OSCILLATIONS AND HOPF BIFURCATION

THE INTUITIVE IDEA OF A HOPF BIFURCATION IS AS FOLLOWS.

CONSIDER A SYSTEM

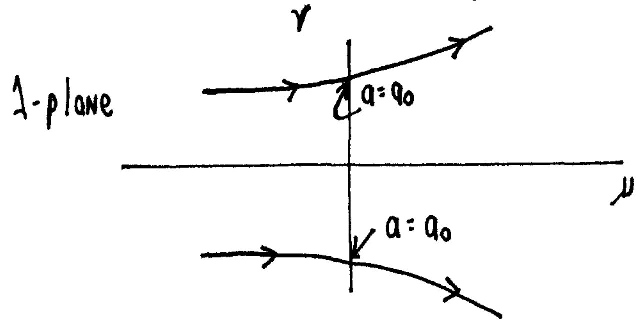
$$\frac{d\underline{x}}{dt} = \underline{f}(\underline{x}; a) \qquad \underline{x}, \underline{f} \in \mathbb{R}^2$$

AND SUPPOSE THAT $\underline{f}(\underline{x}_0; a) = 0 \rightarrow \underline{x}_0 = \underline{x}_0(a)$. NOW LET $J(\underline{x}_0; a)$ DENOTE THE JACOBIAN AT $\underline{x}_0$.

WE NOW SUPPOSE THAT

$$J(x_0; a) = \begin{pmatrix} \mu & \nu \\ -\nu & \mu \end{pmatrix} \qquad \mu = \mu(a), \ \nu = \nu(a)$$

WITH $\mu(a_0) = 0$ AND $\mu'(a_0) > 0$ AT SOME $a = a_0$. NOTICE THAT THE EIGENVALUES OF J ARE $\lambda = \mu \pm i\nu$ AND SO WE GET A PLOT AS SHOWN IN COMPLEX λ PLANE



ARROWS INDICATE BEHAVIOR AS a INCREASES

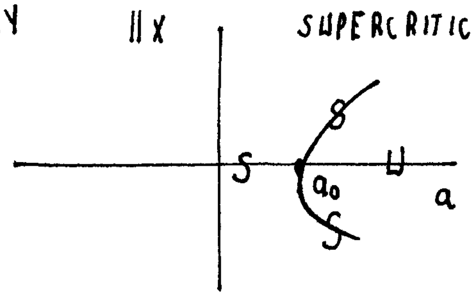
<u>NOTE</u>:

$\underline{x}_0$ unstable IF $a > a_0 \quad |a-a_0| \ll 1$

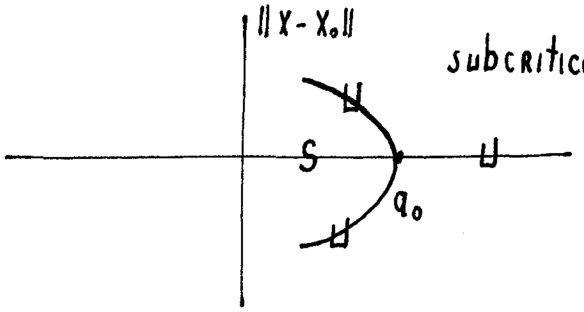
$\underline{x}_0$ stable IF $a < 0 \quad |a-a_0| \ll 1$

THE CONDITION $\mu'(a_0) > 0$ IS CALLED A TRANSVERSAL CROSSING CONDITION. HENCE AS a INCREASES PAST a_0 A PAIR OF COMPLEX CONJUGATE EIGENVALUES ENTER THE RIGHT HALF PLANE. DEPENDING ON THE NONLINEAR TERMS THE BIFURCATION DIAGRAM HAS TWO POSSIBILITIES LOCALLY

SUPERCRITICAL HOPF



subcritical HOPF



NOW AT THE POINT $a = a_0$ WE GET THE BIRTH OF A PERIODIC SOLUTION WITH PERIOD $T = 2\pi / \nu(a_0)$ SINCE $\frac{d\underline{x}}{dt} = \begin{pmatrix} 0 & \nu_0 \\ & 0 \end{pmatrix} \underline{x}$ AT $\nu_0 = \nu(a_0)$ YIELDS $x_1'' + \nu_0^2 x_1 = 0$ SO THAT $x_1 = \cos(\nu_0 t)$ OR $x_1 = \sin(\nu_0 t)$.

FOR EXAMPLE WITH

$$x' = \mu x - y + xy^2 \equiv f(x,y)$$
$$y' = x + \mu y + y^3 \equiv g(x,y)$$

WE NOTICE THAT $(0,0)$ IS AN EQUILIBRIUM. WE THEN COMPUTE J FROM LINEARIZING AROUND THE ZERO SOLUTION AS

$$J \equiv \begin{pmatrix} \mu & -1 \\ 1 & \mu \end{pmatrix} \quad \text{WITH } \lambda = \mu \pm i.$$

AS μ CROSSES THROUGH ZERO THE ORIGIN CHANGES STABILITY FROM A STABLE SPIRAL TO AN UNSTABLE SPIRAL. THIS SUGGESTS A HOPF BIFURCATION AT $\mu = 0$.

<u>THEOREM</u> SUPPOSE THAT

$$x' = f(x,y;\mu)$$
$$y' = g(x,y;\mu)$$

WITH $f(0,0;\mu) = g(0,0;\mu) = 0$. SUPPOSE THAT THE JACOBIAN MATRIX $J \equiv \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}$ evaluated at the ORIGIN when $\mu = 0$ IS OF the form $J\big|_{\mu=0} = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}$.

