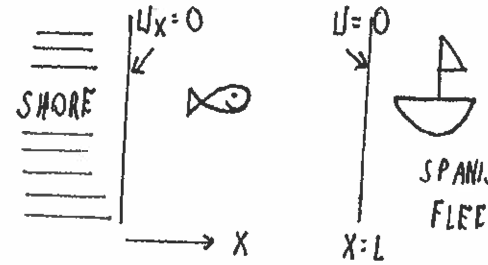
APPLICATION OF EIGENVALUE EXPANSIONS AND SOLVABILITY CONDITIONS

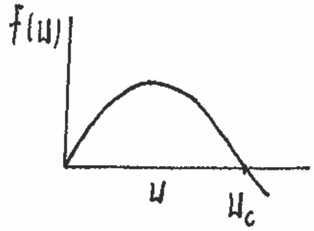
WE CONSIDER THE FISHING PROBLEM FOR THE DENSITY $U(x,t)$ OF FISH

$$U_t = \nu^2 U_{xx} + \Gamma f(U) \qquad 0 < x < L, \quad t > 0$$

$$U_x(0,t) = 0, \quad U(L,t) = 0 \qquad \Gamma > 0$$



WE TAKE THE FUNCTION $f(U)$ TO BE OF LOGISTIC TYPE



$$f(0) = f(U_c) = 0, \qquad f'(0) = 1$$

$$\text{SO} \quad f(U) = U\left(1 - U/U_c\right)$$

QUESTIONS (i) DO THERE EXIST A NON-TRIVIAL EQUILIBRIUM SOLUTION? IS THIS SOLUTION STABLE? HOW SHOULD ONE CHOOSE L SO THAT $U = 0$ IS NOT THE ONLY STABLE EQUILIBRIUM SOLUTION?

(ii) WHAT IS THE BIFURCATION DIAGRAM AND THE STABILITY OF THE VARIOUS BRANCHES OF SOLUTIONS.

NON-DIMENSIONALIZE WE LET $y = x/L$, $t = \omega\tau$, $U = U_c \mathcal{U}$.

THIS YIELDS,

$$\frac{1}{\omega}\mathcal{U}_\tau = \frac{\nu^2}{L^2}\mathcal{U}_{yy} + \Gamma\,\mathcal{U}(1-\mathcal{U})$$

WE THEN LET $\omega = 1/\Gamma$ AND DEFINE $D = \dfrac{\nu^2}{L^2\Gamma}$.

$\nu \ll 1$, OR $L \gg 1$, OR $\Gamma \gg 1 \longrightarrow D \ll 1$

$\nu \gg 1$, OR $L \ll 1$, OR $\Gamma \ll 1 \longrightarrow D \gg 1$.

THEN WE HAVE,

$$\mathcal{U}_\tau = D\,\mathcal{U}_{yy} + \mathcal{U}(1-\mathcal{U}).$$

NOW IF WE RELABEL WE GET

$$U_t = D\,U_{xx} + f(U) \qquad 0 < x < 1 \qquad f(U) = U(1-U).$$

$$U_x = 0 \ \text{ON}\ x = 0, \qquad U = 0 \ \text{ON}\ x = 1$$

WITH SOME $U(x,0) = \phi_0(x)$

EQUILIBRIUM PROBLEM

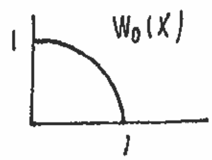
WE LET $\alpha = 1/\sqrt{D} = \sqrt{r}\,L/_{V}$, SO THAT THE EQUILIBRIUM PROBLEM IS

$$W_{XX} + \alpha^2 W(1-W) = 0, \quad 0<X<1$$
$$W_X(0)=0, \quad W(1)=0.$$

$W(X) \geq 0$ NO NEGATIVE FISH CONCENTRATIONS.

IF WE LINEARIZE THEN $W_{0XX} + \alpha^2 W_0 = 0$ SO THAT $W_0 = \cos(\alpha X)$ WITH $W_0(1)=0$, THEN $\alpha X = \pi/2$ AT $X=1 \longrightarrow \alpha = \pi/2$.

THUS THE BIFURCATION POINT IS AT $\alpha = \pi/2$ AND $W_0 = \cos(\pi X/2)$ HAS THE SHAPE



NOW WE PERFORM A WEAKLY NONLINEAR ANALYSIS TO CONSTRUCT THE BIFURCATING SOLUTION BRANCH.

WE LET $\alpha = \frac{\pi}{2}(1+\alpha_1 \varepsilon + \dots)$ $\quad W = \varepsilon W_0 + \varepsilon^2 W_1 + \dots$

THEN
$$\alpha^2 f(w) = \frac{\pi^2}{4}(1+2\alpha_1\varepsilon+\dots)\left[(\varepsilon W_0 + \varepsilon^2 W_1) f'(0) + \frac{\varepsilon^2 W_0^2}{2} f''(0) + \dots\right]$$
$$= \frac{\pi^2}{4}\left[\varepsilon W_0 f'(0) + \varepsilon^2\left(2\alpha_1 W_0 f'(0) + W_1 f'(0) + \frac{W_0^2}{2} f''(0)\right)\right]$$

THIS YIELDS THE FOLLOWING: (WITH $f'(0)=1$, $f''(0)=-2$)

(1) $\left\{ \begin{array}{l} W_{0XX} + \frac{\pi^2}{4} W_0 = 0 \\ W_{0X}(0)=0, \quad W_0(1)=0 \end{array} \right.$ $\qquad \mathcal{L} W_0 \equiv W_{0XX} + \frac{\pi^2}{4} W_0$

(2) $\left\{ \begin{array}{l} W_{1XX} + \frac{\pi^2}{4} W_1 = -\frac{\pi^2}{4}\left(2\alpha_1 W_0 - W_0^2\right) \\ W_{1X}(0)=0, \quad W_1(1)=0 \end{array} \right.$

THE SOLUTION TO (1) IS $W_0 = A\cos(\pi X/2)$.

THEN $$\mathcal{L} W_1 = \frac{\pi^2}{4}\left(A^2\cos^2(\pi x/2) - A\,2\alpha_1\cos(\pi x/2)\right)$$

SINCE $\mathcal{L} W_0 = 0$, THE SOLVABILITY CONDITION YIELDS

$$(W_0, \mathcal{L} W_1) - (W_1, \mathcal{L} W_0) = (W_0 W_{1x} - W_1 W_{0x})\Big|_0^1 = 0.$$

THIS YIELDS THE FOLLOWING: $(W_0, \mathcal{L} W_1) = 0$

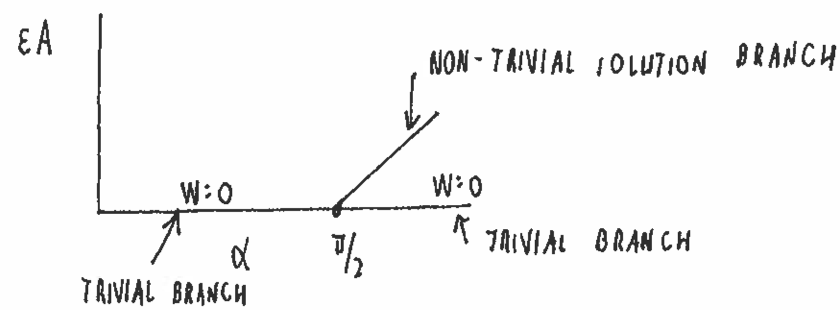
$$\longrightarrow \quad A^2 \int_0^1 \cos^3(\pi x/2)\,dx = 2\alpha_1 A \int_0^1 \cos^2(\pi x/2)\,dx$$

OR $$\alpha_1 = \frac{A}{2}\,\frac{\int_0^1 \cos^3(\pi x/2)\,dx}{\int_0^1 \cos^2(\pi x/2)\,dx} = A\left(\frac{4}{3\pi}\right)$$

HENCE $$\left.\begin{aligned} \alpha &\sim \frac{\pi}{2}\left(1 + \frac{4}{3\pi}(\varepsilon A) + \cdots\right) \\ W &\sim (\varepsilon A)\cos(\pi x/2) + \cdots \end{aligned}\right\} \Longrightarrow \alpha \sim \frac{\pi}{2}\left(1 + \frac{4}{3\pi}W(0) + \cdots\right)$$

WHEN $W(0) \ll 1$.

