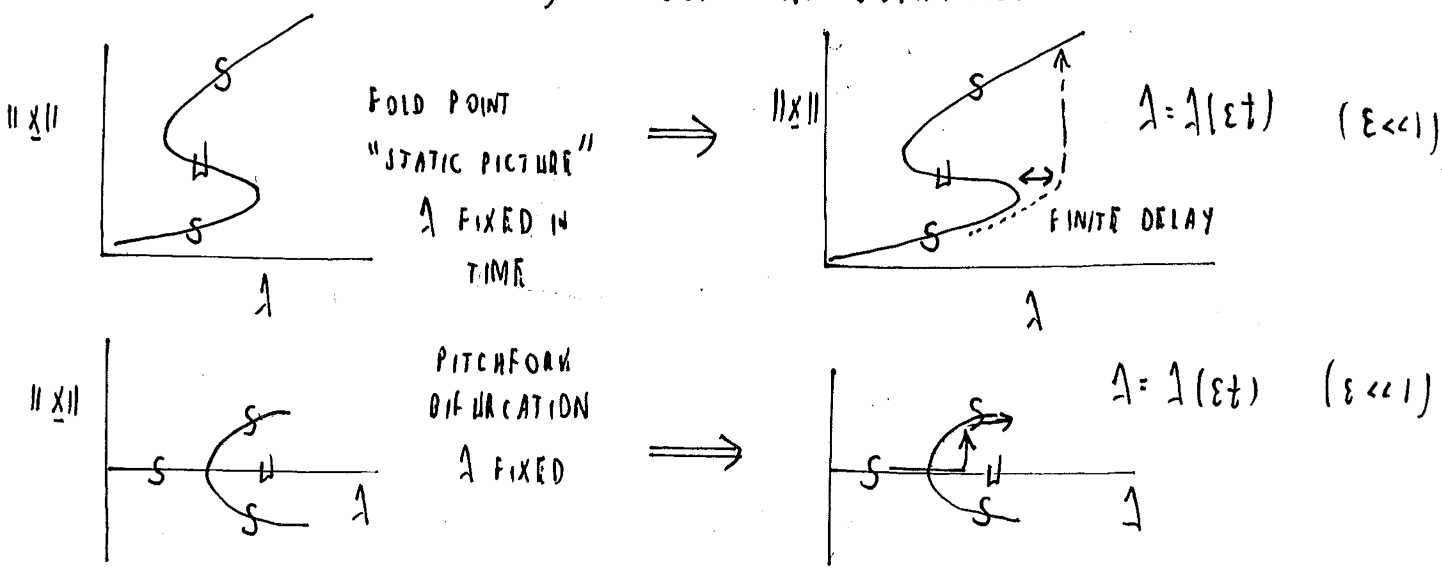
# DELAYED BIFURCATION

WE WILL CONSIDER ODE MODELS WITH A PARAMETER $\lambda$ OF THE FORM

$$(*) \quad \frac{d\underline{x}}{dt} = F(\underline{x}, \lambda)$$

WHERE FOR SOME VALUES OF $\lambda$ THERE IS A BIFURCATION POINT OF EQUILIBRIA FOR $(*)$. WE THEN ASK HOW DOES THE BEHAVIOR OF SOLUTIONS CHANGE NEAR THE BIFURCATION POINT IF $\lambda$ CROSSES SLOWLY THROUGH THE BIFURCATION POINT IN TIME. FOR INSTANCE, CONSIDER TWO SCENARIOS




WE WILL SHOW THAT A SLOW RAMPING THROUGH THE BIFURCATION POINT LEADS TO A "DELAY" IN THE BIFURCATION. THE QUALITATIVE CHANGE IN THE SOLUTION OCCURS NOT AT THE STATIC BIFURCATION POINT BUT INSTEAD AT SOME POINT <u>BEYOND</u> THE STATIC VALUE.

WE WILL ANALYZE DIFFERENT TYPES OF BIFURCATIONS WITH THIS SLOW SWEEP,

(i) TRANSCRITICAL

(ii) PITCHFORK

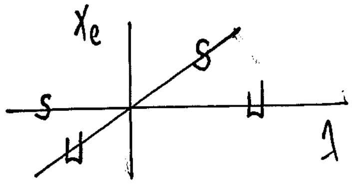
(iii) FOLD POINT

(iv) HOPF BIFURCATION

## TRANSCRITICAL BIFURCATION

CONSIDER $\quad x' = x[\lambda - x]$.

