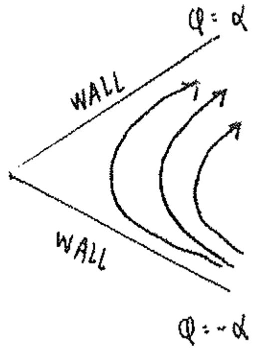
STOKES FLOW



IN 2-D THE NAVIER STOKES EQUATIONS ARE

$$\Delta^2 \psi + RE\, J_r[\psi, \Delta\psi] = 0 \qquad J_r(a,b) = \frac{1}{r}[\partial_r a\, \partial_\varphi b - \partial_\varphi a\, \partial_r b]$$

(LOOK FOR AN EVEN SOLUTION)

RE IS REYNOLDS NUMBER.
ψ STREAM FUNCTION

$$\psi = \psi_n = 0 \text{ ON body}$$

MOFFATT: J. FLUID MECHANICS 1-18, 1964
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WE LET $Re \ll 1$.

IN A SMALL NEIGHBOURHOOD OF THE ORIGIN, THE STREAM FUNCTION ψ SATISFIES

$$\Delta^2 \psi = 0, \qquad \text{ALSO} \quad \Delta\psi = -w, \quad \nabla^2 w = 0 \quad (w \text{ VORTICITY})$$

WITH BOUNDARY CONDITIONS

$\psi = 0$ AT $\varphi = \pm\alpha$, $\psi_\varphi = 0$ AT $\varphi = \pm\alpha$ (wall is a streamline and no flow through wall). THE FLUID VELOCITIES ARE $V_r = 1/r\, \partial\psi/\partial\varphi$, $V_\varphi = -\psi_r$.

NOW WE LET $\psi = r^\lambda f(\varphi)$ WHERE WE WANT $RE\,\lambda > 1$ FOR BOUNDED VELOCITIES.

NEED $\left(\partial_{rr} + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_{\varphi\varphi}\right)\left(\partial_{rr} + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_{\varphi\varphi}\right) r^\lambda f(\varphi) = 0 \longrightarrow \left(\partial_{rr} + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_{\varphi\varphi}\right)\left[r^{\lambda-2}(\lambda^2 f + f'')\right] = 0$

WE LET $g(\varphi) = f''(\varphi) + \lambda^2 f(\varphi)$. THEN, WE GET

$$\left(\partial_{rr} + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_{\varphi\varphi}\right) r^{\lambda-2} g(\varphi) = 0 \longrightarrow g''(\varphi) + (\lambda-2)^2 g(\varphi) = 0.$$

WE WANT AN EVEN SOLUTION TO CORRESPOND TO STREAMLINES SHOWN ABOVE

$$g(\varphi) = \tilde{A}\cos[(\lambda-2)\varphi]$$

THEN

$$f''(\varphi) + \lambda^2 f(\varphi) = \tilde{A}\cos[(\lambda-2)\varphi].$$

AN EVEN SOLUTION HAS THE FORM FOR $\lambda \neq 1, \neq 2$ AND $\lambda \neq 0$. NOTE $V_r \sim r^{\lambda-1} f'(\varphi)$ IS ODD IN φ.

$$f(\varphi) = A_0 \cos(\lambda\varphi) + C\cos[(\lambda-2)\varphi]$$

HENCE,

$$\psi \sim \sum_{n=1}^{\infty} r^{\lambda_n}\left[A_n \cos[\lambda_n\varphi] + C_n\cos[(\lambda_n-2)\varphi]\right]$$

IN THE CORNER. IF $\lambda = 1$ THEN $f(\varphi) = A\cos\varphi + C\varphi\sin\varphi$. THESE ARE IRRELEVANT TO THE PROBLEM SINCE WE WANT BOUNDED VELOCITIES, i.e. $\lambda > 1$.

WE REMARK THAT UPON COMBINING

$$(1) \qquad g''(\varphi) + (\lambda-2)^2 g(\varphi) = 0$$

WITH $g = f'' + \lambda^2 f$ WE OBTAIN THE <u>FOURTH-ORDER</u> EIGENVALUE PROBLEM

$$(*) \quad \begin{cases} f''''(\varphi) + \lambda^2 f''(\varphi) + (\lambda-2)^2 f''(\varphi) + \lambda^2(\lambda-2)^2 f(\varphi) = 0. \\ f(\alpha) = f'(\alpha) = 0, \quad f(-\alpha) = f'(-\alpha) = 0. \end{cases}$$

THIS IS NOT SELF-ADJOINT AND THERE IS NO GUARANTEE OF REAL EIGENVALUES FOR THIS PROBLEM.

WE SOLVE $(*)$ BY FIRST FINDING EVEN SOLUTIONS TO (1) FOR $g(\varphi)$ AND THEN FINDING SOLUTIONS FOR $f(\varphi)$.

THIS GAVE $\qquad f(\varphi) = A\cos(\lambda\varphi) + C\cos[(\lambda-2)\varphi]$

NOW WE SET $f(\alpha) = f'(\alpha) = 0$ (BY SYMMETRY IT FOLLOWS THAT $f(-\alpha) = f'(-\alpha) = 0$).