WE REQUIRE $\varepsilon A > 0$ FOR $W \geq 0$ AND SO FOR $\varepsilon A \ll 1$



NOW WE DETERMINE LARGE AMPLITUDE EQUILIBRIUM SOLUTIONS.

WE WRITE $W_{xx} + \alpha^2 f(W) = 0$, $W_x(0) = 0$, $W(1) = 0$.

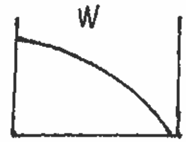
WE MULTIPLY BY W_x AND INTEGRATE

$$\frac{1}{2}W_x^2 + \frac{\alpha^2}{2}F(W) = \text{CONSTANT} \qquad F(W) = 2\int_0^W f(\Lambda)\,d\Lambda.$$

NOW LABEL W_0 BY $W(0) = W_0$. THEN WE HAVE

$$W_x^2 + \alpha^2\left[F(W) - F(W_0)\right] = 0.$$

WE WANT A PICTURE LIKE



. HENCE $W_x < 0$ AND $W(x) < W_0$.

THIS GIVES $W_x = -\alpha \left[F(W_0) - F(W) \right]^{1/2}$, WHICH IS A SEPARABLE EQUATION. THEN WE GET WITH $W(0) = W_0$:

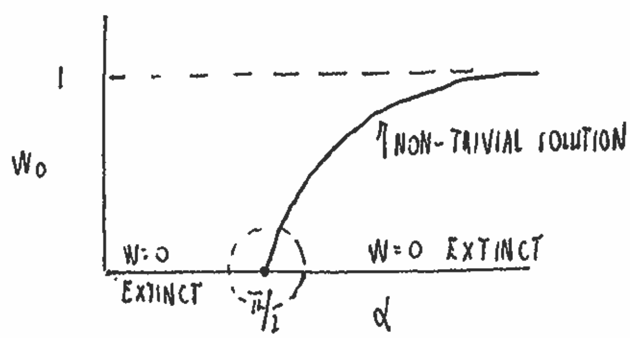
$$-\alpha x = \int_{W_0}^{W} \left[F(W_0) - F(\tilde{W}) \right]^{-1/2} d\tilde{W}$$

EQUIVALENTLY, WE CAN WRITE THIS AS $\alpha x = \int_{W}^{W_0} \left[F(W_0) - F(\Lambda) \right]^{-1/2} d\Lambda$, WHERE $F(u) = 2\int_0^u f(s)\,ds$. FINALLY, WE LET $W(1) = 0$. THIS GIVES A RELATION BETWEEN THE UNKNOWN W_0 AND α AS

$$\alpha = \int_0^{W_0} \left[F(W_0) - F(\Lambda) \right]^{-1/2} d\Lambda \qquad F(u) = 2\int_0^u f(\Lambda)\,d\Lambda .$$

<u>REMARK</u> : SINCE $f(\Lambda) = \Lambda(1-\Lambda)$ THEN $F(u) = 2\left[\frac{u^2}{2} - \frac{u^3}{3} \right] = u^2 - \frac{2}{3}u^3$.

WE CLAIM THAT THE GRAPH OF W_0 VS α LOOKS AS SHOWN



- IN THE REGION NEAR $\alpha = \pi/2$ WE SHOWED EARLIER THAT $\alpha \sim \frac{\pi}{2}\left(1 + \frac{4}{3\pi} W_0 + \cdots\right)$ FOR $W(0) = W_0 \ll 1$.

NOW WE WOULD LIKE TO SHOW THAT $\alpha \to \pi/2$ AS $W_0 \to 0$. THE ONLY WAY THAT $\int_0^{W_0} \to \neq 0$ AS $W_0 \to 0$ IS IF THE INTEGRAL BECOMES SINGULAR AS $W_0 \to 0$.

NOW AS $W_0 \to 0$ WE USE $F(0) = F'(0) = 0$ AND $F''(0) = 2f'(0) = 2$ TO CALCULATE

$$F(W_0) \sim F''(0) W_0^2/2 + \cdots$$

THUS
$$\alpha = \int_0^{W_0} [F(W_0) - F(\Lambda)]^{-1/2} d\Lambda \sim \int_0^{W_0} [W_0^2 - \Lambda^2]^{-1/2} d\Lambda = \sin^{-1}(\Lambda/W_0)\Big|_0^{W_0} = \frac{\pi}{2}$$

HENCE $\alpha \longrightarrow \pi/2$ AS $W_0 \to 0$. $\alpha = \frac{\pi}{2}$ IS A TRANSCRITICAL BIFURCATION POINT.

<u>SUMMARY</u> (i) IF $\alpha < \pi/2 \longrightarrow \frac{\sqrt{\Gamma}L}{\nu} < \frac{\pi}{2} \longrightarrow L < \frac{\pi\nu}{2\sqrt{\Gamma}}$ then

the only solution is the extinct fish solution.

(ii) IF $L > L_c = \frac{\pi\nu}{2\sqrt{\Gamma}}$, then there are two solutions.

AN EXTINCT FISH SOLUTION AND A NON-ZERO SOLUTION.

<u>STABILITY OF SOLUTIONS</u>

WE RETURN TO $U_t = D U_{xx} + f(U) \quad 0 < x < 1 \qquad D = 1/\alpha^2$

WITH $U_x(0,t) = U(1,t) = 0$ AND $f(U) = U(1-U)$.

NOW WE EXAMINE THE STABILITY OF THE ZERO SOLUTION. WE LET

$U = V \ll 1$ AND LINEARIZE TO OBTAIN

$$V_t = D V_{xx} + f'(0) V$$

$$V_x(0,t) = V(1,t) = 0 \qquad V(x,0) = V_0(x)$$

WE SUBSTITUTE $V = e^{\lambda t}\phi$ TO OBTAIN

$$\phi_{xx} + \frac{1}{D}[f'(0) - \lambda]\phi = 0, \quad 0 < x < 1, \quad \phi_x(0) = \phi(1) = 0.$$

THEN $\phi = \cos\left[\sqrt{\frac{1}{D}(f'(0) - \lambda)}\, x\right]$ WITH $\sqrt{\frac{1}{D}(f'(0) - \lambda)} = (2j-1)\frac{\pi}{2} \quad j = 1, 2, \ldots$

THEREFORE $\phi_j(x) = \cos\left(\frac{(2j-1)}{2}\pi x\right) \qquad \lambda_j = 1 - K_j^2 D \quad (f'(0) = 1)$

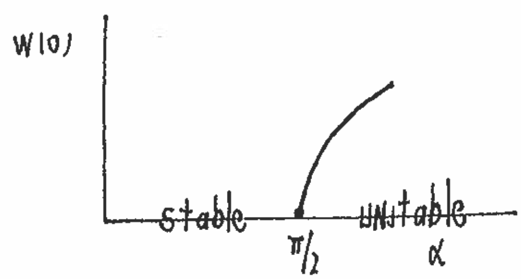
$$K_j = \frac{(2j-1)\pi}{2}, \quad j = 1, 2, \ldots$$

THEREFORE $\qquad V(x,t) = \sum_{j=1}^{\infty} A_j e^{\lambda_j t} \cos\left(\frac{(2j-1)\pi}{2} x\right)$

WE CONCLUDE THAT $V(x,t) \to 0$ FOR ANY $V(x,0) = V_0(x)$ IF AND ONLY IF $\lambda_j < 0$ FOR $j = 1, 2, ..$

NOTICE THAT $\lambda_j < \lambda_1$ FOR ALL $j = 1, 2, ...$ HENCE A NECESSARY AND SUFFICIENT CONDITION FOR LINEAR STABILITY IS THAT $\lambda_1 < 0$. THIS HOLDS WHEN $1 - \frac{\pi^2}{4} D < 0$ OR $D > 4/\pi^2 \longrightarrow \alpha < \pi/2$.