IF $\lambda$ IS A CONSTANT PARAMETER WE GET




$\lambda = 0$ IS TRANSCRITICAL BIFURCATION POINT.

WE NOW LET $\lambda = \lambda(\varepsilon t)$ WITH $\lambda(0) = \lambda_i < 0$. LET $\tau = \varepsilon t$ AND $\dot{x} = d/d\tau$.

WE GET

$$\varepsilon \dot{x} = x[\lambda(\tau) - x], \quad \text{WITH} \quad x(0) = x_0 \neq 0$$

NOW PUT $x = 1/u$ SO THAT $\dfrac{-\varepsilon \dot{u}}{u^2} = \dfrac{\lambda}{u} - \dfrac{1}{u^2}$. THIS YIELDS

$$\varepsilon \dot{u} + \lambda u = 1. \longrightarrow \quad \dot{u} + \frac{\lambda(\tau)}{\varepsilon} u = \frac{1}{\varepsilon} \quad \text{WITH} \quad u(0) = \frac{1}{x_0}.$$

NOW MULTIPLY BY THE INTEGRATING FACTOR $\Phi = e^{\frac{1}{\varepsilon}\int_0^\tau \lambda(s)\,ds} = \Phi(\tau)$

$$\left( u e^{\frac{1}{\varepsilon}\int_0^\tau \lambda(s)\,ds} \right)' = \frac{1}{\varepsilon} e^{\frac{1}{\varepsilon}\int_0^\tau \lambda(s)\,ds}.$$

WE GET $\quad u\,\Phi(\tau) = \dfrac{1}{\varepsilon}\displaystyle\int_0^\tau \Phi(\Lambda)\,d\Lambda + \dfrac{1}{x_0}.$

THIS YIELDS $\quad u = \dfrac{1}{\Phi(\tau)}\left[ \dfrac{1}{\varepsilon}\displaystyle\int_0^\tau \Phi(\Lambda)\,d\Lambda + \dfrac{1}{x_0} \right]$

OR

$$(1) \quad \left\{ \; x = \frac{x_0 \Phi(\tau)}{\frac{x_0}{\varepsilon}\int_0^\tau \Phi(\Lambda)\,d\Lambda + 1} \quad \text{WITH} \quad \Phi(\Lambda) = \exp\left[ \frac{1}{\varepsilon}\int_0^\Lambda \lambda(s)\,ds \right]. \right.$$

THIS IS THE EXACT SOLUTION. WHAT IS ITS BEHAVIOR FOR $0 < \varepsilon \ll 1$ AND $\lambda(0) = \lambda_i < 0$?

WE SUPPOSE THAT $\lambda(\tau) = \lambda_i + \tau$ SO THAT $\lambda$ IS A LINEAR RAMP THAT CROSSES THROUGH THE BIFURCATION POINT AT TIME $\tau = -\lambda_i$.

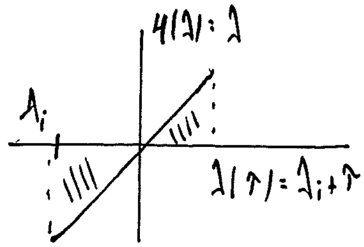
NOW WE DEFINE $\Phi(\tau) = \int_0^\tau \lambda(s)\,ds$, SO THAT $f(\tau) = \exp\left[\frac{1}{\varepsilon}\Phi(\tau)\right]$.

WE OBSERVE THAT IF $\Phi(\tau) < 0 \rightarrow f(\tau)$ IS EXPONENTIALLY SMALL AS $\varepsilon \to 0^+$.

AND $\int_0^\tau f(\eta)\,d\eta$ IS EXPONENTIALLY SMALL

IF $\Phi(\eta) < 0$ ON $0 < \eta < \tau$.

NOW
$$\Phi(\tau) = \int_{\lambda(0)}^{\lambda(\tau)} \lambda \frac{ds}{d\lambda}\,d\lambda = \int_{\lambda(0)}^{\lambda(\tau)} \lambda\,d\lambda = \int_{\lambda_i}^{\lambda(\tau)} \lambda\,d\lambda.$$




WE CLAIM THAT $\Phi(\tau) < 0$ FOR

$\lambda(0) < \lambda < -\lambda_i$; $\quad (-\lambda_i > 0)$

I.E. $0 < \tau < -2\lambda_i$

THIS YIELDS THAT $x \triangleq 0$ FOR $0 < \tau < -2\lambda_i$. (stage 2).

