1)]^2/2\lambda_0''(1).}\Big]$

NOW AT $\mu_0 = 1$ WE HAVE

$$\Delta u_{0\mu}^* + \lambda_0 e^{u_0^*}\mu_{0\mu}^* = -\lambda_{0\mu}\big|_{\mu=1} e^{u_0^*} = 0.$$

THUS AT $\mu_0 = 1$, $\quad L u_{0\mu} \equiv \Delta u_{0\mu} + \lambda_0 e^{u_0} u_{0\mu} = 0.$

THUS AT $\mu = 1$, WE HAVE $\quad \displaystyle\int_\Omega (u_1 L u_{0\mu} - u_{0\mu} L u_1)\, d\underline{x} = \int_{\partial\Omega} u_1 \partial_n u_{0\mu}\, dx$

$$= \int_\Omega \lambda_1 u_{0\mu} e^{u_0}\, d\underline{x}$$

THUS $$\lambda_1 \int_\Omega u_{0\mu} e^{u_0} \, d\underline{x} = + \int_{\partial\Omega} H(\varphi) \, \partial_r u_{0\mu}\Big|_{r=1} d\underline{x} \quad \text{AT } \mu_0 = 1.$$

THUS $\lambda_c = 2 + \varepsilon \lambda_1$ IS THE FOLD POINT LOCATION.

NOW NOTICE THAT $$\int_\Omega u_{0\mu} e^{u_0} \, dx = -\frac{1}{\lambda_0} \int_\Omega \Delta u_{0\mu} \, d\underline{x} = -\frac{1}{\lambda_0} \int_{\partial\Omega} \partial_r u_{0\mu}\Big|_{r=1} d\underline{x}.$$

THUS $$-\frac{\lambda_1}{\lambda_0} \int_{\partial\Omega} \partial_r u_{0\mu}\Big|_{r=1} ds = + \int_{\partial\Omega} H(\varphi) \, \partial_r u_{0\mu}\Big|_{r=1} ds$$

THIS YIELDS,

$$-\frac{\lambda_1}{\lambda_0} \left(\partial_r u_{0\mu}\Big|_{r=1} \right) 2\pi = + \partial_r u_{0\mu}\Big|_{r=1} \int_0^{2\pi} H(\varphi) \, d\varphi$$

WE CONCLUDE THAT

$$\lambda_1(1) = -\lambda_0(1) \frac{1}{2\pi} \int_0^{2\pi} H(\varphi) \, d\varphi = \lambda_0(1) \bar{H}$$

IN SUMMARY, FOR $\varepsilon \ll 1$ WE HAVE

$$\lambda_c = 2 \left[1 - \varepsilon \bar{H}(\varphi) \right]$$

$$\bar{H}(\varphi) = \frac{1}{2\pi} \int_0^{2\pi} H(\varphi) \, d\varphi.$$