SO THE EXTINCT FISH SOLUTION IS STABLE WHEN THE EFFECTIVE DIFFUSIVITY D IS LARGE ENOUGH i.e. $D > 4/\pi^2$.
HOWEVER, WHEN $D < 4/\pi^2$ THEN $\lambda_1 > 0$ AND SO WE HAVE INSTABILITY OF $U \equiv 0$.
NOW THIS GIVES A PICTURE OF THE FORM



NEXT, WE CALCULATE <u>LOCALLY</u> THE STABILITY OF THE NON-TRIVIAL EQUILIBRIUM FISH SOLUTION. WE WRITE $D = 1/\alpha^2$

$$\alpha^2 U_t = U_{xx} + \alpha^2 F(U) \qquad 0 < x < 1, \quad U_x(0,t) = U(1,t) = 0.$$

WE LINEARIZE AROUND THE EQUILIBRIUM SOLUTION BRANCH $W(x)$ SATISFYING

$$W_{xx} + \alpha^2 F(W) = 0 \quad , \quad 0 < x < 1$$

$$W_x(0) = W(1) = 0$$

WITH $W(0)$ VS α GIVEN BY THE INTEGRAL RELATION ON THE PREVIOUS PAGE.

THUS, IN EFFECT $W = W(x; W_0)$, $\alpha = \alpha(W_0)$ $\quad W_0 = W(0)$.

WE THEN LET $u = w + e^{\lambda t}\phi$ WITH $\phi \ll 1$.

THIS LEADS TO THE EIGENVALUE PROBLEM

$$\alpha^2 \lambda e^{\lambda t}\phi = w_{xx} + e^{\lambda t}\phi_{xx} + \alpha^2 (F(w) + F'(w) e^{\lambda t}\phi)$$

THIS LEADS TO WITH $\sigma = \alpha^2 \lambda$,

$$\begin{cases} \phi_{xx} + \alpha^2 F'(w)\phi = \sigma\phi \\ \phi_x(0) = \phi(1) = 0 \end{cases}$$

NOTICE THAT THIS IS AN EIGENVALUE PROBLEM OF THE FORM

$$\phi_{xx} + g(x)\phi = \sigma\phi$$
$$\phi_x(0) = \phi(1) = 0$$

WITH A GIVEN $g(x)$. WE WANT TO DETERMINE THE EIGENVALUES AND IN FACT SHOW THAT ANY EIGENVALUE SATISFIES $\sigma < 0$. THIS IS IN GENERAL A DIFFICULT PROBLEM. (SEE PAGE **16**)

HOWEVER, WE CAN CALCULATE σ NEAR THE BIFURCATION POINT. RECALL THAT THE LOCAL DESCRIPTION OF THE BIFURCATING BRANCH IS $w = \varepsilon\cos(\pi x/2) + \ldots$ $\alpha = \frac{\pi}{2}\left(1 + \frac{4\varepsilon}{3\pi} + \ldots\right)$

THIS LEADS TO $\phi_{xx} + \frac{\pi^2}{4}(1 + \varepsilon a(x))\phi = \sigma\phi$

FOR SOME $a(x)$. IF WE SET $\varepsilon = 0$ THEN THE EIGENVALUES σ_j SATISFY $\sigma_j = \frac{\pi^2}{4}\left[1 - (2j-1)^2\right]$, HENCE $\sigma_j < 0$ FOR $j \geq 2$ AND $\sigma_1 = 0$.

FOR $\varepsilon \ll 1$ WE MUST ONLY CALCULATE THE EIGENVALUE WITH $\sigma_1 = O(\varepsilon)$ TO SHOW STABILITY. SEE THE HOMEWORK.

SOLUTION 5 $\qquad \alpha^2 u_t = u_{xx} + \alpha^2 u(1-u)$

$$u_x(0,t) = u(1,t) = 0, \quad u(x,0) = \tfrac{1}{2}(1-x^2).$$

(i) RECALL THAT THE BIFURCATION DIAGRAM OF NON-TRIVIAL SOLUTIONS $W_0 = W_0(\alpha)$ SATISFIES

$$\alpha = \int_0^{W_0} [F(W_0) - F(\Lambda)]^{-1/2}\, d\Lambda \qquad F(u) = 2\int_0^u s(1-s)\,ds$$

WE CALCULATE THAT $F(u) = u^2/_2 - 2u^3/_3$.

THIS YIELDS $\quad \alpha = \int_0^{W_0} \left[(W_0^2 - \Lambda^2) - \frac{2}{3}(W_0^3 - \Lambda^3) \right]^{-1/2} d\Lambda.$

WE PLOT THIS CURVE ON THE NEXT PAGE.

WE SHOWED IN CLASS THAT $\alpha \longrightarrow \pi/_2$ AS $W_0 \to 0$.

WE LET $\Lambda = W_0 s$ SO THAT

$$\alpha = \int_0^1 \frac{ds}{\left[(1-s^2) - \frac{2W_0}{3}(1-s^3) \right]^{1/2}}$$

NOW AS $W_0 \longrightarrow 0$ IT IS CLEAR THAT $\alpha \longrightarrow \int_0^1 \frac{1}{\sqrt{1-s^2}}\, ds = \sin^{-1}(1) = \frac{\pi}{2}$.

(ii) NOW DEFINE $\quad y(s; W_0) = (1-s^2) - \frac{2W_0}{3}(1-s^3)$.

WE CALCULATE $\quad y(1; W_0) = 0, \quad y_s(1; W_0) = 2(W_0 - 1), \quad y_{ss}(1; W_0) = 4W_0 - 2.$

THIS YIELDS $\quad y(s; W_0) \sim 2(W_0-1)(s-1) + (2W_0-1)(s-1)^2 + \ldots$

AS $s \longrightarrow 1$. THIS GIVES FOR SOME Λ SMALL FIXED THAT

$$\alpha = \int_0^{1-\Lambda} \frac{ds}{[y(s;W_0)]^{1/2}} + \int_{1-\Lambda}^1 \frac{ds}{[2(W_0-1)(s-1) + (2W_0-1)(s-1)^2 + \ldots]^{1/2}}$$

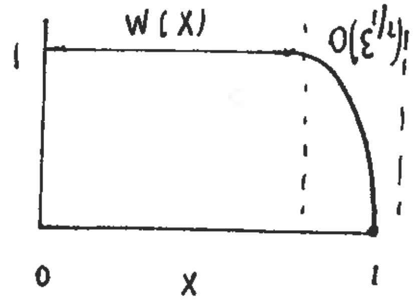
THE SECOND INTEGRAL DIVERGES AS $W_0 \to 1$ SINCE WHEN $W_0 = 1$

WE HAVE

FOR $\alpha >> 1$ WE HAVE $\varepsilon W_{XX} + W(1-W) = 0$ $\varepsilon = 1/\alpha^2 << 1.$

$$W_X(0) = W(1) = 0$$

THE SOLUTION WILL LOOK LIKE



LET $y = (1-X)/\varepsilon^{1/2} = \alpha(-X+1)$

THEN WE OBTAIN BL PROFILE.

$$\hat{W}_{yy} + \hat{W}(1-\hat{W}) = 0, \quad 0 \le y < \infty$$

$$\hat{W}(\infty) = 1, \quad \hat{W}(0) = 0$$

(iii) WE SOLVE BY THE METHOD OF LINES ON $-1 < X < 1$:

LET $\Delta X = 2/N$ AND $X_j = -1 + j\Delta X$ $j = 0, .., N.$

THEN $$\alpha^2 \left(\frac{U_n^{m+1} - U_n^m}{\tau} \right) = \frac{U_{n+1}^m - 2U_n^m + U_{n-1}^m}{(\Delta X)^2} + \alpha^2 U_n^m (1 - U_n^m)$$