NOW SUPPOSE $\tau > -2\lambda_i$ OR $\lambda(\tau) > -\lambda_i$. THEN $\Phi(\tau) > 0$ AND

SO $f(\tau)$ IS EXPONENTIALLY LARGE.

BY LAPLACE'S METHOD:
$$\int_0^\tau f(\eta)\,d\eta = \int_0^\tau \exp\left[\frac{1}{\varepsilon}\int_0^\eta \lambda(s)\,ds\right] d\eta.$$

THE DOMINANT CONTRIBUTION TO THE INTEGRAL COMES FROM UPPER ENDPOINT. RECALLING THAT FOR $X \to +\infty$
$$\int_a^b g(\eta)\, e^{X f(\eta)}\,d\eta \sim \frac{g(\eta)}{X f'(\eta)}\, e^{X f(\eta)}\Big|_{\eta=b} + O\left(\frac{1}{X^2}\right) \quad \text{AS } X \to \infty$$

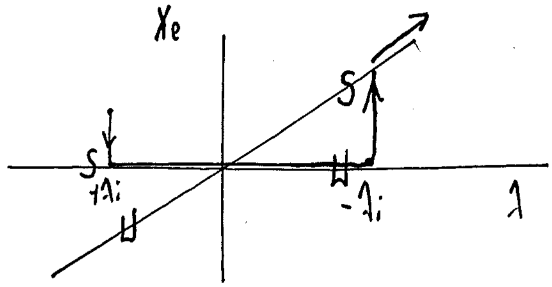
WHEN $f$ HAS ITS MAXIMUM AT $X = b$ WE HAVE THAT
$$\left.\begin{aligned}\int_0^\tau f(\eta)\,d\eta = \int_0^\tau \exp\left[\frac{1}{\varepsilon}\int_0^\eta \lambda(s)\,ds\right] d\eta \sim \frac{\varepsilon}{\lambda(\eta)} \exp\left[\frac{1}{\varepsilon}\int_0^\eta \lambda(s)\,ds\right]\Bigg|_{\eta=\tau} = \frac{\varepsilon}{\lambda(\tau)} f(\tau)\end{aligned}\right\} (2)$$

THEREFORE WE HAVE FROM (1) THAT FOR $\tau > 0$

$$x \simeq \left.\frac{x_0 \, F(\tau)}{\frac{x_0}{\varepsilon}\,\frac{\varepsilon}{\lambda(\tau)}\, F(\tau) + 1}\right\} \qquad (3)$$

HENCE IF $F(\tau)$ IS EXPONENTIALLY LARGE, I.E. IF $\lambda > -\lambda_i$ WE HAVE THAT (3) BECOMES $x \simeq \lambda(\tau)$.

THIS LEADS TO THE FOLLOWING PICTURE FOR A SLOW RAMP THROUGH THE BIFURCATION POINT.




NOTICE THAT FOR $\varepsilon \ll 1$ THE JUMP OCCURS AT $\lambda = -\lambda_i > 0$ WHICH IS AT $\tau = -2\lambda_i$ OR $t = -2\lambda_i/\varepsilon$ AND <u>NOT</u> AT "STATIC" BIFURCATION TIME $t_s = -\lambda_i/\varepsilon$. THERE IS A FINITE $\varepsilon$-INDEPENDENT DELAY IN THE VALUE OF $\lambda$ AT WHICH THE JUMP OCCURS.

<u>REMARK</u> THE KEY QUANTITY WHICH MONITORS WHETHER $F(\tau)$ IS EXPONENTIALLY SMALL OR LARGE AS $\varepsilon \to 0$ IS THE TERM

$$\Phi(\tau) = \int_0^\tau \lambda(s)\, ds = \int_{\lambda_i}^{\lambda(\tau)} \lambda \, d\lambda$$

IF $\Phi > 0 \longrightarrow F(\tau) = \exp\left[\frac{1}{\varepsilon}\int_0^\tau \lambda(s)\, ds\right]$ IS EXPONENTIALLY LARGE AND THE JUMP HAS ALREADY OCCURED.