SUPPOSE ALSO THAT

(i) $f_{\mu x} + g_{\mu y} \neq 0$ AT $(0,0;0)$

(ii) AND $a \neq 0$ where

$$a = \frac{1}{16}\left[f_{xxx} + g_{xxy} + f_{xyy} + g_{yyy} \right] + \frac{1}{16\omega}\left[f_{xy}\left(f_{xx} + f_{yy} \right) - g_{xy}\left(g_{xx} + g_{yy} \right) - f_{xx}\, g_{xx} + f_{yy}\, g_{yy} \right]$$

EVALUATED AT THE POINT $(0,0;0)$.

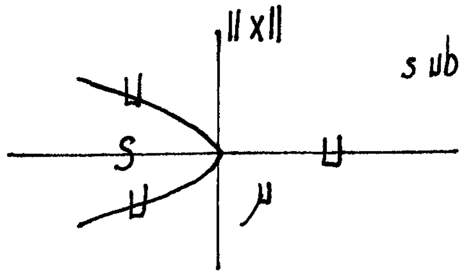
THEN

(i) $\exists$ A CURVE OF PERIODIC SOLUTIONS BIFURCATING FROM THE ORIGIN INTO $\mu < 0$ IF $a\left(f_{\mu x} + g_{\mu y} \right) > 0$ OR INTO $\mu > 0$ IF $a\left(f_{\mu x} + g_{\mu y} \right) < 0$.

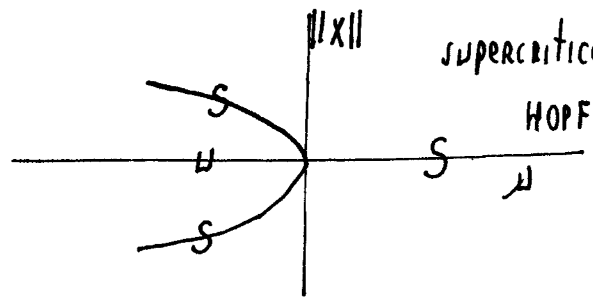
(ii) THE ORIGIN IS STABLE FOR $\mu > 0$ AND UNSTABLE FOR $\mu < 0$ IF WE HAVE $f_{\mu x} + g_{\mu y} < 0$.

THE ORIGIN IS STABLE FOR $\mu < 0$ AND UNSTABLE FOR $\mu > 0$ IF WE HAVE $f_{\mu x} + g_{\mu y} > 0$.

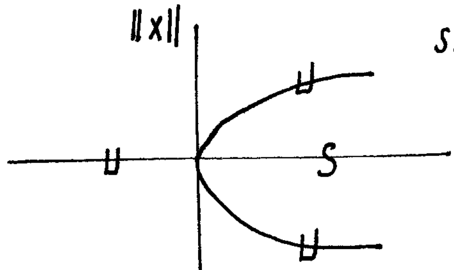
(iii) PERIODIC SOLUTIONS ARE STABLE IF THE ORIGIN IS UNSTABLE ON THE SIDE OF $\mu = 0$ WHERE PERIODIC SOLUTIONS EXIST



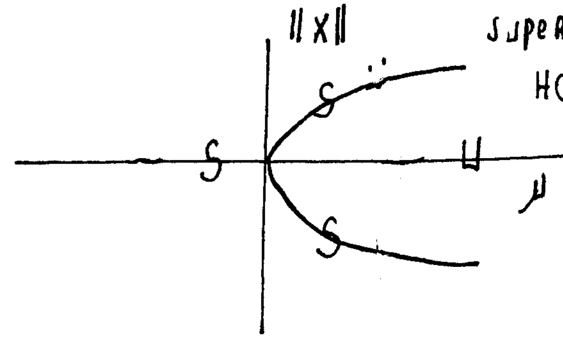
subcritical HOPF



supercritical HOPF



subcritical HOPF



supercritical HOPF

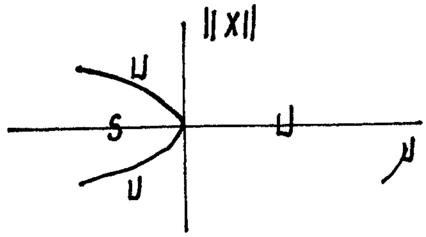
EXAMPLE

$$x' = \mu x - y + x y^2 = f(x,y)$$
$$y' = x + \mu y + y^3 = g(x,y)$$

WE HAVE THAT $\mu = 0$ IS A HOPF BIFURCATION POINT. WE CALCULATE

$$f_{\mu x} = 1, \quad f_{xxx} = 0, \quad f_{xyy} = 2, \quad g_{yyy} = 6, \quad g_{\mu y} = 1, \quad f_{xyy} = 2$$

WE GET $\quad a = \frac{1}{16}\left(2 + 6\right) = \frac{1}{2} > 0 \quad$ AND $\quad a\left(f_{\mu x} + g_{\mu y}\right) > 0.$



subcatical HOPF BIFURCATION.

PROOF WE CAN ALWAYS CHANGE VARIABLES TO WRITE THE SYSTEM IN THE FORM

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \alpha(\mu) & \beta(\mu) \\ -\beta(\mu) & \alpha(\mu) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} F \\ G \end{pmatrix} \qquad \|x\| = (x^2 + y^2)^{1/2}$$

WHERE F, G ARE $O\left(\|x\|^2\right)$ AS $\|x\| \to 0$. AT $\mu = 0$ WE HAVE THAT $\beta(0) = \omega > 0$, AND $\alpha(0) = 0$.

NOTICE THAT $f_{\mu x} = \alpha'(\mu)$, $g_{\mu y} = \alpha'(\mu)$ SO THAT $f_{\mu x} + g_{\mu y} = 2\alpha'(\mu)$ AND WE SUPPOSE THAT $\alpha'(0) \neq 0$.