WE OBTAIN THAT

$$A\cos(\lambda\alpha) + C\cos[(\lambda-2)\alpha] = 0$$

$$A\lambda\sin(\lambda\alpha) + C(\lambda-2)\sin[(\lambda-2)\alpha] = 0$$

$\exists$ A NONTRIVIAL SOLUTION IFF λ SATISFIES THE EIGENVALUE EQUATION

$$(2) \qquad \det\begin{bmatrix} \cos\lambda\alpha & \cos((\lambda-2)\alpha) \\ \lambda\sin\lambda\alpha & (\lambda-2)\sin[(\lambda-2)\alpha] \end{bmatrix} = 0.$$

ONCE A ROOT TO (2) IS FOUND WE CAN SET $A = \underline{K}\cos[(\lambda-2)\alpha]$, $C = -\underline{K}\cos(\lambda\alpha)$, SO THAT

$$(3) \qquad \psi \sim \underline{K}\, r^{\lambda}\left[\cos(\lambda\varphi)\cos((\lambda-2)\alpha) - \cos(\lambda\alpha)\cos((\lambda-2)\varphi)\right]$$

NEXT, WE SIMPLIFY THE EIGENVALUE RELATION

$$(\lambda-2)\sin[(\lambda-2)\alpha]\cos(\lambda\alpha)=\lambda\cos((\lambda-2)\alpha)\sin(\lambda\alpha)$$

SUBTRACT AND ADD $2\cos[(\lambda-2)\alpha]\sin(\lambda\alpha)$ TO THE RIGHT SIDE TO OBTAIN

$$(\lambda-2)\sin[(\lambda-2)\alpha]\cos(\lambda\alpha)=(\lambda-2)\cos((\lambda-2)\alpha)\sin(\lambda\alpha)+2\cos((\lambda-2)\alpha)\sin(\lambda\alpha)$$

THIS YIELDS

$$(\lambda-2)\left[\sin((\lambda-2)\alpha)\cos(\lambda\alpha)-\cos((\lambda-2)\alpha)\sin(\lambda\alpha)\right]=2\cos((\lambda-2)\alpha)\sin(\lambda\alpha)$$

USING THE TRIG IDENTITY GIVES

$$-(\lambda-2)\sin(2\alpha)=2\cos[(\lambda-2)\alpha]\sin(\lambda\alpha)$$

NEXT USE $\sin A\cos B=\frac{1}{2}\sin(A-B)+\frac{1}{2}\sin(A+B)$ WITH $A=\lambda\alpha,\ B=(\lambda-2)\alpha$

TO GET

$$-(\lambda-2)\sin(2\alpha)=2\left(\frac{1}{2}\right)\left[\sin(2\alpha)+\sin(2(\lambda-1)\alpha)\right]$$

OR

$$-(\lambda-2)\sin(2\alpha)=\sin(2\alpha)+\sin[2(\lambda-1)\alpha]$$

THIS FINALLY YIELDS THAT λ SATISFIES

$$(4)\qquad -(\lambda-1)\sin(2\alpha)=\sin[2(\lambda-1)\alpha]$$

THUS UPON WRITING $\mu=\lambda-1$ WE OBTAIN THAT μ SATISFIES

$$(5)\qquad \sin(2\mu\alpha)=-\mu\sin(2\alpha)$$

IN TERMS OF μ WE HAVE FROM (3)

$$(6)\qquad \psi\sim \underline{K}\,r^{1+\mu}\left[\cos((1+\mu)\varphi)\cos((\mu-1)\alpha)-\cos((1+\mu)\alpha)\cos((\mu-1)\varphi)\right]$$

OUR GOAL IS FOR FIXED α IN $0<\alpha<\pi$ TO FIND THE SMALLEST ROOT OF (5) SATISFIES $\mu>0$ SO THAT $\psi \to 0$ AS $r\to 0$, AND VELOCITIES $\to 0$ AS $r\to 0$.

NOTICE THAT $\mu=0$ IS A ROOT $\forall\alpha$. THIS GIVES $\lambda=1$. HOWEVER, CAN WE FIND A ROOT TO (5) WITH μ REAL AND $\mu>0$

THUS WE SET $y=+2\mu\alpha$ TO OBTAIN

(7) $\sin y = -yB$ WITH $B=\dfrac{\sin(2\alpha)}{2\alpha}$

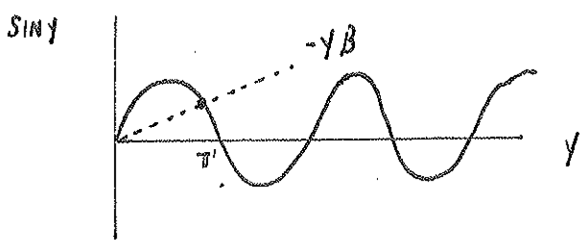
NOTICE THAT

$B>0$ ON $0<\alpha<\pi/2 \Rightarrow B\to 1$ AS $\alpha\to 0^+$. CASE I

$B<0$ ON $\frac{\pi}{2}<\alpha<\pi \Rightarrow B\to 0$ AS $\alpha\to\frac{\pi}{2}^+$ AND $\alpha\to\pi$.

NOTICE $|B|<1$. CASE II

II) NOW THIS GIVES THE PICTURE FOR $B<0$ i.e. $\pi/2<\alpha<\pi$. SINCE $-1<B<0$ WE HAVE



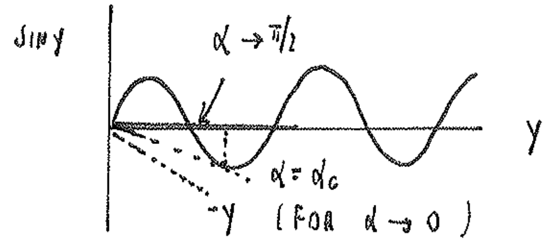
- THUS $\exists$ y_0 IN $0<y<\pi/2$ FOR WHICH THERE IS A ROOT. THIS YIELDS

$\mu=+y_0/2\alpha$ WITH $\mu>0$ AND $\frac{y_0}{2\alpha}<\frac{\pi}{\pi}<1$ SO $\mu<1$.

WE CONCLUDE THAT WHEN $\pi/2<\alpha<\pi$, $\exists$ ROOT μ_0 REAL TO (5) THAT SATISFIES $\mu_0<1$.

(I) NEXT, SUPPOSE $0<\alpha<\pi/2$ SO THAT $B>0$ WITH $0<B<1$.