THE DELAYED BIFURCATION VALUE IS THE VALUE $\lambda_F = \lambda(\tau_F)$ FOR WHICH $\int_{\lambda_i}^{\lambda_F} \lambda \, d\lambda = 0$ "EQUAL-AREA" RULE

NOW WITH THIS INSIGHT WE REVISIT

$$\varepsilon \dot{x} = x[\lambda(\tau) - x] \quad \text{WITH} \quad x(0) = x_0.$$

- FOR $\tau = O(\varepsilon)$ WE HAVE A QUICK TRANSITION FROM $x_0$ TO $0$. (stage I)
- FOR $\tau = O(1)$ WE HAVE STAGE II WHERE $x$ IS NEAR $x=0$. WE LINEARIZE

TO GET $\varepsilon \dot{x} \hat{=} x[\lambda(\tau)]$

$$\text{SO THAT} \quad x = c\exp\left[\frac{1}{\varepsilon}\int_0^\tau \lambda(s)\,ds\right].$$

WE HAVE THAT $x \ll 1$ AS $\varepsilon \to 0^+$ ONLY WHEN $\Phi(\tau) \equiv \int_0^\tau \lambda(s)\,ds < 0$.

THE LINEARIZATION IS INCONSISTENT WHEN $\tau > \tau_F$ WHERE $\Phi(\tau_F) = 0$.

<u>DELAYED BIFURCATION TIME</u> IS $\tau_F$ WITH $\int_0^{\tau_F} \lambda(s)\,ds = 0$.

LET $\lambda = \lambda_i + s$ WE GET $\tau_F = -2\lambda_i$, WITH $\lambda_i < 0$.

<u>REMARK</u> AS AN EXACTLY SOLVABLE EXAMPLE CONSIDER

$$x' = (-1 + \varepsilon t)x \quad \text{WITH} \quad x(0) = x_0$$

WHERE THE "EIGENVALUE" IS $\lambda(\varepsilon t) = -1 + \varepsilon t$. WE MIGHT NAIVELY EXPECT THAT WHEN $\lambda \geq 0$, i.e. WHEN $t \geq 1/\varepsilon$ THAT THE SOLUTION BEGINS TO GROW EXPONENTIALLY.

HOWEVER THE EXACT SOLUTION IS

$$x(t) = x_0 \exp\left[\frac{(\varepsilon t - 1)^2 - 1}{2\varepsilon}\right]$$

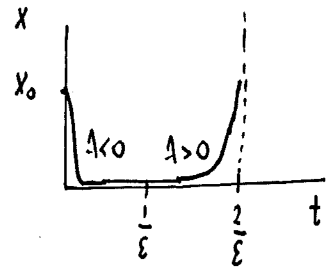
NOTICE THAT FOR $(\varepsilon t - 1)^2 - 1 > 0$ THE SOLUTION GROWS EXPONENTIALLY. THE ONSET OF THIS IS $t = \frac{2}{\varepsilon}$ AND NOT $t = 1/\varepsilon$ AS NAIVELY OBTAINED BY THE SIGN OF THE EIGENVALUE.

IF $(\varepsilon t - 1)^2 - 1 < 0$ THE SOLUTION DECAYS EXPONENTIALLY. THIS OCCURS FOR $O(1) \ll t < 2/\varepsilon$.

NOTICE THAT $\lambda < 0$ IF $t < 1/\varepsilon$

$\lambda > 0$ IF $t > 1/\varepsilon$

WE THEN GET A PICTURE AS SHOWN




NOW FROM A LINEARIZED ANALYSIS CAN WE PREDICT THE TIME FOR EXPONENTIAL GROWTH? SUPPOSE $x \ll 1$. THEN

$$x' \doteq \lambda(\varepsilon t)\, x \qquad\qquad \lambda < 0 \text{ IF } \tau < 1$$

WE LET $\tau = \varepsilon t$ SO THAT $\varepsilon \dot{x} = \lambda(\tau)\, x \qquad \lambda > 0$ IF $\tau > 1$

WE SOLVE TO OBTAIN $x \doteq x_0 \exp\left[\frac{1}{\varepsilon}\Phi(\tau)\right]$