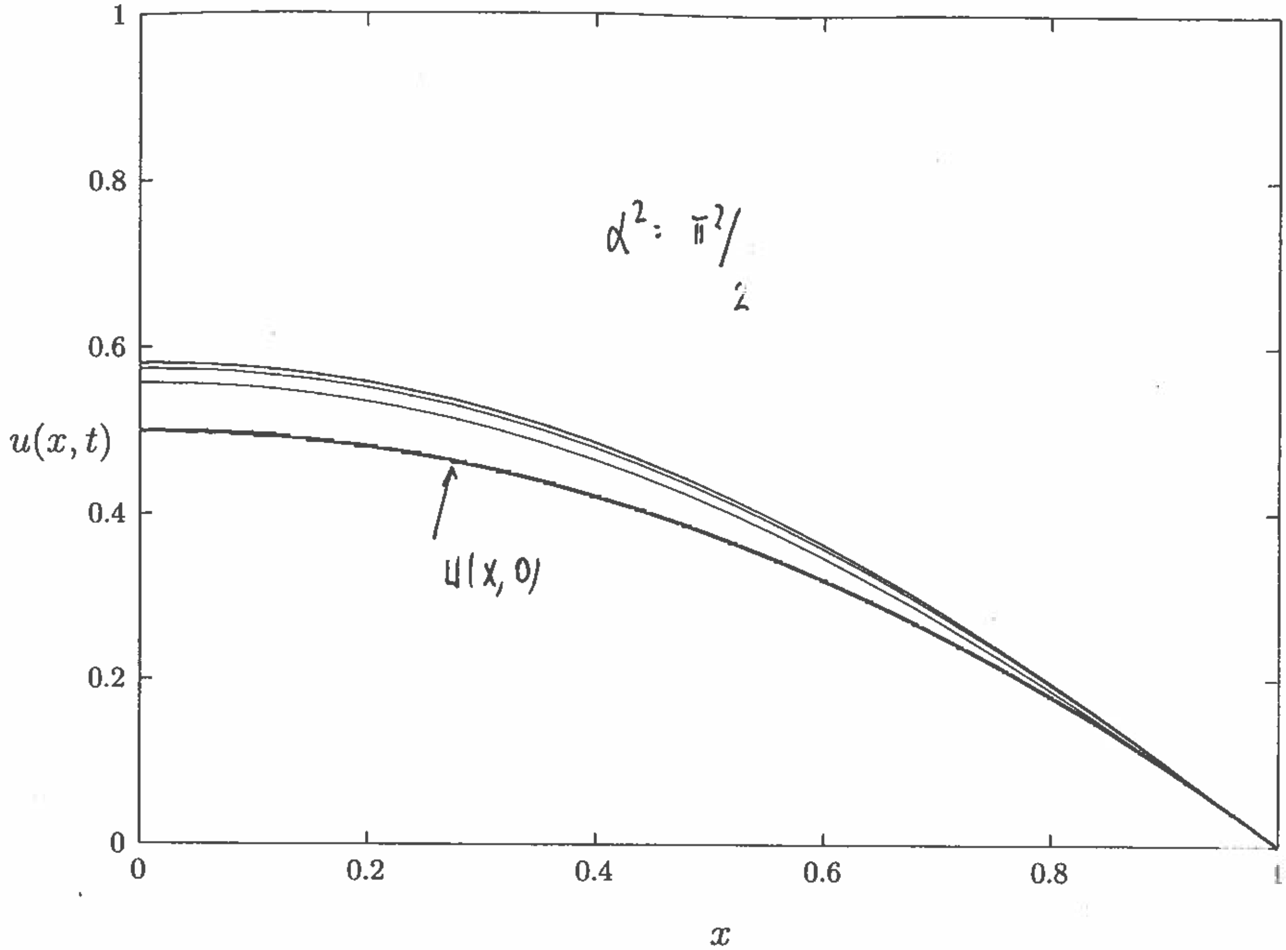
THUS YIELDS $$U_n^{m+1} = U_n^m + \frac{\tau}{\alpha^2 (\Delta X)^2} \left[U_{n+1}^m - 2U_n^m + U_{n-1}^m \right] + \tau U_n^m (1 - U_n^m)$$

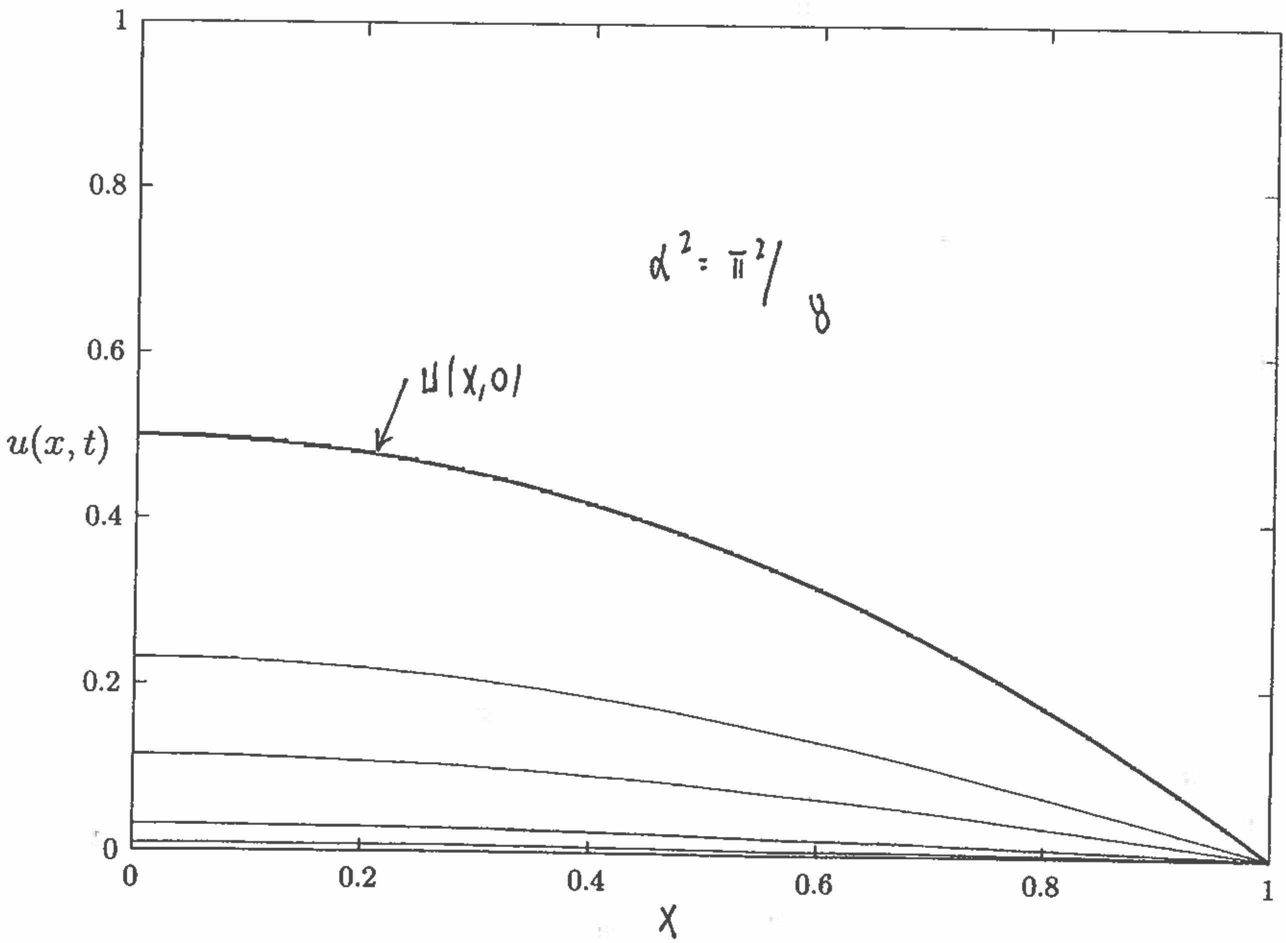
WITH $n = 1, .., N-1$ AND $U_1 = U_N = 0$