THEN WE HAVE



WE CONCLUDE THAT:

(i) IF $\alpha < \alpha_c$ THERE ARE NO SOLUTIONS WITH y REAL TO $\sin y = -By$ AND HENCE NO REAL μ. HERE α_c SATISFIES THE TANGENCY PROBLEM

$$\sin y = -yB$$

$$\cos y = -B$$

WHICH YIELDS $\tan y = y$ IN $\pi/2 < y < 3\pi/2$ WHERE $\sin y < 0$, $\cos y < 0$.

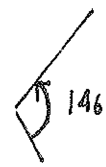
WE CALCULATE $y_0 \cong 4.4934$ BY NEWTON SO $\cos y_0 = -.217233$.

THEN $2\alpha_c$ SATISFIES

$$\frac{\sin[2\alpha_c]}{2\alpha_c} = -\cos(y_0) = .217233$$

WHICH YIELDS $2\alpha_c = 2.5535$ (RADIANS)

OR (8) $\alpha_c = \dfrac{2.5535}{2}\left(\dfrac{180}{\pi}\right) = 73.15^\circ$.



$146.3^\circ = 2\alpha_c$

WE CONCLUDE THAT IF $\alpha < \alpha_c$, THEN NO REAL μ EXIST.

REMARK THERE ARE REAL SOLUTIONS IF $\alpha > \alpha_c = 73.15^\circ$ WITH A $\mu < 0$. AND WE ARE LOOKING FOR A SOLUTION WITH $\mu > 0$.

WE CALCULATE $\mu = \dfrac{y_0}{2\alpha}$ FOR $\alpha > \alpha_c$.

AT $\alpha = \alpha_c$, $\mu_c = \dfrac{4.4934}{2.5535} \cong 1.759$.

NOTICE AS $\alpha \to \pi/2$, $\mu \to \dfrac{\pi}{\pi} = 1$.

NOTICE THAT AS $\alpha \to \alpha_c^+$, TWO NEARBY REAL ROOTS BECOME COMPLEX AND DISAPPEAR INTO COMPLEX PLANE FOR $\alpha < \alpha_c$.

NOW LOOK FOR COMPLEX ROOTS μ OF

$$\sin(2\mu\alpha) = -\mu \sin(2\alpha)$$

WHEN $\alpha < \alpha_c$. WE WANT TO FIND THE ROOT μ^* WITH THE SMALLEST $Re(\mu^*)$ IN $Re(\mu^*) > 0$.

WE LET $\mu = p + iq$ AND LET $\xi = 2\alpha p$, $\eta = 2\alpha q$.

$$\sin[\xi + i\eta] = -(p + iq)\sin(2\alpha)$$

THUS,

$$\sin(\xi + i\eta) = -(\xi + i\eta) K \qquad K \equiv \frac{\sin(2\alpha)}{2\alpha}.$$

WE USE $\sin(\xi + i\eta) = \sin\xi \cosh\eta + i\cos\xi \sinh\eta$.

THIS GIVES THAT ξ, η SATISFY THE COUPLED SYSTEM IN $\xi > 0$

$$(9) \quad \begin{cases} \sin\xi \cosh\eta = -K\xi \\ \cos\xi \sinh\eta = -K\eta \end{cases} \qquad K = \frac{\sin(2\alpha)}{2\alpha}, \quad 0 < 2\alpha < 146.3^\circ$$

NOTICE $K > 0$ SINCE $2\alpha < \pi$.

WLOG WE TAKE $\eta > 0$.

WE SEEK THE ROOT OF (9) FOR WHICH $\xi > 0$ IS THE SMALLEST.

IN TERMS OF THE ROOTS ξ_m, η_m, $m = 1, 2, 3, \ldots$ WE HAVE

$$(10) \qquad \lambda_m = 1 + \frac{1}{2\alpha}(\xi_m + i\eta_m), \qquad m = 1, 2, 3, 4, \ldots$$

THE DOMINANT BEHAVIOR OCCURS FOR $m = 1$ WHICH GIVES THE ROOT FOR WHICH $Re(\lambda_m)$ IS <u>SMALLEST</u>.

CLEARLY, ANY ROOT TO (9) IN $\eta > 0$ MUST OCCUR WHEN $\sin\xi < 0$ AND $\cos\xi < 0$. THUS WE HAVE

$$(2m-1)\pi < \xi_m < (2m - \tfrac{1}{2})\pi.$$

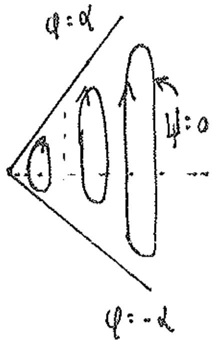
IN PARTICULAR, WE COMPUTE NUMERICALLY THAT:

2α	ξ_1	η_1	RE(λ_1)
90°	4.30	1.77	$1 + \frac{\xi_1}{2\alpha} = 2.737$
70°	4.26	1.97	3.519
140°	4.46	0.64	1.759

$p = \xi_1/2\alpha$, $q = \eta_1/2\alpha$

THE <u>KEY ISSUE</u> IS WHAT DOES COMPLEX λ SIGNIFY?

IT INDICATES ZONES OF RECIRCULATION OR EDDIES IN THE LOCAL FLOWFIELD. INFINITE SEQUENCE OF CLOSED STREAMLINE EDDIES IN ANY WEDGE WITH ANGLE $0 < 2\alpha < 146^\circ$.