WE INTRODUCE COMPLEX VARIABLES $z = x + iy = e^{i\varphi}$, $\bar{z} = x - iy = e^{-i\varphi}$ SO THAT

$$(x + iy)' = (\alpha - i\beta)x + (i\alpha + \beta)y + \underbrace{F + iG}_{N}.$$

THIS CAN BE WRITTEN AS

$$(*) \Big\} \quad z' = (\alpha - i\beta)z + N(z, \bar{z}; \mu)$$

WHERE $\quad x = \frac{1}{2}(z + \bar{z}), \quad y = \frac{-i}{2}(z - \bar{z}).$

OUTLINE OF PROOF

(i) A NORMAL FORM ON (*) IS USED TO GET CANONICAL ODE.

(ii) INJPECTION OF NORMAL FORM EQUATION.

LEMMA $\exists$ A NEAR-IDENTITY TRANSFORMATION THAT IS ANALYTIC AND HAS THE FORM

$$\zeta = z + S(z, \bar{z}; \mu) \quad \text{WITH} \quad S = O(|z|^2) \quad \text{AS} \quad z \to 0$$

SUCH THAT (*) ADOPTS THE NORMAL OR CANONICAL FORM

$$(+) \quad \zeta' = [\alpha(\mu) - i\beta(\mu)]\zeta + [\gamma(\mu) + i\delta(\mu)]\zeta|\zeta|^2 + O(|\zeta|^4)$$

FOR SOME FUNCTIONS $\gamma(\mu)$ AND $\delta(\mu)$. WE DEFINE $m_2 \equiv \gamma(\mu) + i\delta(\mu)$

WE GIVE THE PROOF OF THIS RESULT BELOW. LET US FIRST APPLY THE RESULT.

LET $\zeta = Re^{i\phi}$. THEN SUBSTITUTING INTO (+) WE GET

$$R' + i\phi' R = (\alpha - i\beta)R + RE[m_2]R^3 + i\,IM(m_2)R^3.$$

THIS YIELDS THAT

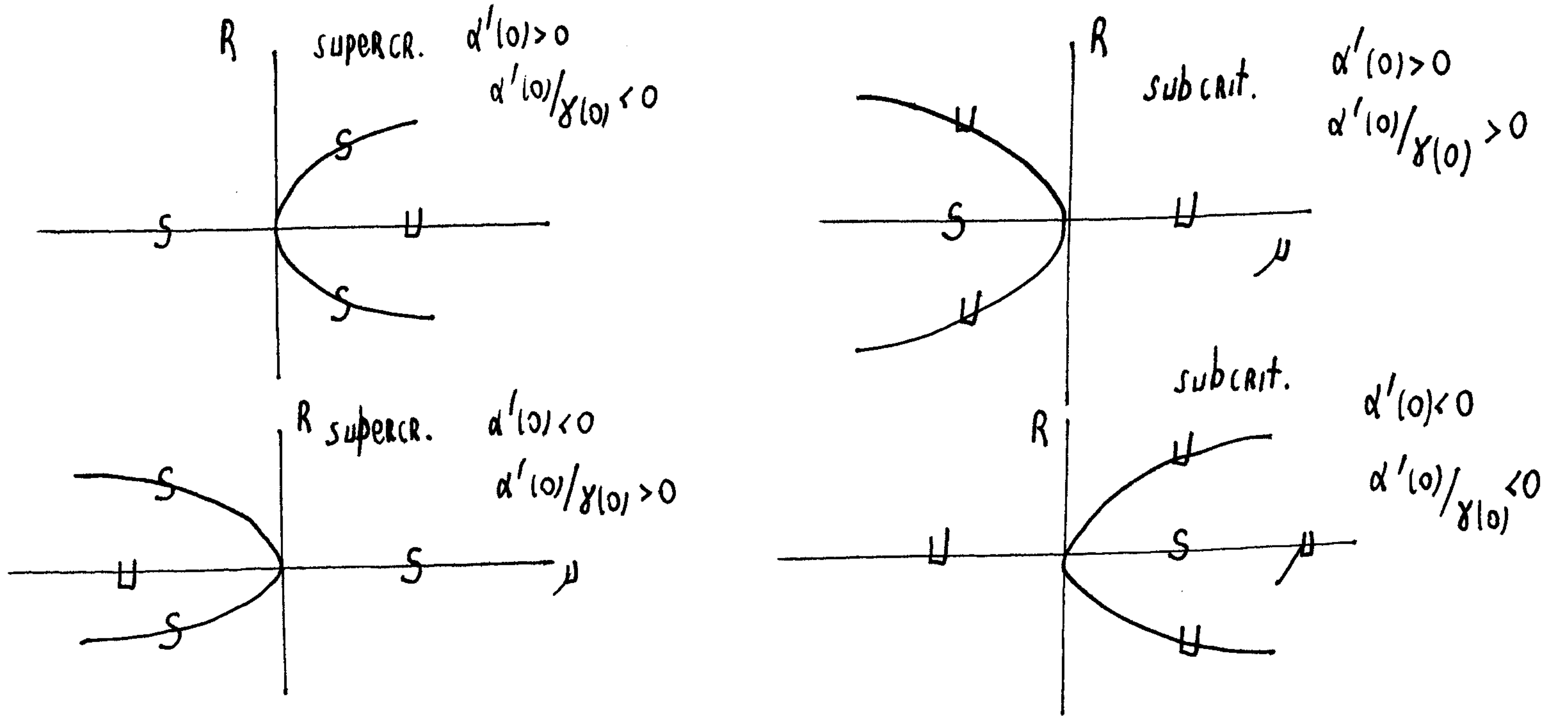
$$R' = \alpha R + RE[m_2]R^3$$

$$\phi' = -\beta + IM[m_2]R^2.$$

LET $\gamma = RE(m_2)$ AND ASSUME THAT $\gamma(0) \neq 0$. THEN NEAR $\mu = 0$ WE HAVE FROM $\alpha(0) = 0$ THAT THE EQUILIBRIA ARE $R = 0$ AND

$$R^2 = -\alpha(\mu)/\gamma(\mu) = -\frac{\alpha'(0)\mu}{\gamma(0)} + O(\mu^2).$$

THEREFORE, WE HAVE FOUR SEPARATE PICTURES AS SHOWN.