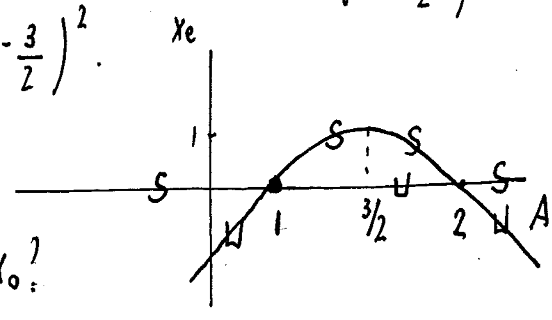
WITH $\Phi(\tau) = \int_0^\tau \lambda(s)\, ds. \qquad \lambda(s) = -1 + s$

NOW IF $\int_0^{\tau_F} \lambda(s)\, ds = 0$ THEN $x$ WILL BEGIN TO GROW EXPONENTIALLY. WE CALCULATE $\int_0^{\tau_F} (-1+s)\, ds = -\tau_F + \frac{\tau_F^2}{2} = 0$ SO $\tau_F = 2$.

$$\rightarrow\ t = \frac{\tau_F}{\varepsilon} = \frac{2}{\varepsilon}.$$

<u>REMARK 1</u> SUPPOSE $x' = -x\left[x - f(A)\right]$ WITH $f(A) = 1 - 4\left(A - \frac{3}{2}\right)^2$.

THE EQUILIBRIA ARE $x_e = 0$ AND $x_e = 1 - 4\left(A - \frac{3}{2}\right)^2$.




<u>Q1</u>: SUPPOSE $A(0) = A_i < 1$. WHEN WILL THE SOLUTION JUMP TO THE NONTRIVIAL BRANCH IF $A(t) = A_i + \varepsilon t$ AND $x(0) = x_0$?

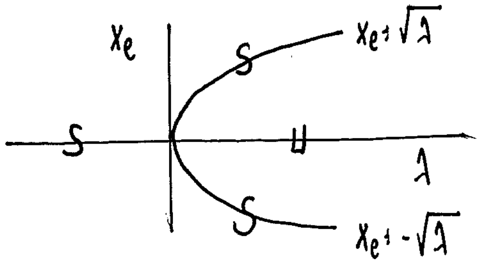
<u>Q2</u>: CAN WE TAKE $A_i$ SUFFICIENTLY NEGATIVE TO "JUMP" ACROSS THE UNSTABLE ZONE $1 < A < 2$ WITH A RAMP $A = A_i + \varepsilon t$?

REMARK 2 (PITCHFORK BIFURCATION)

CONSIDER $x' = \lambda(\varepsilon t)\, x - x^3$

WITH $\lambda$ FIXED WE HAVE THE EQUILIBRIA $x_e = 0$ AND $x_e = \pm\sqrt{\lambda}$.

THIS YIELDS THE BIFURCATION DIAGRAM WITH $\lambda$ CONSTANT.




NOW SUPPOSE $\lambda(\varepsilon t)$ OSCILLATES SLOWLY ACROSS THE PITCHFORK BIFURCATION VALUE $\lambda = 0$. WHAT TYPE OF DYNAMICS WILL OCCUR?

EXAMPLE CONSIDER THE ODE

$$x' = x\,[-1 + A f(x)] \quad \text{WITH} \quad f(0) = 1 \quad \text{AND} \quad x > 0.$$

SUPPOSE THAT $A = A(\varepsilon t)$ WITH $\tau = \varepsilon t$ AND $A(0) < 1$, $\varepsilon \ll 1$.

(i) EQUILIBRIA ARE $x_e = 0$ AND $A f(x_e) = 1$ SO $A = 1$ IS A TRANSCRITICAL BIFURCATION POINT.

(ii) FOR $x \cong 0$, $f(0) = 1$ AND $x' = (A-1)x$.

THUS FOR $A$ CONSTANT AND $A > 1 \rightarrow x = 0$ UNSTABLE

$A < 1 \rightarrow x = 0$ STABLE.