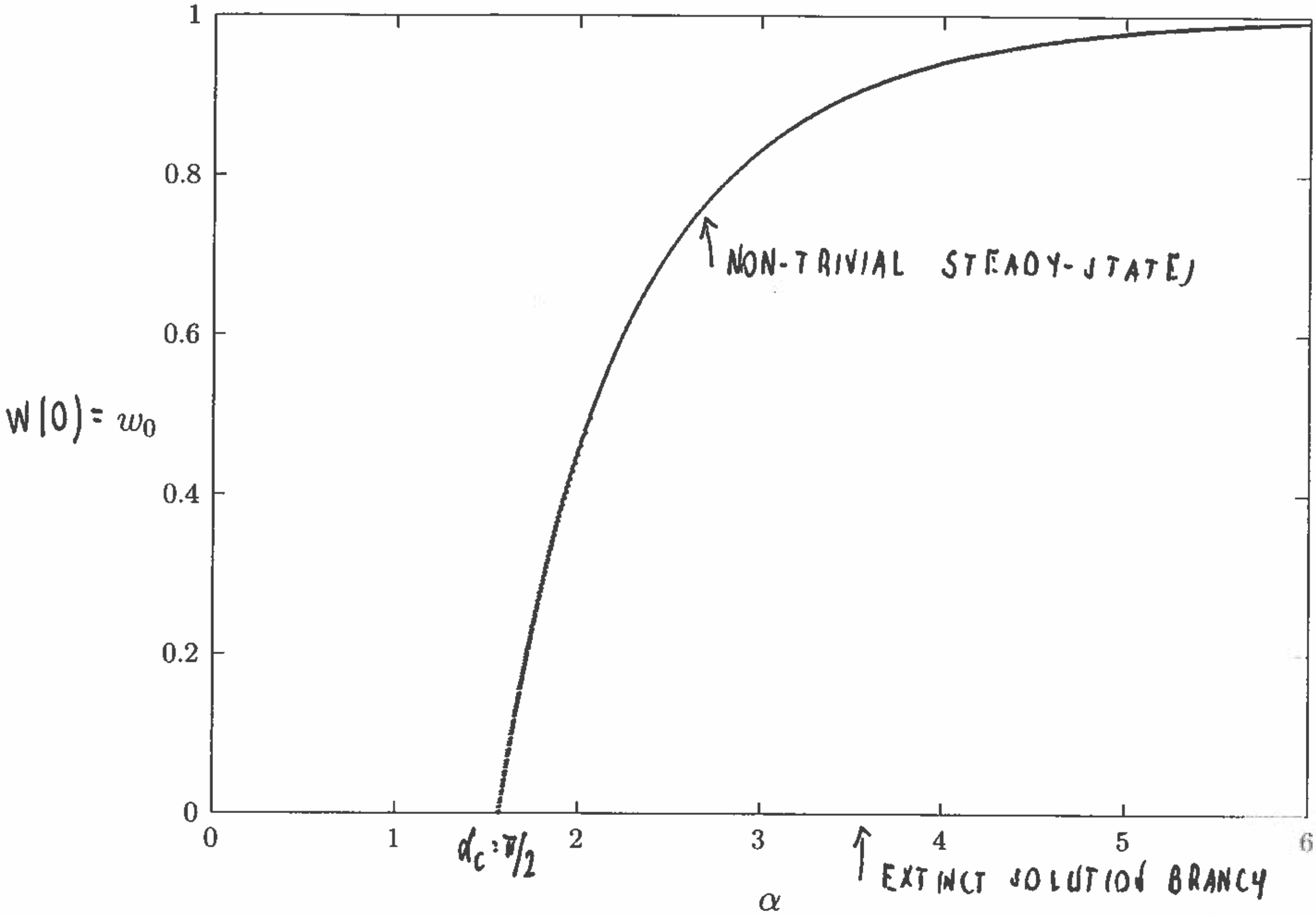
WE THEN BY SYMMETRY ONLY PLOT SOLUTION ON $0 \le X \le 1$

- IF $\alpha^2 = \pi^2/8$ THEN $\alpha^2 < \frac{\pi^2}{4}$ AND $U(X,t) \to 0$ AS $t \to \infty$
- IF $\alpha^2 = \pi^2/2$ THEN $\alpha^2 > \pi^2/4$ AND $U(X,t) \to W(X)$ AS $t \to \infty$

(SEE ATTACHED PLOTS)


$\alpha^2 = \pi^2/2$
$u(x,t)$
$u(x,0)$
0
0.2
0.4
0.6
0.8
1
x


$\alpha^2 = \pi^2/8$
$u(x,0)$
$u(x,t)$
0
0.2
0.4
0.6
0.8
1
X



PLOT OF

$$W(0) = W_0 \quad \text{VERSUS} \quad \alpha = \int_0^1 \frac{ds}{\left[(1-s^2) - \frac{2W_0}{3}(1-s^3) \right]^{1/2}}$$

FOR EQUILIBRIUM SOLUTIONS.

FISH PROBLEM

LINEAR STABILITY

LET $W(X)$ BE A STEADY-STATE OF

$$\alpha^2 U_t = U_{XX} + \alpha^2 f(U)$$

$$U_X = 0, \quad X = 0 \; ; \quad U = 1 \quad \text{AT} \quad X = 1$$

$$U(X,0) = U_0(X)$$

I.e. IF $|U(X,0) - W(X)|$ IS SMALL DOES $|U(X,t) - W(X)|$ REMAIN SMALL AS $t \to \infty$?

WE CONSIDER A LINEARIZED STABILITY THEORY:

LET $U(X,t) = W(X) + V(X,t)$ WITH $V \ll 1$.

WE OBTAIN: $\alpha^2 V_t = W_{XX} + \alpha^2 f(W+V) + V_{XX} = W_{XX} + \alpha^2 f(W) + \alpha^2 f'(W) V + V_{XX}$.

WE OBTAIN THAT
$$\left\{ \begin{array}{l} \alpha^2 V_t = V_{XX} + \alpha^2 f'(W) V \\ V_X(0,t) = 0, \quad V(1,t) = 0 \\ V(X,0) \text{ ARBITRARY.} \end{array} \right.$$

BY SEPARATING VARIABLES $V(X,t) = e^{\lambda t} \Phi(X)$

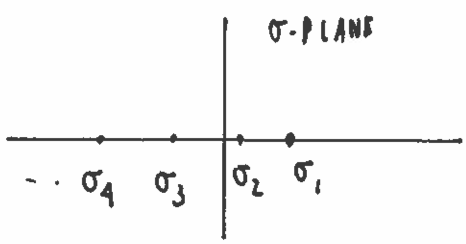
WE GET
$$\alpha^2 \lambda \Phi = \Phi'' + \alpha^2 f'(W) \Phi$$

WE THEN LABEL $\sigma = \alpha^2 \lambda$ SO THAT WE GET SL EIGENVALUE PROBLEM:

$$(*) \left\{ \begin{array}{l} \Phi'' + [\alpha^2 f'(W) - \sigma] \Phi = 0 \quad \text{IN } 0 \le X \le 1 \\ \Phi'(0) = 0, \quad \Phi(1) = 0. \end{array} \right.$$

THE EIGENVALUES ARE $\sigma_1, \sigma_2, \sigma_3, \ldots$ WITH $\Phi_1, \Phi_2, \ldots$

WE HAVE $\sigma_{j+1} < \sigma_j \quad \forall j$ AND $\sigma_j \to -\infty$ AS $j \to +\infty$, σ_j REAL.



THEN BY COMPLETENESS, WE HAVE
$$V = \sum_{j=1}^{\infty} c_j \, e^{\sigma_j / \alpha^2 t} \, \Phi_j(X)$$

WE HAVE $V \to 0$ AS $t \to \infty$ FOR ANY $V(X,0)$

IFF $\sigma_j < 0 \quad \forall j = 1, 2, \ldots$,

IF $\sigma_j < 0$, $\forall j = 1, \ldots, \infty$, WE SAY THAT THE STEADY-STATE W IS LINEARLY STABLE.