WE WRITE $\psi \sim K r^{\lambda} [g(\varphi, \lambda)]$

WITH $g(\varphi,\lambda) = \cos\lambda\varphi \cos[(\lambda-2)\alpha] - \cos(\lambda\alpha)\cos((\lambda-2)\varphi)$

FOR $\lambda = 1 + p_1 + i q_1$ THEN $r^{1+p_1+iq_1} = r^{1+p_1}[\cos(q_1 \ln r) + i\sin(q_1 \ln r)]$

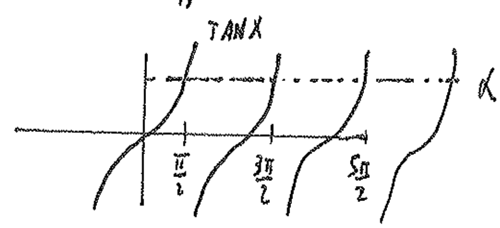
WE DECOMPOSE $g(\varphi, 1+p_1+iq_1) = g_R(\varphi) + i g_I(\varphi)$

THEN $\psi_R = \mathrm{RE}[\psi] \simeq K r^{1+p_1}[\cos(q_1 \ln r) g_R(\varphi) - \sin(q_1 \ln r) g_I(\varphi)]$ (11)

WLOG WE TAKE $q_1 > 0$.

THIS FUNCTION $\psi_R(r,\varphi)$ HAS FOR EACH φ IN $|\varphi| < 2\alpha$ AN INFINITE NUMBER OF ZERO CROSSINGS AS $r \to 0$. THEY ARE AT ROOTS r_n OF

$$\tan(-q_1 \ln r) = -\frac{g_I(\varphi)}{g_R(\varphi)}$$



LET $X = -q_1 \ln r \to +\infty$ AS $r \to 0^+$. $\tan X = \alpha$

SO $$-q_1 \ln r_n \longrightarrow \tan^{-1}\left[-\frac{g_I}{g_R}\right] + \left(n + \frac{1}{2}\right)\pi \qquad n = 0, 1, 2, \ldots$$

THUS WE HAVE $\psi_R = 0$ AT THESE INFINITE # POINTS r_n WITH $r_n \to 0$ AS $n \to \infty$.

THE FUNCTION ψ_R OSCILLATES BETWEEN THESE ZEROES.

REMARK TO EXAMINE THIS FURTHER LET'S ANALYZE THE TRANSVERSE VELOCITY COMPONENT ALONG THE MIDLINE $\varphi = 0$.

$$V_\varphi\big|_{\varphi=0} = -\frac{\partial \psi_R}{\partial r}\bigg|_{\varphi=0}.$$

NOW $$\frac{\partial \psi}{\partial r}\bigg|_{\varphi=0} = \lambda K_1\, r^{\lambda-1}\, g(0,\lambda) = r^{p_1+iq_1}\, \tilde{g}(0,\lambda) \qquad \hat{g}(0,\lambda) = \lambda K_1\, g(0,\lambda)$$

WE OBTAIN THE REAL PART

$$RE\left[\frac{\partial \psi}{\partial r}\bigg|_{\varphi=0}\right] = r^{p_1}\left[\cos(q_1 \ln r) + i\sin(q_1 \ln r)\right]\left[\hat{g}_R + i\hat{g}_I\right]$$

$$= r^{p_1}\left[\cos(q_1 \ln r)\,\hat{g}_R - \sin(q_1 \ln r)\,\hat{g}_I\right]$$

$$= \frac{r^{p_1}}{(\hat{g}_R^2 + \tilde{g}_I^2)} \sin\left[q_1 \ln r + B\right] \qquad \text{FOR SOME } B.$$

THUS $$V_\varphi\big|_{\varphi=0} = C\, r^{p_1} \sin\left[q_1 \ln r + B\right] \qquad \text{FOR SOME } B, C.$$

NOTICE $V_\varphi\big|_{\varphi=0} = 0$ WHEN $q_1 \ln r + B = -n\pi \quad n = 0,1,2,\dots$

$$r_n = e^{-(n\pi+B)/q_1}$$

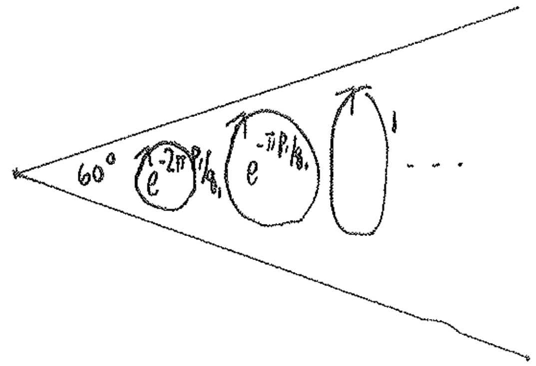
WITH $$r_{n+1}/r_n = e^{-\pi/q_1}$$

SO $r_n \rightarrow 0$ AS $n \rightarrow \infty$ geometric PROGRESSION.

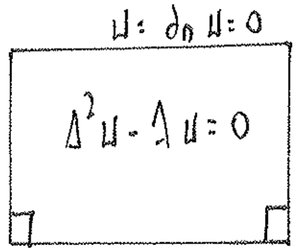
THESE POINTS ARE AT THE CENTERS OF THE EDDIES. THE MAXIMA OF $V_\varphi\big|_{\varphi=0}$ OCCUR AT $q_1 \ln r + B = -(2m+1)\pi/2 \quad m = 0,1,2,\dots \quad r_m = e^{-((2m+1)\pi/2 + B)/q_1}$

$$\frac{\left[(V_\varphi)\big|_{\varphi=0}\right]_{m+1}}{\left[(V_\varphi)\big|_{\varphi=0}\right]_m} = \frac{(r_{m+1})^P}{(r_m)^P} = e^{-\pi P_1/q_1}.$$

THUS THE INTENSITY OF THE EDDIES DECREASE EXPONENTIALLY AS $r \rightarrow 0$.



REMARK FROM A COMPLETELY DIFFERENT APPLICATION WE CAN USE THIS RESULT TO PROVE THAT THE FUNDAMENTAL MODE OF VIBRATION OF A PLATE IN A SQUARE DOMAIN IS NOT OF ONE SIGN, WHEN THE BOUNDARIES ARE CLAMPED.