THUS IF $\alpha'(0) > 0$, we get supercritical bifurcation IF $\gamma(0) < 0$ AND subcritical IF $\gamma(0) > 0$.

NOTICE THAT THE FREQUENCY IS $T \approx 2\pi/\beta(0)$ locally AND THAT $R = O(\sqrt{|\mu|})$. WE CAN SHOW THAT THE TRANSFORMATION IS SUCH THAT

$$R = r + O(r^2) \quad \text{WHERE} \quad r = (x^2 + y^2)^{1/2}.$$

ALSO THAT $\operatorname{sign}(a) = \operatorname{sign}(\gamma(0))$.

WE PROCEED TO PROVE THE LEMMA. WE WILL NOT DERIVE THE EXPLICIT FORMULA FOR a TO AVOID TERRIBLY MESSY ALGEBRA (REFERENCES ARE AVAILABLE THOUGH.

WE FIRST WRITE (*) AS

$$z' = (\alpha - i\beta) z + N(z, z'; \mu)$$

WITH $$N(z, \bar{z}; \mu) = \frac{1}{2} n_1 z^2 + n_2 z\bar{z} + \frac{n_3}{2} \bar{z}^2 + O(|z|^3)$$

WE FIRST LET $W = z + Q(z, \bar{z}; \mu)$

WITH $$Q(z, \bar{z}; \mu) = \frac{1}{2} q_1 z^2 + q_2 z\bar{z} + \frac{1}{2} q_3 \bar{z}^2$$

AND CHOOSE g_1, g_2, g_3 TO ELIMINATE THE QUADRATIC TERMS IN THE SYSTEM FOR W.

THE INVERSE IS $\quad z = w - Q(w, \bar{w}; \mu) + O(|w|^3)$

WE NOW CALCULATE THAT

$$w' = z' + (g_1 z + g_2 \bar{z}) z' + (g_2 z + g_3 \bar{z}) \bar{z}'.$$

SUBSTITUTING WE OBTAIN,

$$w' = (\alpha - i\beta) z + \frac{1}{2} \Lambda_1 z^2 + \Lambda_2 z \bar{z} + \frac{1}{2} \Lambda_3 \bar{z}^2 + (g_1 z + g_2 \bar{z})(\alpha - i\beta) z$$
$$+ (g_2 z + g_3 \bar{z})(\alpha + i\beta) \bar{z} + O(|z|^3).$$

NOW USE THE INVERSE MAPPING:

$$w' = (\alpha - i\beta)\left(w - \frac{1}{2} g_1 w^2 - g_2 w \bar{w} - \frac{1}{2} g_3 \bar{w}^2 \right) + \frac{1}{2} \Lambda_1 w^2 + \Lambda_2 w \bar{w} + \frac{1}{2} \Lambda_3 \bar{w}^2$$
$$+ (g_1 w + g_2 \bar{w})(\alpha - i\beta) w + (g_2 w + g_3 \bar{w})(\alpha + i\beta) \bar{w} + O(|w|^3)$$

THIS CAN BE WRITTEN AS:

$$w' = (\alpha - i\beta) w + \frac{1}{2} \hat{\Lambda}_1 w^2 + \hat{\Lambda}_2 w \bar{w} + \frac{1}{2} \hat{\Lambda}_3 \bar{w}^2 + O(|w|^3).$$

WHERE

$$\hat{\Lambda}_1 = \Lambda_1 + (\alpha - i\beta) g_1$$

$$\hat{\Lambda}_2 = \Lambda_2 + (\alpha + i\beta) g_2$$

$$\hat{\Lambda}_3 = \Lambda_3 + (\alpha + 3i\beta) g_3$$

NOW SINCE AT THE BIFURCATION POINT $\mu = 0$ WE HAVE $\alpha(0) = 0$ BUT $\beta(0) \neq 0$, WE CAN CHOOSE g_1, g_2, g_3 TO SET $\hat{\Lambda}_i = 0$ FOR $i = 1, 2, 3$.

THIS DETERMINES $g_i = g_i(\mu)$ ANALYTIC.

AT THIS STAGE WE HAVE THAT

$$(+) \qquad w' = (\alpha - i\beta)w + M(w, \bar{w}; \mu)$$

WITH $M = O(|w|^3)$ AS $|w| \to 0$. HENCE, M HAS THE FORM

$$M(w, \bar{w}; \mu) = \frac{1}{3} m_1 w^3 + m_2 w^2 \bar{w} + m_3 w \bar{w}^2 + \frac{1}{3} m_4 \bar{w}^3 + O(|w|^4).$$

NEXT, WE INTRODUCE A SECOND NEAR IDENTITY TRANSFORMATION IN THE FORM

$$\zeta = w + R(w, \bar{w}; \mu)$$

WHERE
$$R(w, \bar{w}; \mu) = \frac{1}{3} \Gamma_1 w^3 + \Gamma_2 w^2 \bar{w} + \Gamma_3 w \bar{w}^2 + \frac{1}{3} \Gamma_4 \bar{w}^3.$$

WE WANT TO CHOOSE Γ_i TO ELIMINATE AS MANY CUBIC TERMS AS POSSIBLE. THE INVERSE IS

$$w = \zeta - R(\zeta, \bar{\zeta}; \mu) + O(|\zeta|^4).$$

WE THEN CALCULATE THAT

$$\zeta' = w' + (\Gamma_1 w^2 + 2\Gamma_2 w\bar{w} + \Gamma_3 \bar{w}^2) w' + (\Gamma_2 w^2 + 2\Gamma_3 w \bar{w} + \Gamma_4 \bar{w}^2) \bar{w}'.$$