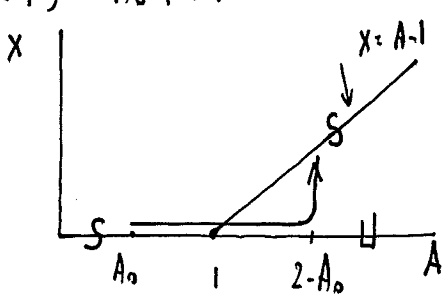
NOW LET $A(\tau) = A_0 + \tau$, WITH $A_0 < 1$ AND $\tau = \varepsilon t$. THE LINEARIZED PROBLEM IS $x' \cong [(A_0 - 1) + \tau]\, x$ IS $\frac{dx}{x} = [A_0 - 1 + \varepsilon t]\, dt$ SO THAT

$$\ln\left(\frac{x}{x_0}\right) = [A_0 - 1]\, t + \varepsilon t^2/2 \longrightarrow x = x_0 \exp\left[\left((A_0 - 1) + \frac{\varepsilon t}{2}\right) t\right].$$

INSTABILITY OCCURS WHEN $t = t^*$ WHERE $A_0 - 1 + \varepsilon t^*/2 = 0$ SO THAT

$$t^* = \frac{2}{\varepsilon}(1 - A_0) \quad \text{AND} \quad A(t^*) = A_0 + \varepsilon t^* = A_0 + \varepsilon \frac{2}{\varepsilon}(1 - A_0) = 2 - A_0.$$

IF $A_0 < 1$, $A_0(t^*) = 2 - A_0 > 1$. NOTICE THAT $A_0(t^*)$ INDEPENDENT OF $\varepsilon$




LET $F(x) = \frac{1}{x+1} \rightarrow x = -1 + A$ EQUILIBRIA

# SLOW PASSAGE THROUGH A FOLD POINT

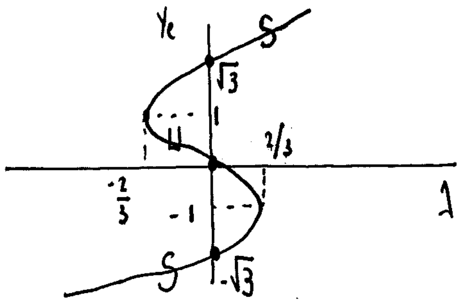
WE NOW CONSIDER THE CASE OF A <u>SADDLE-NODE</u> BIFURCATION POINT.

WE CONSIDER THE SCALAR PROBLEM

$$y' = f(y, \lambda)$$

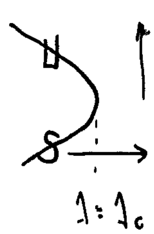
FOR INSTANCE IF $f(y,\lambda) = y - \frac{y^3}{3} + \lambda$. EQUILIBRIA ARE $\lambda = -y_e + y_e^3/3$.

THE PLOT IS AS SHOWN. NOW $d\lambda/dy = 0$ AT $y_e^2 = 1$ SO $y_e = \pm 1 \rightarrow \lambda = \mp \frac{2}{3}$




WE HAVE A FOLD POINT AT $y_e = 1$, $\lambda = -2/3$ AND AT $y_e = -1$, $\lambda = 2/3$.

WE DEFINE $\lambda_c = 2/3$, $y_c = -1$. WE SUPPOSE THAT $\lambda = \lambda_c + \varepsilon t$ SO THAT $\lambda$ SLOWLY INCREASES <u>BEYOND</u> THE SADDLE-NODE POINT. WHEN DOES THE TRANSITION TO THE UPPER BRANCH OCCUR. A ZOOM OF REGION NEAR SADDLE-NODE POINT IS




NOW WE LET $y = y_c + \varepsilon^p y_1 + \cdots$ AND $t = \varepsilon^q \tau$. WE SUBSTITUTE TO GET

$$y' = f(y,\lambda) \cong f^0 + f_y^0 (y - y_c) + \frac{f_{yy}^0}{2}(y-y_c)^2 + f_\lambda^0 (\lambda - \lambda_c) + \cdots$$

HERE $f^0 \equiv f(y_c, \lambda_c)$, $f_{yy}^0 \equiv f_{yy}(y_c, \lambda_c)$.