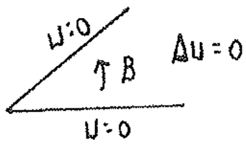


WE HAVE A 90° CORNER SO THAT $2\alpha < 146.3^\circ \Rightarrow$ MOFFAT EDDIES EXIST IN EVERY CORNER.

LAPLACE'S EQUATION: CORNER SINGULARITIES

TWO-DIMENSIONS

FOR LAPLACE'S EQUATION IN A WEDGE WE HAVE



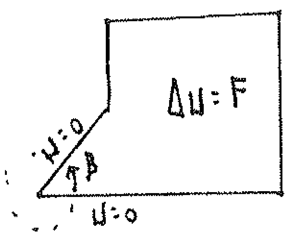
WE LOOK FOR $u = r^{\lambda} g(\varphi)$ AND TRY TO CALCULATE THE SMALLEST VALUE OF λ WITH $\lambda > 0$. WE NEED u BOUNDED IN THE CORNER

WE SUBSTITUTE TO GET

$$\left.\begin{aligned} g''(\varphi) + \lambda^2 g(\varphi) = 0 \\ g(0) = g(B) = 0 \end{aligned}\right\} \Longrightarrow \quad g = K \sin\left(\pi \varphi / B\right) \qquad \lambda = \frac{\pi}{B} > 0$$

is smallest root.

HENCE FOR A PROBLEM OF THE FORM



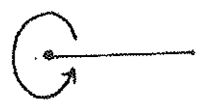
WHERE F IS SMOOTH, WE EXPECT THAT

$$u \sim K r^{\pi/B} \sin\left(\pi\varphi/B\right) + o\left(r^{\pi/B}\right)$$

AS $r \to 0$. (CORNER REGION)

CALCULATING K IS A GLOBAL PROBLEM AND CANNOT IN GENERAL BE DONE.

- IF $B = 2\pi$

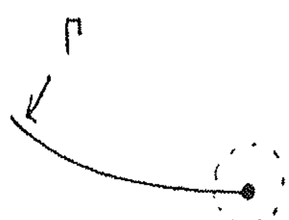


$$u \sim K r^{1/2} \sin\left(\varphi/2\right)$$

u_r is infinite as $r \to 0$.

- NOTE u_r is infinite whenever $B > \pi$.

NOW THE QUASI-STEADY PROPAGATION OF A CRACK IS GOVERNED BY THE FOLLOWING PROBLEM



$$\Delta^2 \psi = 0 \quad \text{IN} \quad \Omega \setminus \Gamma \qquad \Omega \in \mathbb{R}^2$$

$$\psi = \psi_n = 0 \quad \text{ON BOTH SIDES OF } \Gamma.$$

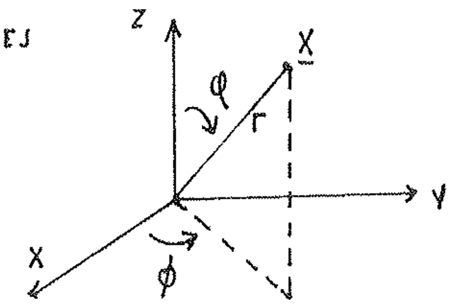
WHAT IS THE SINGULARITY IN THE CORNER REGION? TO LEADING ORDER

$$\psi \sim A_1 r^{3/2} B_1(\varphi) + A_2 r^{3/2} B_2(\varphi)$$

A_1, A_2 stress intensity FACTORS.

CONSIDER <u>LAPLACE'S EQUATION IN 3-DIMENSIONS:</u>

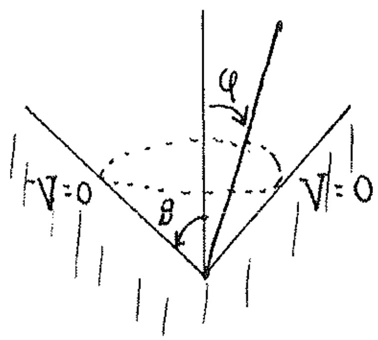
TRODUCE SPHERICAL COORDINATES



THEN LAPLACE'S EQUATION $\Delta V = 0$ IS

$$V_{rr} + \frac{2}{r} V_r + \frac{1}{r^2 \sin\varphi} \frac{\partial}{\partial\varphi}\left(\sin\varphi \frac{\partial V}{\partial\varphi}\right) + \frac{1}{r^2 \sin^2\varphi} \frac{\partial^2 V}{\partial\phi^2} = 0$$

OW SUPPOSE WE HAVE AZIMUTHAL SYMMETRY SO THAT $V(r, \varphi)$



I.e. $0 \le \varphi \le B$

WE WRITE $V = A r^{\mu} Q(\varphi)$

SUBSTITUTING WE GET

$$\left[\mu(\mu-1) + 2\mu\right] Q + \frac{1}{\sin\varphi} \frac{\partial}{\partial\varphi}\left(\sin\varphi \frac{\partial Q}{\partial\varphi}\right) = 0$$

THEREFORE $$\frac{1}{\sin\varphi} \frac{d}{d\varphi}\left[\sin\varphi \frac{dQ}{d\varphi}\right] + \mu(\mu+1) Q = 0 \qquad (*) \text{ (SEE BELOW)}$$

NOW LET $x = \cos\varphi$. THEN THE EQUATION TRANSFORMS TO LEGENDRE'S

QUATION

$$(1) \qquad \frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right] + \mu(\mu+1) P = 0. \qquad \cos B < x < 1.$$

WE DENOTE THE SOLUTION BY $P_{\mu}(x)$.