NOW WE USE THE EQUATION

$$\zeta' = (\alpha - i\beta) w + \frac{1}{3} m_1 w^3 + m_2 w^2 \bar{w} + m_3 w \bar{w}^2 + \frac{1}{2} m_4 \bar{w}^3$$
$$+ (\Gamma_1 w^2 + 2\Gamma_2 w \bar{w} + \Gamma_3 \bar{w}^2)(\alpha - i\beta) w + (\Gamma_2 w^2 + 2\Gamma_3 w\bar{w} + \Gamma_4 \bar{w}^2)(\alpha + i\beta)\bar{w} + .$$

NOW USE THE INVERSE TRANSFORMATION TO GET

$$\zeta' = (\alpha - i\beta)\left[\zeta - \frac{1}{3}\Gamma_1 \zeta^3 - \Gamma_2 \zeta^2 \bar{\zeta} - \Gamma_3 \zeta \bar{\zeta}^2 - \frac{1}{3}\Gamma_4 \bar{\zeta}^3\right] + \frac{1}{3} m_1 \zeta^3 + m_2 \zeta^2 \bar{\zeta} + m_3 \zeta \bar{\zeta}^2 + \frac{1}{3} m_4 \bar{\zeta}$$
$$+ (\Gamma_1 \zeta^2 + 2\Gamma_2 \zeta\bar{\zeta} + \Gamma_3 \bar{\zeta}^2)(\alpha - i\beta)\zeta$$
$$+ (\Gamma_2 \zeta^2 + 2\Gamma_3 \zeta \bar{\zeta} + \Gamma_4 \bar{\zeta}^2)(\alpha + i\beta)\bar{\zeta} + \ldots$$

WE CAN THEN WRITE THIS SYSTEM AS

$$\zeta' = (\alpha - i\beta)\zeta + \frac{1}{3}\hat{m}_1 \zeta^3 + \hat{m}_2 \zeta^2 \bar{\zeta} + \hat{m}_3 \zeta \bar{\zeta}^2 + \frac{1}{3}\hat{m}_4 \bar{\zeta}^3 + O(|\zeta|^4).$$

HERE WE HAVE

$$\hat{m}_1 = m_1 + 2\Gamma_1(\alpha - i\beta)$$

$$\hat{m}_2 = m_2 + 2\alpha\Gamma_2$$

$$\hat{m}_3 = m_3 + 2(\alpha + i\beta)\Gamma_3$$

$$\hat{m}_4 = m_4 + 2(\alpha + 2i\beta)\Gamma_4$$

NOW SINCE $\alpha(0) = 0$, WE CAN CHOOSE Γ_1, Γ_3 AND Γ_4 TO ELIMINATE $\hat{m}_1 = \hat{m}_3 = \hat{m}_4 = 0$ (RECALL ALSO $\beta(0) > 0$).

HOWEVER, WE CANNOT ELIMINATE $\hat{m}_2$. THEREFORE WE SET $\Gamma_2 = 0$ AND THIS LEADS TO THE NORMAL FORM

$$\zeta' = (\alpha - i\beta)\zeta + m_2 |\zeta|^2 \zeta.$$

IF WE SET $m_2 = \gamma + i\delta$ we get the result FROM the LEMMA.

REMARK

(i) NOTICE THAT THE m_2 TERM IS PRESERVED IN THE SECOND NEAR IDENTITY TRANSFORMATION IF WE HAVE $\Gamma_2 = 0$. HENCE, PERHAPS WE CAN ISOLATE THE m_2 TERM AFTER ONE STAGE OF THE NEAR IDENTITY TRANSFORMATION.

(ii) DERIVING A FORMULA FOR m_2 IN TERMS OF f, g IS LENGTHY. THE SIGN OF $RE[m_2]$ DETERMINES THE LOCAL STRUCTURE.

(iv) CONSIDER THE ORIGINAL SYSTEM WITH

$$z' = (\alpha - i\beta) z + N(z, \bar{z}; \mu)$$

AND $$N(z,\bar{z};\mu) = \frac{1}{2}\Lambda_1 z^2 + \Lambda_2 z\bar{z} + \frac{1}{2}\Lambda_3 \bar{z}^2 + \frac{1}{3}\Lambda_4 z^3 + \Lambda_5 z^2\bar{z} + \Lambda_6 z\bar{z}^2 + \frac{1}{3}\Lambda_7 \bar{z}^3$$

$$+ \text{ quartic terms.}$$

notice that the linearized version has the solution

$$z = z_0 e^{i\omega_0 t} \quad \text{WITH} \quad \omega_0 = -\beta(0).$$

OUT OF ALL OF THE NONLINEAR TERMS WRITTEN FOR $N(z,\bar{z},\mu)$ (i.e. up to order of cubics), the only term with same time factor of $e^{i\omega_0 t}$ is the $z^2\bar{z}$ term. This term, written as $|z|^2 z$, is called the RESONANT TERM.

THE NEAR-identity transformations eliminate all the NON-RESONANT TERMS IN the calculation.

<u>EXAMPLE</u> CONSIDER THE ODE

$$x' = \mu x + y$$

$$y' = -x + \mu y - x^2 y$$

where the equilibrium point is $(0,0)$. The JACOBIAN J is

$$J = \begin{pmatrix} \mu & 1 \\ -1 & \mu \end{pmatrix}$$

SO THAT $\alpha(\mu) = \mu$ AND $\beta = 1$. SINCE $\alpha'(0) \neq 0$ AND $\alpha(0) = 0$, THERE IS A HOPF BIFURCATION AT $\mu = 0$ (NOTICE $a \neq 0$).

<u>NOTE</u>: $\mu < 0$ ORIGIN IS A STABLE SPIRAL POINT

$\mu > 0$ ORIGIN IS UNSTABLE SPIRAL.