STEADY-STATE

WE HAVE $W_{xx} + \alpha^2 f(W) = 0$ WITH $f(W) = W(1-W)$. $\alpha = \frac{L\sqrt{r}}{\sqrt{\ }}$

$W_x(0) = 0, \quad W(1) = 0.$

WE SHOWED THAT

$$\alpha = \int_0^{W_0} [F(W_0) - F(\Lambda)]^{-1/2} d\Lambda \quad \text{WITH} \quad F(U) = 2\int_0^U s(1-s)\,ds.$$

NOW WE CALCULATE $F(U) = 2(U^2/2 - U^3/3)$ AND $F(W_0) - F(\Lambda) = W_0^2 - \Lambda^2 - \frac{2}{3}(W_0^3 - \Lambda^3)$

WE CONCLUDE THAT

$$\alpha = \int_0^{W_0} \left[W_0^2 - \Lambda^2 - \frac{2}{3}(W_0^3 - \Lambda^3)\right]^{-1/2} d\Lambda.$$

NOW WE LET $\Lambda = W_0 S$, $d\Lambda = W_0\, dS$, SO THAT

$$\alpha = \int_0^1 \left[1 - S^2 - \frac{2W_0}{3}(1 - S^3)\right]^{-1/2} dS. \qquad = \int_0^1 \frac{dS}{\sqrt{(1-S)\left[(1+S) - \frac{2W_0}{3}(S^2+S+1)\right]}}$$

↓ WHEN $S=1$ AND $W_0 = 1$ IT VANISHES

NOW AS $W_0 \to 0$ IT FOLLOWS THAT $\alpha \to \int_0^1 \frac{1}{\sqrt{1-S^2}} dS = \sin^{-1}(S)\Big|_0^1 = \pi/2.$

NOW WE CALCULATE

$$\frac{d\alpha}{dW_0} = \int_0^1 -\frac{1}{2}\left(-\frac{2}{3}\right)(1-S^3)\frac{1}{\left[(1-S^2) - \frac{2W_0}{3}(1-S^3)\right]^{3/2}} dS$$

$$= \frac{2}{6}\int_0^1 (1-S^3)\left[(1-S^2) - \frac{2W_0}{3}(1-S^3)\right]^{-3/2} dS > 0.$$

THUS $\alpha \uparrow$ AS $W_0 \uparrow$.

NOW AS $W_0 \to 1$ WE CLAIM THAT $\alpha \to \infty$. DEFINE

$$y(S; W_0) = \left[1 - S^2 - \frac{2W_0}{3}(1-S^3)\right], \quad \text{SO} \quad \alpha = \int_0^1 [y(S, W_0)]^{-1/2} dS.$$

NOW AS $S \to 1$, $\quad y \sim (-2S + 2W_0 S^2)\Big|_{S=1}(S-1) + (-2 + 4W_0 S)\Big|_{S=1}\frac{(S-1)^2}{2}.$

SO $y(S, W_0) \sim 2(W_0 - 1)(S-1) + (2W_0 - 1)(S-1)^2 + \dots$ AS $S \to 1$.

NOTICE THAT $y(S,1) \sim (S-1)^2$ AS $S \to 1$ AND THIS IS NOT INTEGRABLE

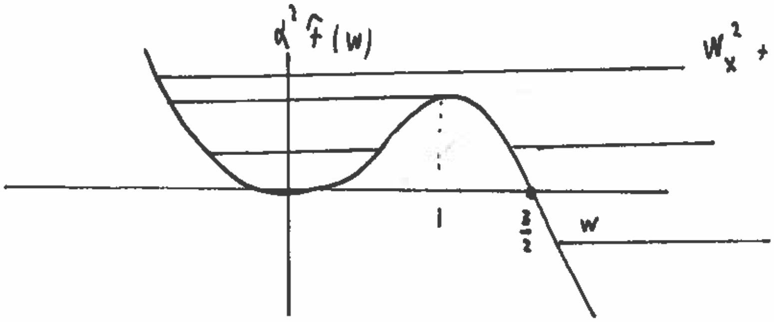
IN $\alpha = \int_0^1 [y(S,1)]^{-1/2} dS = \infty$.

NOW GEOMETRICAL PICTURE IN PHASE PLANE

$$W_x^2 + \alpha^2 [F(W)] = \text{CONSTANT}$$

NOTICE $W = 0, 1$ EQUILIBRIA $F'(0) = 0$ (center), $F'(1) = 0$ (saddle).

(15.75)

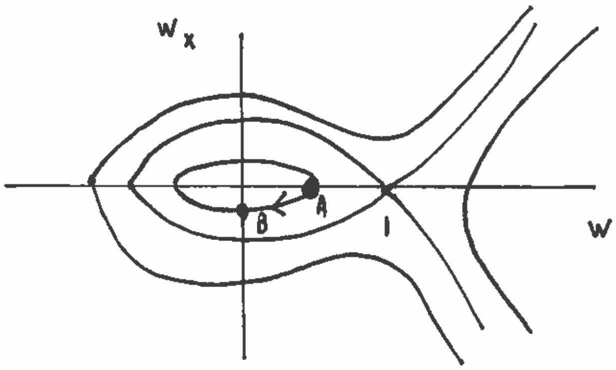


$w_x^2 + \alpha^2 \hat{f}(w) = E$

$$\hat{f}(w) \equiv w^2 - \frac{2w^3}{3}$$

$$\hat{f}'(w) = 2w - 2w^2 = 0 \text{ AT}$$

$w = 0, 1$



NOW WANT TRAJECTORY WITH $w_x(0)=0$, $w(1)=0$, $w_x < 0$.

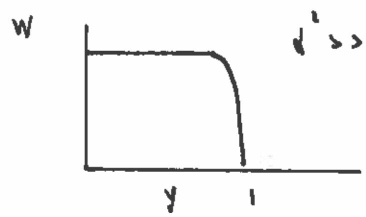
point A , point B

TRAJECTORIES CAN'T GROW SO AS WE NEED $w(0) < 1$.

NOW AS $\alpha^2 \to \infty$, WE HAVE THAT OUTER SOLUTION IS $w_{out} = 1$.

NOW NEAR $x = 0$, WE HAVE WITH $y = \alpha(1-x)$ SO $\partial_x = -\partial_y \alpha$, HENCE

$$\begin{cases} V_{yy} + V(1-V) = 0, & 0 < y < \infty \\ V(0) = 0, \quad V(\infty) = 1. \end{cases}$$



$\alpha^2 \gg 1$.

LINEARIZED STABILITY

TRIVIAL STEADY-STATE $W = 0$

WITH $F(u) = u(1-u)$, $F'(u) = 1$ SO $F'(0) = 1$.

THEN (*) BECOMES

$$\Phi'' + [\alpha^2 - \sigma]\Phi = 0 \; ; \quad \Phi'(0) = 0, \; \Phi(1) = 0.$$

WE OBTAIN $\Phi(x) = \cos\left(\sqrt{\alpha^2 - \sigma}\, x\right)$ SO $\sqrt{\alpha^2 - \sigma} = \dfrac{(2j+1)\pi}{2}$ FOR $j = 0, 1, 2, \ldots$

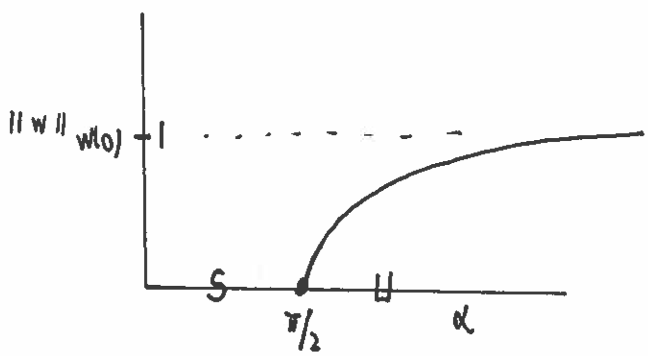
WE CONCLUDE $\alpha^2 - \sigma_j = (2j+1)^2 \dfrac{\pi^2}{4}$

THUS $\sigma_j = \alpha^2 - (2j+1)^2 \dfrac{\pi^2}{4}$, $j = 1, 2, 3, \ldots$

NOW IF $\sigma_1 < 0$, THEN $\sigma_j < 0 \;\; \forall j = 1, 2, \ldots$ $\sigma_1 < 0$ WHEN $\alpha^2 < \dfrac{\pi^2}{4}$, OR $\alpha < \dfrac{\pi}{2}$.