HENCE $$V \sim A r^{\mu} P_{\mu}(\cos\varphi)$$

where μ is the smallest value with $\mu > 0$ SUCH THAT THE SOLUTION

$$P_\mu(\cos B) = 0.$$

THIS IS THE EIGENVALUE RELATION FOR μ IN TERMS OF B.

NOW NOTICE THAT $0 < B < \pi$.

SUPPOSE THAT $0 \leq \theta \leq B$ THEN $\cos B \leq X \leq 1$. THE PROBLEM (1) IS THAT

$$(*)\quad \begin{cases} \dfrac{d}{dx}\left[(1-x^2)\dfrac{dP}{dx}\right] + \mu(\mu+1)P = 0 & \cos B \leq X \leq 1 \\ P \text{ is finite at } X = 1 \\ P(\cos B) = 0. \end{cases}$$

THIS IS AN EIGENVALUE PROBLEM FOR $\mu = \mu(B)$.

REMARKS

1. NOTICE THAT $X=1$ IS A SINGULAR POINT. ONE SOLUTION IS singular NEAR $X=1$ THE OTHER is FINITE AS $X \to 1$.

2. CONSIDER
$$(1-x)(1+x)P'' - 2xP' + \mu(\mu+1)P = 0.$$

let $\xi = x-1$ WITH ξ SMALL.

$$-2\xi P'' - 2P' + \mu(\mu+1)P \approx 0$$

$$\xi^2 P'' + \xi P' - \frac{\mu}{2}(\mu+1)\xi P = 0$$

let $P = \xi^q \;\rightarrow\; q(q-1) + q = 0 \;\rightarrow\; q = 0$ repeated ROOT.

thus one solution is regular at the origin, the other diverges logarithmically.

3. IF $\mu = N$ N non-negative integer, then $P_N(X)$ is A POLYNOMIAL.

$P_0(X) = 1$ $\qquad P_2(X) = \frac{1}{2}(3X^2-1)$

$P_1(X) = X$ $\qquad \vdots$

THIS CORRESPONDS TO WHERE WE REPLACE $P(\cos B) = 0$ BY P FINITE AT $X=-1$.

RECURRENCE RELATION

WE WRITE $\left[(1-x^2)y'\right]' + \mu(\mu+1)y = 0 \qquad \cos\beta < x < 1.$

$y(1)$ FINITE $\quad y(a) = 0 \quad$ WITH $\quad a = \cos\beta$

WE LET $t = x-1$ SO THAT

$$\left[(t^2+2t)y'\right]' - \mu(\mu+1)\,y = 0$$

WE EXPAND $\quad y = \sum_{K=0}^{\infty} a_K\, t^K$

WE SUBSTITUTE TO FIND THE RECURRENCE RELATION

$$a_K = \frac{(\mu-(K-1))(\mu+K)}{2K^2}\, a_{K-1}, \qquad K \geq 1$$

THIS YIELDS WITH $a_0 = 1$ THAT

$$a_K = \frac{(\mu+K)(\mu+K-1)\cdots(\mu+2)(\mu+1)\,\mu\,(\mu-1)\cdots(\mu-(K-1))}{2^K (K!)^2}$$

IN THIS WAY WE DEFINE

$$P_\mu(x) = \sum_{K=0}^{\infty} \frac{(\mu+K)(\mu+K-1)\cdots(\mu+1)\,\mu\,(\mu-1)\cdots(\mu-(K-1))}{(2^K)(K!)^2}\,(x-1)^K$$

WHICH IS THE LEGENDRE FUNCTION OF THE FIRST KIND OF ORDER μ.

NOTICE $\quad a_1 = \dfrac{\mu(\mu+1)}{2(1)^2} \qquad a_2 = \dfrac{\mu(\mu+1)(\mu-1)(\mu+2)}{(2(1)^2)(2(2)^2)}$

THIS SERIES CONVERGES FOR $|x-1| < 1$, i.e. ON THE RANGE $\cos\beta < x < 1$.

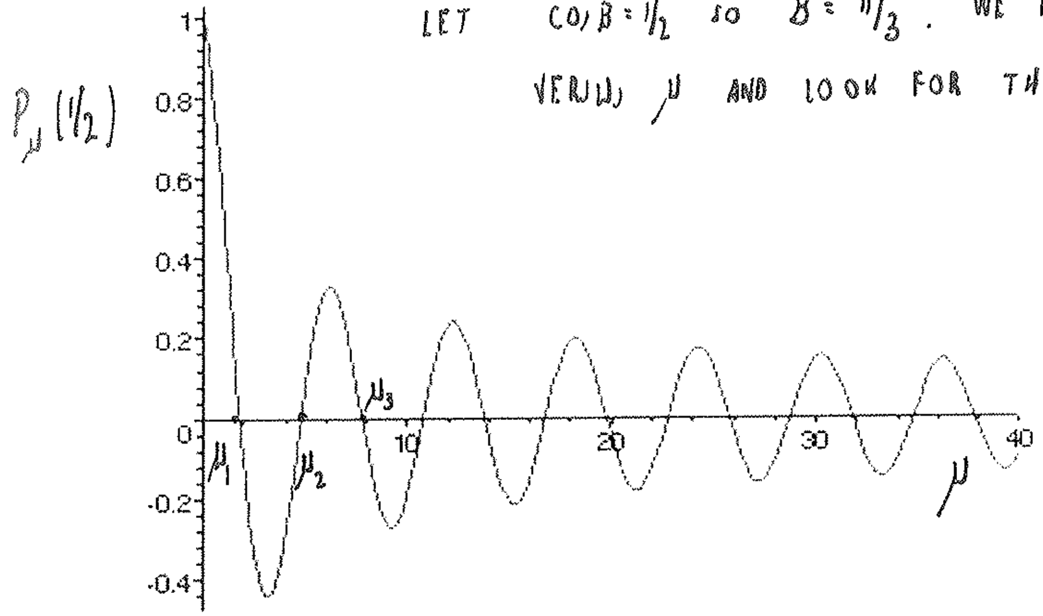
THUS, WE DEFINE $\quad P_\mu(x) = \sum_{K=0}^{\infty} a_K(\mu)\,(x-1)^K \quad$ FOR SOME $a_K = a_K(\mu)$ WITH $a_0 = 1$.