NOW WE LET $z = x + iy$ SO THAT

$$z' = (\mu - i)\, z - i x^2 y$$

BUT $x = \frac{1}{2}(z + \bar{z})$ AND $y = \frac{-i}{2}(z - \bar{z})$. THIS YIELDS THAT

$$z' = (\mu - i)\, z - \frac{1}{8}\left(z^3 + z^2 \bar{z} - z \bar{z}^2 - \bar{z}^3 \right)$$

SINCE THIS IS A CUBIC ALREADY, ALL WE NEED IS THE SECOND IDENTITY TRANSFORMATION IN THE FORM

$$\zeta = z + R(z, \bar{z}; \mu)$$

WITH $\quad R(z, \bar{z}; \mu) = \frac{1}{3} \Gamma_1 z^3 + \Gamma_3 z \bar{z}^2 + \frac{1}{3} \Gamma_4 \bar{z}^3 .$

THIS WILL LEAVE THE COEFFICIENT OF $z^2 \bar{z}$ INVARIANT.

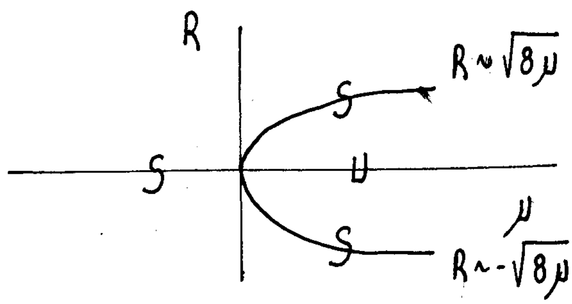
THUS THE NORMAL FORM IS SIMPLY

$$\zeta' = (\mu - i)\, \zeta - \frac{1}{8} |\zeta|^2 \zeta \qquad M_2 = -1/8$$

SINCE $\zeta = R e^{i\varphi}$, we get taking Real parts that

$$R' = \mu R - \frac{1}{8} R^3 .$$

supercritical HOPF BIFURCATION AT $\mu = 0$.



EXAMPLES OF HOPF BIFURCATIONS

NOT ALL PROBLEMS ARE IMMEDIATELY CAST IN THE FORM

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \alpha(\mu) & B(\mu) \\ -B(\mu) & \alpha(\mu) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} F \\ G \end{pmatrix}$$

TO BEGIN WITH. HENCE, OTHER TECHNIQUES CAN DETECT HOPF BIFURCATION POINTS

WE CONSIDER A FEW EXAMPLES

EXAMPLE 1 THE SCHNAKENBURG MODEL IS

$$\begin{aligned} x' &= a - x + x^2 y = f(x,y) \qquad a, b > 0 \qquad x, y \geq 0. \\ y' &= b - x^2 y = g(x,y) \end{aligned}$$

THE EQUILIBRIUM POINTS SATISFY $x^2 y = b$ AND $a - x + x^2 y = 0$. THIS YIELDS THAT

$$(x_E, y_E) = \left(a+b, \frac{b}{(a+b)^2}\right)$$

NOW WE LINEARIZE AROUND THE EQUILIBRIUM POINT TO GET THE JACOBIAN

$$J = \begin{pmatrix} -1+2xy & x^2 \\ -2xy & -x^2 \end{pmatrix} = \begin{pmatrix} -1+2b/(a+b) & (a+b)^2 \\ -2b/(a+b) & -(a+b)^2 \end{pmatrix}$$

A SIMPLE CALCULATION YIELDS THAT

$$\det J = -\frac{(b-a)}{(a+b)}(a+b)^2 + 2b(a+b) = (a+b)(2b-(b-a)) = (a+b)^2$$

$$\operatorname{trace} J = \frac{b-a}{b+a} - (b+a)^2 = \frac{(b-a)-(b+a)^3}{(b+a)}$$

RECALL THAT $\det J = \lambda_1 \lambda_2$ $\quad \operatorname{trace} J = \lambda_1 + \lambda_2$.

IF $\lambda_1 = i\nu$ AND $\lambda_2 = -i\nu$ AND we have strict transversal crossing then we have a HOPF BIFURCATION. this will occur when $\det J > 0$ AND $\text{trace } J = 0$ (together with crossing condition).

THE EQUILIBRIUM IS UNSTABLE WHEN trace $J > 0 \longrightarrow (b-a) > (b+a)^3$
" " " STABLE WHEN trace $J < 0 \longrightarrow (b-a) < (b+a)^3$.

NOW when $(b-a) = (a+b)^2$ we have $\lambda_{\pm} = i(a+b)$ AND A STRICT TRANSVERSALITY OCCURS. THUS, WE HAVE A HOPF BIFURCATION AND A PERIODIC SOLUTION AT LEAST NEAR THE BOUNDARY WHERE trace $J = 0$

NOW WE CAN PLOT THE STABILITY REGION IN THE (a,b) PLANE.

WE WRITE $b - a = (b+a)^3$ SO THAT

$$b = \frac{(b+a)^3 + (b+a)}{2}$$

WE LABEL $b + a = X_E$ SO THAT $b = \frac{X_E^3}{2} + \frac{X_E}{2}$ AND $a = X_E - b$.

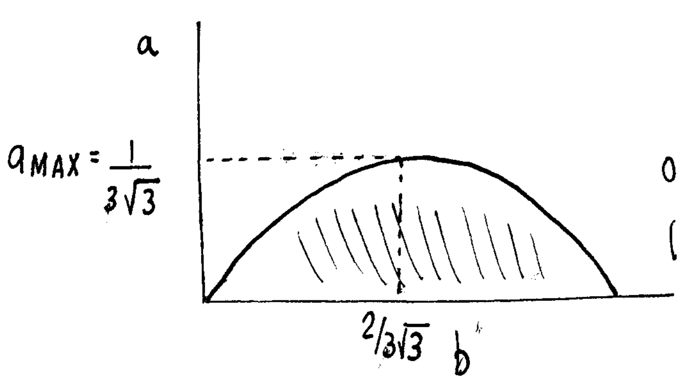
THUS, $a = X_E - \frac{X_E}{2} - \frac{X_E^3}{2}$, SO THAT THE CURVE IN PARAMETRIC FORM IS

$$b = \frac{X_E^3}{2} + \frac{X_E}{2}, \quad a = \frac{X_E}{2}(1 - X_E^2) \qquad 0 < X_E < 1.$$

NOW $da/dX_E = 0$ WHEN $\frac{1}{2} - \frac{3X_E^2}{2} = 0$ SO THAT $X_E = 1/\sqrt{3}$.