THUS EXTINCT FISH SOLUTION IS LINEARLY STABLE IFF $\alpha < \pi/2$,

WHERE $\alpha \equiv \dfrac{L\sqrt{r}}{\nu}$. THUS EXTINCT FISH IS <u>STABLE</u> IFF $L < L_c = \dfrac{\pi}{2}\dfrac{\nu}{\sqrt{r}}$.



L_c IS CALLED CRITICAL PATCH SIZE.

NOW CONSIDER <u>NON-TRIVIAL SOLUTION</u> BRANCH:

$$\begin{cases} \Phi'' + [\alpha^2 f'(w) - \sigma]\Phi = 0 \\ \Phi'(0) = 0, \; \Phi(1) = 0 \end{cases}$$

WE HAVE

$$\begin{cases} w'' + \alpha^2 f(w) = 0 \\ w'(0) = 0, \; w(1) = 0. \end{cases}$$

WE WANT TO SHOW THAT $\sigma_1 > 0$. FROM SL THEORY $\Phi_1(x) > 0$.

NOW $$\int_0^1 (w\Phi_1'' - \Phi_1 w'')\, dx = \int_0^1 \left(w\left[(\sigma_1 - \alpha^2 f'(w))\Phi_1\right] + \Phi_1 \alpha^2 f(w)\right) dx$$

$$0 = (w\Phi_1'|_0^1 - \Phi_1 w')|_0^1 = \sigma_1 \int_0^1 w\Phi_1\, dx + \int_0^1 \Phi_1\left[-\alpha^2 w f'(w) + \alpha^2 f(w)\right] dx.$$

NOW THIS YIELDS THAT

$$\sigma_1 \int_0^1 w \Phi_1 \, dx = -\int_0^1 \Phi_1 \alpha^2 \left[f(w) - w f'(w) \right] dx .$$

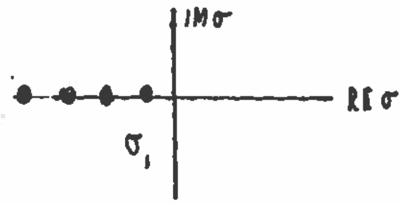
NOW $f(w) = w(1-w)$, $f'(w) = 1 - 2w$

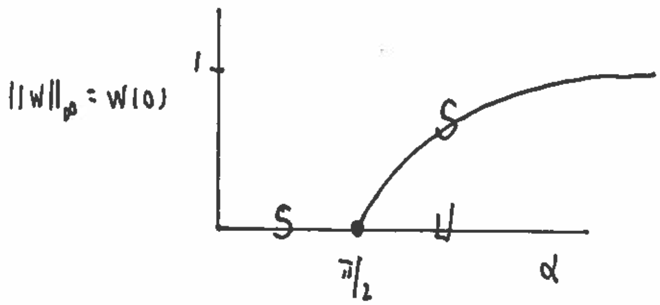
$$f(w) - w f'(w) = w(1-w) - w(1-2w) = -w^2 + 2w^2 = w^2 .$$

THUS $$\sigma_1 \int_0^1 w \Phi_1 \, dx = -\int_0^1 \Phi_1 \alpha^2 w^2 \, dx .$$

NOW SINCE $w > 0$ AND $\Phi_1 > 0$ WE OBTAIN $\sigma_1 < 0$.

BY ORDERING PRINCIPLE $\sigma_{j+1} < \sigma_j < \ldots < \sigma_1 < 0$.





WE CONCLUDE THAT THE ENTIRE NON-TRIVIAL STEADY-STATE BRANCH IS LINEARLY STABLE.

LOCAL ANALYSIS

IN GENERAL WE CANNOT DETERMINE THE GLOBAL BRANCH ANALYTICALLY AND DETERMINE THE LINEAR STABILITY OF THE ENTIRE BRANCH, AS WE HAVE DONE.

NOW RECALL THAT OUR CONSTRUCTION OF THE STEADY-STATE NEAR $\alpha = \pi/2$. WE OBTAIN NEAR $\alpha = \pi/2$ THAT

$$\alpha = \frac{\pi}{2}\left(1 + \frac{4\varepsilon}{3\pi} + \ldots\right), \quad w = \varepsilon w_0 + \varepsilon^2 w_1 + \ldots$$

$$w_0 = \cos\left(\pi x/2\right)$$

WE OBTAIN THAT

$$(*) \quad \begin{cases} \Phi'' + \alpha^2 f'(w) \Phi = \sigma \Phi \\ \Phi'(0) = 0, \quad \Phi(1) = 0 \end{cases}$$

WHEN $\alpha = \pi/2$, THEN $\sigma_1 = 0$. $\rightarrow$ THE EIGENVALUE IS ZERO. WE NOW PUT

$$\sigma = \varepsilon \mu_0 + \varepsilon^2 \mu_1 + \ldots$$

$$\Phi = \Phi_0 + \varepsilon \Phi_1 + \ldots$$

OUR GOAL IS TO SUBSTITUTE THIS INTO (*) TO GET

$$\Phi_0'' + \varepsilon \Phi_1'' + \dots + \frac{\pi^2}{4}\left(1+\frac{4\varepsilon}{3\pi}+\dots\right)^2 f'(\varepsilon w_0+\varepsilon^2 w_1+\dots)(\Phi_0+\varepsilon\Phi_1+\dots) = \varepsilon \mu_0 \Phi_0 .$$

WE OBTAIN WITH $f'(0)=1$ THAT $f'(\varepsilon w_0+\varepsilon^2 w_1) \cong f'(0)+\varepsilon w_0 f''(0)+\dots \quad (f''(0)=-2)$

$$\Phi_0'' + \varepsilon \Phi_1'' + \dots + \frac{\pi^2}{4}\left(1+\frac{8\varepsilon}{3\pi}+\dots\right)\left[1+f''(0)\,\varepsilon w_0+\dots\right]\left[\Phi_0+\varepsilon\Phi_1+\dots\right] = \varepsilon \mu_0 \Phi_0 + \dots$$

BUT $w_0 = \cos(\pi x/2)$ SO THAT

$$\Phi_0'' + \varepsilon \Phi_1'' + \dots + \frac{\pi^2}{4}\left(1+\frac{8\varepsilon}{3\pi}+\dots\right)\left(1-2\varepsilon\cos\left(\frac{\pi x}{2}\right)\right)\left(\Phi_0+\varepsilon\Phi_1+\dots\right) = \varepsilon \mu_0 \Phi_0 + \dots$$

THIS GIVES,

$$\Phi_0'' + \frac{\pi^2}{4}\Phi_0 = 0, \quad \Phi_0'(0)=0, \quad \Phi_0(1)=0 \quad \to \quad \Phi_0 = \cos(\pi x/2)$$

$$L\Phi_1 \equiv \Phi_1'' + \frac{\pi^2}{4}\Phi_1 = -\frac{\pi^2}{4}\left(\frac{8}{3\pi} - 2\cos\left(\frac{\pi x}{2}\right)\right)\Phi_0 + \mu_0 \Phi_0 \equiv g(x;\mu_0)$$

NOW THE SOLVABILITY CONDITION IS THAT $0 = \int_0^1 \Phi_0 L\Phi_1\,dx = \int_0^1 \Phi_1 L\Phi_0\,dx = \int_0^1 \Phi_0 g\,dx$,

SO THAT $\displaystyle\int_0^1 \Phi_0\, g(x;\mu_0)\,dx = 0 \longrightarrow$ EQUATION FOR μ_0.