THE SOLUTION WITH $y(1) = 1$ IS $y = P_\mu(x)$.

NOW LET'S SUPPOSE FOR EXAMPLE THAT $\cos\beta = 1/2$, SO THAT $\beta = \pi/3$.

THEN WE USE RECURRENCE RELATION OR MATLAB'S BUILT-IN FUNCTION TO GET A PLOT OF $P_\mu(1/2)$ VERSUS μ.

LET $\cos\beta = 1/2$ SO $\beta = \pi/3$. WE PLOT $P_\mu(1/2)$ VERSUS μ AND LOOK FOR THE FIRST ZERO.



$$\mu_1 \cong 1.777, \qquad \mu_2 = 4.762$$

NOW FROM TABLES OF SPECIAL FUNCTIONS, FOR SOME $A > 0$

$$P_\mu(\cos\varphi) \sim A J_0\left[(2\mu+1)\sin\varphi/2\right] \quad \text{AS} \quad \mu \to \infty.$$

NOW IF WE HAVE $\mu \to \infty$ THEN $(2\mu+1)\sin\beta/2 \sim Z_0$ WITH $J_0(Z_0) = 0$ AND Z_0 SMALLEST ROOT OF J_0. THUS FOR $\mu \to \infty$, WHICH CORRESPONDS TO $\beta \to 0$, WE HAVE USING $\sin\beta \sim \beta$ THAT $(2\mu+1)\beta/2 \sim Z_0$ OR

$$\mu \sim Z_0/\beta - 1/2 \qquad \text{WE KNOW} \quad Z_0 \cong 2.4048$$

THE SECOND RELATION FROM ABRAMOWITZ-STEGUN P. 335 FORMULA (8.6.20)

IS THAT

$$P_\mu(\cos\varphi) \sim P_0(\cos\varphi) + 2\log\left[\cos(\varphi/2)\right]\mu \quad \text{AS} \quad \mu \to 0.$$

(SEE REMARK AT END OF THE NOTES)

NOW SINCE $P_0(\cos\varphi) = 1$ WE LET $B \to \pi^-$ TO OBTAIN

$$\cos(\varphi/2) \sim \cos(\pi/2) - \tfrac{1}{2}\sin(\pi/2)(B - \pi) + \dots = -\tfrac{1}{2}(B - \pi) \quad \text{AS} \quad B \to \pi^-.$$

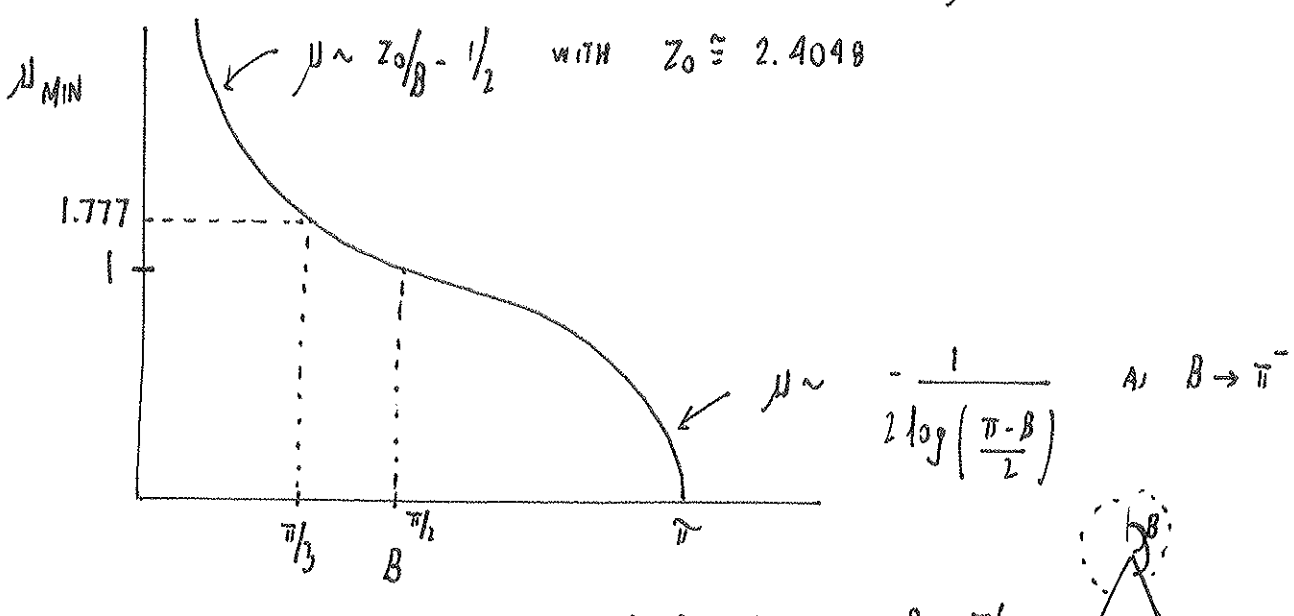
THUS FOR $B \to \pi^-$, $\quad P_\mu(\cos B) \sim 1 + 2\mu\log\left[\dfrac{\pi - B}{2}\right]$ AS $\mu \to 0$.

SETTING $P_\mu(\cos B) = 0$ YIELDS

$$\mu \sim -\frac{1}{2\log\left[\frac{(\pi - B)}{2}\right]} \qquad \text{FOR} \quad B \to \pi^-.$$

USING $B = 170^\circ$ WE CALCULATE $\mu \cong 0.2$.