THIS YIELDS THAT $a_{MAX} = 1/3\sqrt{3}$ WITH $b = \frac{2}{3\sqrt{3}}$.

HENCE, WE GET A CURVE AS SHOWN



IN shaded region it turns out that we get periodic solutions (since we can construct trapping region). HB theorem guarantees periodic solutions near where trace $J = 0$.

EXAMPLE 2 (GLYCOLYSIS OSCILLATIONS)

CONSIDER THE SYSTEM

$$x' = -x + ay + x^2 y = f(x,y), \quad x, y \geq 0$$
$$y' = b - ay - x^2 y = g(x,y), \quad a, b > 0.$$

THE EQUILIBRIA ARE SIMPLY $\quad y = \dfrac{x}{a+x^2} \qquad y = \dfrac{b}{a+x^2}$

HENCE $\quad x_E = b \quad$ AND $\quad y_E = b/(a+b^2)^2$

NOW WE CALCULATE THE JACOBIAN AS

$$J = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} = \begin{pmatrix} -1 + 2xy & a + x^2 \\ -2xy & -(a+x^2) \end{pmatrix}$$

SO THAT AT THE EQUILIBRIUM POINT WE HAVE

$$J = \begin{pmatrix} \dfrac{b^2-a}{b^2+a} & b^2 + a \\ -2b^2/b^2+a & -(a+b^2) \end{pmatrix}$$

WE THEN CALCULATE THAT $\quad \det J = a - b^2 + 2b^2 = a + b^2 > 0$

$$\text{trace } J = \frac{(b^2-a) - (b^2+a)^2}{b^2+a}.$$

NOW WE GET THAT THE EQUILIBRIUM POINT IS UNSTABLE IF trace $J > 0$ AND IS STABLE IF trace $J < 0$. WHEN TRACE $J = 0$ WE HAVE

$$\lambda_{\pm} = \pm i [\det J]^{1/2}.$$

NOW THE HB theorem says that a periodic solution is born along the curve trace $J = 0$.

NOW WE FIND where trace $J > 0$. WE HAVE THAT

$$(b^2 - a) > (b^2 + a)^2.$$

WE LET $\varsigma = b^2$ SO THAT

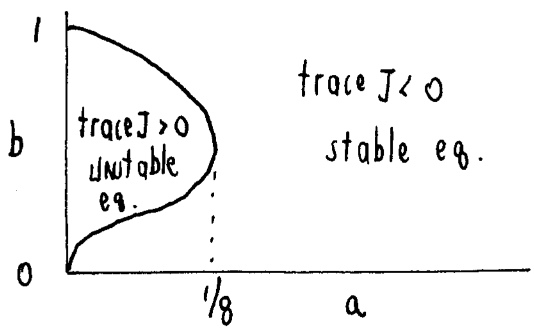
$$\varsigma^2 + (2a-1)\varsigma + a^2 + a < 0$$

SOLVING WITH EQUALITY WE GET

$$\varsigma = \frac{1-2a}{2} \pm \frac{1}{2}\sqrt{(2a-1)^2 - 4(a^2+a)}$$

THIS YIELDS THAT $$b^2 = \frac{(1-2a)}{2} \pm \frac{1}{2}\sqrt{1-8a} \qquad 0 < a < 1/8 .$$

THE REGION IS SHOWN BELOW



A HOPF BIFURCATION IS GUARANTEED TO OCCUR ALONG THE BOUNDARY trace $J = 0$.