WE GET

$$\mu_0 \int_0^1 \Phi_0^2\,dx = \frac{2\pi}{3}\int_0^1 \Phi_0^2\,dx - \frac{\pi^2}{2}\int_0^1 \Phi_0^2 \cos\left(\frac{\pi x}{2}\right)dx$$

$$= \frac{2\pi}{3}\int_0^1 \Phi_0^2\,dx - \frac{\pi^2}{2}\int_0^1 \Phi_0^3\,dx .$$

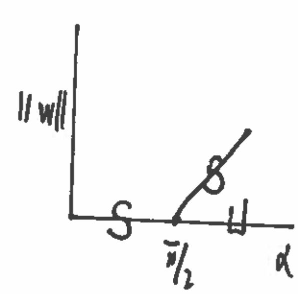
SO

$$\mu_0 = \frac{2\pi}{3} - \frac{\pi^2}{2}\,\frac{\int_0^1 \cos^3(\pi x/2)\,dx}{\int_0^1 \cos^2(\pi x/2)\,dx} = \frac{2\pi}{3} - \frac{\pi^2}{2}\left(\frac{8}{3\pi}\right) = -\frac{2\pi}{3} .$$

NOW SINCE THE PRINCIPAL EIGENVALUE σ_1 SATISFIES

$$\sigma_1 \sim -2\pi\varepsilon/3 < 0$$

IT FOLLOWS THAT, AT <u>LEAST LOCALLY</u>, THE BIFURCATING STEADY-STATE BRANCH IS STABLE.



AMPLITUDE EQUATION (DESCRIBES DYNAMICS <u>NEAR</u> THE BIFURCATION)

WE INTRODUCE A SMALL PARAMETER $\varepsilon > 0$ BY

$$\alpha^2 = \frac{\pi^2}{4}(1+\varepsilon)$$

WE LOOK FOR A SOLUTION IN THE FORM

$$u(x,\tau) = \varepsilon\, u_0(x,\tau) + \varepsilon^2 u_1(x,\tau) + \ldots \qquad \text{WITH } \tau = \varepsilon t$$

AND $u_0(x,\tau) = A(\tau)\cos\left(\frac{\pi x}{2}\right)$ WHERE $A(\tau)$ IS TO BE FOUND.

$$\alpha^2 u_{0_t} = u_{0xx} + \alpha^2 f(u)$$

$$\frac{\pi^2}{4}(1+\varepsilon\ldots)\,\varepsilon^2 A' \cos\left(\frac{\pi x}{2}\right) = \varepsilon\left[u_{0xx} + \frac{\pi^2}{4} f'(0)\, u_0\right]$$

$$+\,\varepsilon^2\left[u_{1xx} + \frac{\pi^2}{4} f'(0)\, u_1 + \frac{\pi^2}{4} f'(0)\, u_0 + \frac{\pi^2}{8} f''(0)\, u_0^2\right.$$

WE OBTAIN

$$\mathcal{L} u_1 \equiv u_{1xx} + \frac{\pi^2}{4} f'(0)\, u_1 = \frac{\pi^2}{4} u_{0\tau} - \frac{\pi^2}{4} f'(0)\, u_0 - \frac{\pi^2}{8} f''(0)\, u_0^2$$

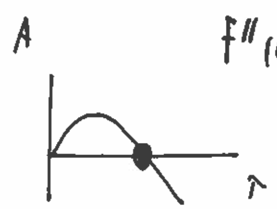
$$u_{0\tau} = A' \cos(\pi x/2)$$

$f'(0) = 1$

$f''(0) = -2$

$$\longrightarrow \quad A' = A\left(1 - \frac{8A}{3\pi}\right)$$

$$\text{S.S.} \qquad \alpha^2 = \frac{\pi^2}{4}(1+\varepsilon)$$



<u>REMARK</u> IF $u_0 = A\,\phi_0$ WITH $\phi_0 = \cos(\pi x/2)$, THE SOLVABILITY CONDITION GIVES $\int_0^1 \phi_0\, \mathcal{L} u_1\, dx = 0$

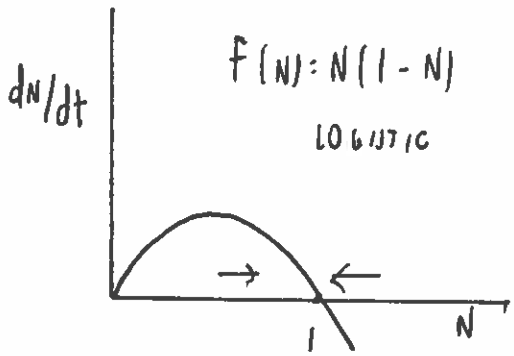
$$\rightarrow \frac{\pi^2}{4} A_0' \int \Phi_0^2\, dx = \frac{\pi^2}{4}\left(\int \Phi_0^2\, dx\right) A + \frac{\pi^2}{8} f''(0) A^2 \int \Phi_0^3$$

$$\rightarrow \frac{\pi^2}{4} A_0' = \frac{\pi^2}{4} A_0 + \left(\frac{\pi^2}{8}\right)(-2) A^2 \frac{\int \Phi_0^3}{\int \Phi_0^2} \rightarrow A_0' = A_0 - A_0^2 \int \Phi_0^3 \Big/ \int \Phi_0^2$$

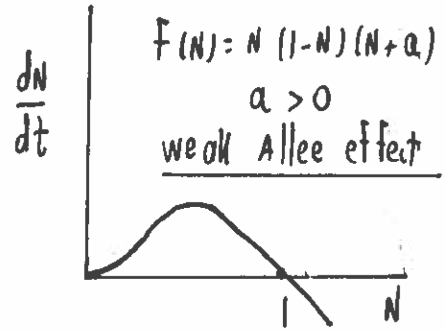
$$\rightarrow A_0' = A_0 - A_0^2\left(8/3\pi\right).$$

EXTENSIONS TO THE THEORY

FOR OTHER TYPES OF ASSUMPTIONS REGARDING FISH POPULATIONS WE HAVE DIFFERENT NONLINEARITIES



$f'(0) > 0, \quad f''(0) < 0$

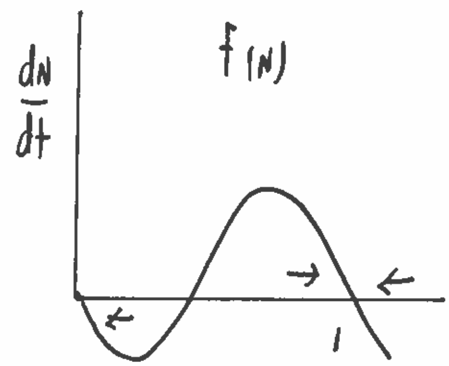


$f'(0) = a > 0$

$f''(0) = 1 - a > 0$

ONLY IF $0 < a < 1$

HARDER TO FIND A MATE WHEN $N \ll 1$



STRONG ALLEE EFFECT, IF $f'(0) < 0$.

POPULATION IN FACT DECREASES IF N TOO SMALL.

IN EACH CASE THE DIFFUSE LOGISTIC MODEL HAS THE FORM

$$\alpha^2 u_t = u_{xx} + \alpha^2 f(u)$$

$$u_x(0,t) = 0, \quad u(1,t) = 0.$$

THE TRIVIAL STEADY-STATE SOLUTION IS $w \equiv 0 \ \forall x$

AND NON-TRIVIAL STEADY-STATES SATISFY

$$w_{xx} + \alpha^2 f(w) = 0$$

$$w_x(0) = 0, \quad w(1) = 0.$$

NOW WITH AN ALLEE EFFECT THE BIFURCATION FROM THE SPATIALLY HOMOGENEOUS STEADY-STATE IS AFFECTED.

IN FACT WITH A STRONG ALLEE EFFECT THERE IS NO BIFURCATING BRANCH OF SOLUTIONS.