THIS GIVES THE FOLLOWING CURVE AFTER COMPUTING $P_\mu(\cos B) = 0$ BY MAPLE.



REMARK A RE-ENTRANT CORNER IS ONE FOR WHICH $B > \pi/2$.

A LIGHTNING ROD WOULD HAVE $B \to \pi^-$. WE HAVE

$$V \sim A r^\mu P_\mu(\cos\varphi) \quad \text{SO} \quad V_r \sim A r^{\mu - 1} P_\mu(\cos\varphi).$$

AS $B \to \pi^-$ THEN $\mu \to 0$. FOR $B > \pi/2$ WE HAVE $\mu < 1$ SO $V_r \to +\infty$ AS $r \to 0$.

FINALLY WE CALCULATE ELECTRIC FIELDS

$$E_r = -\frac{\partial V}{\partial r} \qquad E_\varphi = -\frac{1}{r}\frac{\partial V}{\partial \varphi}$$

ITH $V = A r^{\mu} P_\mu(\cos\varphi)$

$$E_r \sim -\mu A r^{\mu-1} P_\mu(\cos\varphi)$$

$$E_\varphi \sim A r^{\mu-1} \sin\varphi \; P_\mu'(\cos\varphi)$$

E_r AND E_φ behave like $r^{\mu-1}$ NEAR the tip.

HENCE, IF $0 < \mu < 1 \longrightarrow \beta > \pi/2$ we have very strong field NEAR THE CORNER.

<u>REMARK (*)</u> $Q'' + \cot\varphi \; Q' + \mu(\mu+1) Q = 0$ WITH $Q = Q(\varphi)$.

LET $x = \cos\varphi$ AND $\dot{Q} = dQ/dx$.

THEN $dQ/d\varphi = \dot{Q}(-\sin\varphi)$

$$d^2Q/d\varphi^2 = \ddot{Q}\sin^2\varphi - \cos\varphi\,\dot{Q}$$

SO $\ddot{Q}\sin^2\varphi - \cos\varphi\,\dot{Q} + \cot\varphi(-\sin\varphi)\dot{Q} + \mu(\mu+1)Q = 0$

YIELDS, $\ddot{Q}\sin^2\varphi - 2\cos\varphi\,\dot{Q} + \mu(\mu+1)Q = 0$

$$\ddot{Q}(1-x^2) - 2x\dot{Q} + \mu(\mu+1)Q = 0$$

SO $\left[(1-x^2)\frac{dQ}{dx}\right] + \mu(\mu+1)Q = 0.$

REMARK THE LEGENDRE FUNCTION $y = P_\mu(x)$ IS THE SOLUTION TO

$$[(1-x^2)y']' + \mu(\mu+1)y = 0, \quad x < 1$$

THAT SATISFIES $y(1) = 1$ (NORMALIZATION CONDITION).

FOR $\mu \to 0$ WE USED $P_\mu(\cos\varphi) \sim 1 + 2\mu \log[\cos(\varphi/2)]$ AS $\mu \to 0$ WHEN $0 < \varphi < \pi$. THIS FORMULA FROM ABRAMOWITZ AND STEGUN CAN BE DERIVED DIRECTLY.

WE EXPAND $y = 1 + \mu y_1 + \dots$ FOR $\mu \ll 1$.

UPON SUBSTITUTING AND KEEPING $O(\mu)$ TERMS,

$$[(1-x^2)y_1']' = -1$$

$$y_1(1) = 0.$$

WE INTEGRATE $(1-x^2)y_1' = -x + C$

TO ELIMINATE THE SINGULARITY AT $x = 1$ SET $C = 1$

THUS $y_1' = \dfrac{x-1}{x^2-1} = \dfrac{1}{x+1}$

SO $y_1 = \log(x+1) + B$.

PUT $y_1(1) = 0 \to B = -\log 2$. THUS FOR $\mu \ll 1$,

$$y \sim 1 + \mu \log\left(\frac{x+1}{2}\right) + O(\mu^2) \text{ ON } -1 < x < 1$$

NOW THIS GIVES $P_\mu(x) \sim 1 + \mu \log\left(\dfrac{x+1}{2}\right)$ AS $\mu \to 0$.

BUT $x = \cos\varphi = \cos^2(\varphi/2) - \sin^2\varphi/2 = 2\cos^2(\varphi/2) - 1$. SO $\dfrac{x+1}{2} = \cos^2(\varphi/2)$

UPON SUBSTITUTING WE GET

$$P_\mu(\cos\varphi) \sim 1 + 2\mu \log(\cos(\varphi/2)) \text{ AS } \mu \to 0.$$

REMARK THE LEGENDRE FUNCTION CAN ALSO BE EXPRESSED AS AN INTEGRAL.

IN FACT, THE MEHLER-DIRICHLET FORMULA GIVES THE INTEGRAL REPRESENTATION

$$P_\mu(\cos\varphi) = \frac{\sqrt{2}}{\pi}\int_0^{\varphi} \frac{\cos\left[(\mu + 1/2)t\right]}{\sqrt{\cos t - \cos\varphi}}\, dt$$

BY USING A STATIONARY PHASE APPROXIMATION WE CAN CALCULATE $P_\mu(\cos\varphi)$ AS $\mu \to +\infty$. THIS INTEGRAL RELATION CAN BE USED FOR COMPUTATIONS.

THE FULL NUMERICS (HEAVY SOLID CURVE), THE SMALL μ APPROXIMATION $\mu \sim -1/2\log\left(\frac{\pi - \beta}{2}\right)$ AS $\beta \to \pi^-$ (SOLID CURVE), AND THE LARGE μ APPROXIMATION $\mu \sim z_0/\beta - 1/2$ WITH $z_0 = 2.4048$ (DOTTED CURVE) ARE SHOWN BELOW.