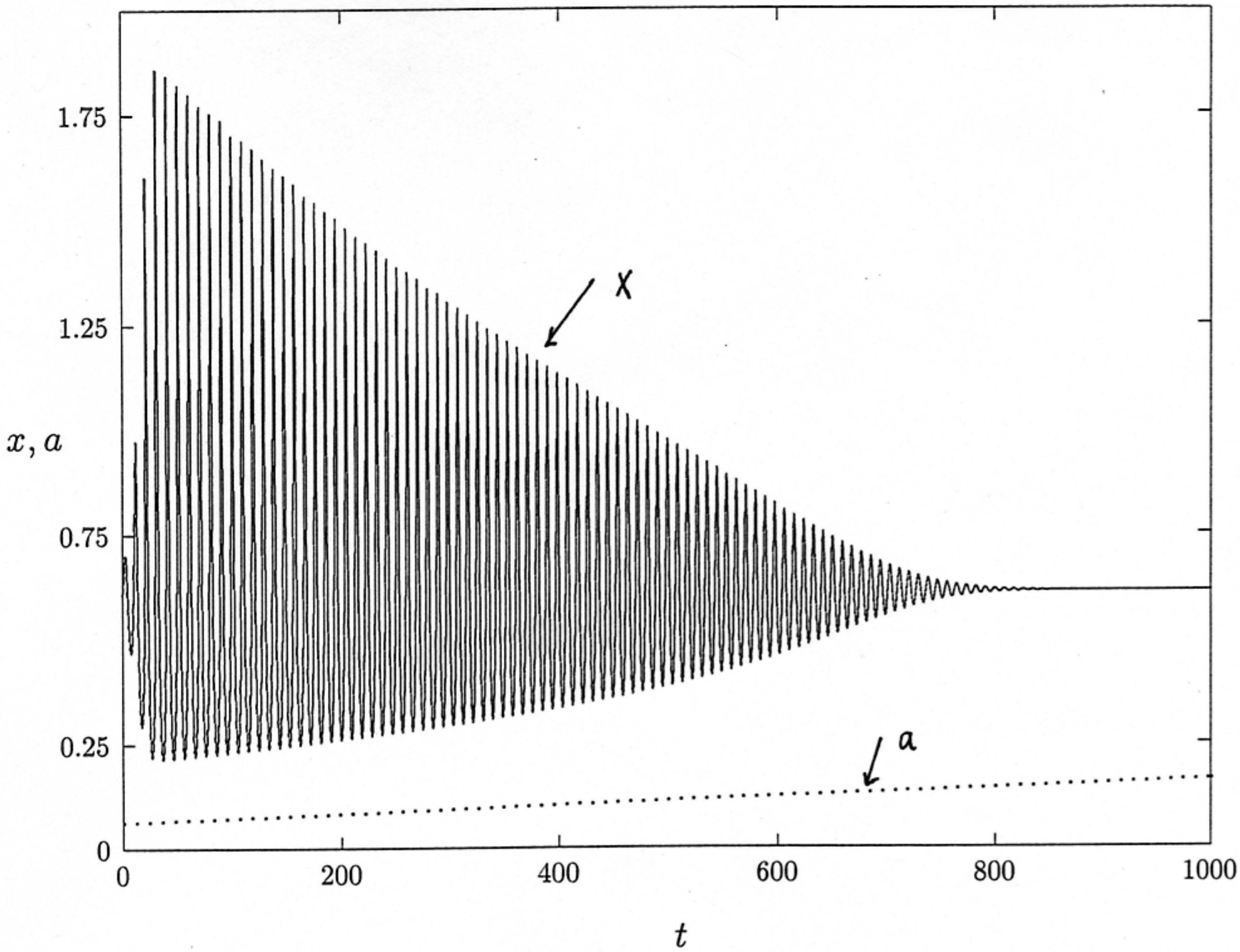
REMARKS

(i) CONSIDER A NUMERICAL EXPERIMENT WITH

$$b = \sqrt{3/8} \qquad \text{AND} \qquad a = \frac{1}{16} + \varepsilon t \qquad \varepsilon = .0001$$

WE FIND THAT THERE IS A PERIODIC SOLUTION EVEN STRICTLY INSIDE THE REGION TRACE $J > 0$ WHICH THEN DIES OUT ONCE a BECOMES TOO LARGE. BASED ON STEADY-STATE ANALYSIS WE PREDICT NO OSCILLATIONS WHEN $a > 1/8$. HOWEVER, THERE MAY BE A SLIGHT DELAY DUE TO SLOWLY VARYING CONTROL PARAMETER (see Figure on next page).

EXAMPLE 1



$$x' = -x + ay + x^2 y \qquad x(0) = .60$$

$$y' = b - ay - x^2 y \qquad y(0) = 1.5$$

$$b = \sqrt{3/8} \qquad a = \frac{1}{16} + \varepsilon t \qquad \varepsilon = .0001$$

theory predicts NO Oscillations after $a > 1/8$

(possible delay in bifurcation)

TO PROVE A MORE GLOBAL RESULT ABOUT HOPF BIFURCATIONS WE APPEAL TO THE POINCARÉ-BENDIXSON THEOREM:

<u>THEOREM</u>: (POINCARE-BENDIXSON). SUPPOSE THAT

$$\underline{x}' = f(\underline{x}) \quad \text{IN} \quad \mathbb{R}^2$$

AND

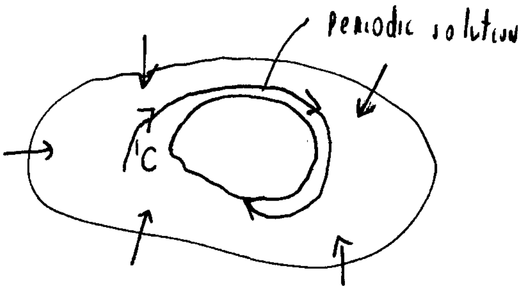
(i) D IS A CLOSED BOUNDED REGION IN $\mathbb{R}^2$

(ii) $f(\underline{x})$ IS SMOOTH ON D

(iii) D CONTAINS NO EQUILIBRIUM POINTS OF $f(\underline{x})$

(iv) D IS A TRAPPING REGION, SO THAT ALL TRAJECTORIES ARE CONFINED IN D.

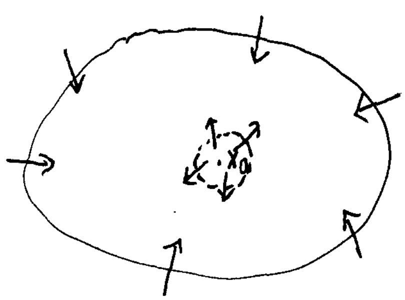
THEN D CONTAINS A PERIODIC SOLUTION AND ANY TRAJECTORY C CONFINED IN D WILL TEND TO A PERIODIC SOLUTION.



<u>PROOF</u>: ANY ADVANCED ODE BOOK

<u>REMARKS</u> CRITICAL THAT WE HAVE $\mathbb{R}^2$. HENCE NO CHAOS IN $\mathbb{R}^2$.

TO APPLY THIS RESULT, SUPPOSE THAT $\underline{x}' = f(\underline{x})$ HAS AN EQUILIBRIUM POINT THAT IS UNIQUE AND IS A REPELLOR, AND THAT WE CAN FIND A REGION WHERE FLOW IS INWARDS SUCH AS SHOWN



$f(\underline{x}_0) = 0$, $\quad \text{trace } J(x_0) > 0$

$\det J(x_0) > 0$

let D be a donught-shaped region with flow as shown. Then by the PB theorem, $\exists$ a periodic solution inside the donught.

Hence, for our problem we must find a region where flow is inwards to get existence of periodic solution when $\text{trace } J > 0$, $\det J > 0$.