NOW AT THE FOLD POINT WE HAVE $f^0 = f_y^0 = 0$. THUS WE GET

$$y' \cong \frac{f_{yy}^0}{2}(y-y_c)^2 + f_\lambda^0 (\lambda - \lambda_c) + \cdots$$

NOW $$\frac{dy}{dt} = \varepsilon^{p-q} \frac{dy_1}{d\tau} = \frac{f_{yy}^0}{2}\left(\varepsilon^{2p} y_1^2\right) + f_\lambda^0 \, \varepsilon t.$$

HOWEVER $t = \varepsilon^{q} \tau$ SO THAT

$$\varepsilon^{p-q} \frac{dy_1}{d\tau} = \frac{f_{yy}^{\circ}}{2} \left( \varepsilon^{2p} y_1^2 \right) + \varepsilon^{1+q} \tau \, f_\lambda^{\circ} .$$

TO BALANCE THE TERMS WE GET

$$p - q = 2p = 1 + q \longrightarrow p = 1/3 , \; q = -1/3$$

THUS, WE OBTAIN

$$(0) \left. \right\} \; y = y_c + \varepsilon^{1/3} y_1 + \cdots , \qquad t = \varepsilon^{-1/3} \tau$$

WHICH YIELDS THE APPROXIMATE EQUATION

$$(1) \left. \right\} \; \dot{y}_1 = \frac{f_{yy}^{\circ}}{2} y_1^2 + f_\lambda^{\circ} \tau .$$

NOW DERIVE A CANONICAL ODE: LET $y_1 = BV$, $\tau = \delta S$, AND $V = V(S)$.

WE OBTAIN,

$$\frac{B}{\delta} V' = \frac{f_{yy}^{\circ} B^2}{2} V^2 + f_\lambda^{\circ} \delta S \qquad \text{WITH} \qquad V' = dV/dS .$$

NOW IN OUR EXAMPLE $f_{yy}^{\circ} = -2 y_c = 2 > 0$ (SINCE $f = y - \frac{y^3}{3} + \lambda$) AND $f_\lambda^{\circ} = 1 > 0$.

SINCE $f_{yy}^{\circ} > 0$ AND $f_\lambda^{\circ} > 0$ WE WRITE

$$\frac{B}{\delta^2 f_\lambda^{\circ}} V' = \frac{f_{yy}^{\circ} B^2}{2 f_\lambda^{\circ} \delta} V^2 + S \left. \right\} (2)$$

WE SET $\frac{B}{\delta^2 f_\lambda^{\circ}} = 1$ AND $\frac{f_{yy}^{\circ} B^2}{2 f_\lambda^{\circ} \delta} = -1$.

NOW DIVIDE

$$\frac{\;\frac{f_{yy}^{\circ} B^2}{2 f_\lambda^{\circ} \delta}\;}{\frac{B^2}{\delta^4 (f_\lambda^{\circ})^2}} = -1 \qquad \text{SO} \qquad \frac{f_{yy}^{\circ} \delta^3}{2} f_\lambda^{\circ} = -1$$

THIS YIELDS THAT

$$\delta^3 = -\frac{2}{f_{yy}^{\circ} f_\lambda^{\circ}} \longrightarrow \delta = -\left( \frac{2}{f_{yy}^{\circ} f_\lambda^{\circ}} \right)^{1/3} .$$

THEN $\qquad B = f_\lambda^\circ \left( \dfrac{2}{f_{yy}^\circ f_\lambda^\circ} \right)^{2/3}$

IN SUMMARY, WE GET

$$(3) \quad \left\{ \; Y_1 = B\, V(S), \quad \tau = \delta S \quad \text{WITH} \quad B = (f_\lambda^\circ)^{1/3} \left( \frac{2}{f_{yy}^\circ} \right)^{2/3}, \quad \delta = -\left( \frac{2}{f_{yy}^\circ f_\lambda^\circ} \right)^{1/3} \right.$$

AND OUR ODE (2) BECOMES

$$(4) \qquad V' = -V^2 + S.$$

NOTICE THAT SINCE $\delta < 0$ WE HAVE $\tau \uparrow \rightarrow S \downarrow$. SINCE $\tau > 0$ CORRESPONDS TO A SLOW RAMP BEYOND THE FOLD POINT, $S$ DECREASING BELOW ZERO IS THE EFFECT IN $S$-PLANE.

RECALL ALSO THAT

$$Y = Y_c + \varepsilon^{1/3} Y_1 + \dots \qquad t = \varepsilon^{-1/3} \tau \qquad \text{WITH RAMP } \lambda = \lambda_c + \varepsilon t$$

$$\text{SO} \quad \lambda - \lambda_c = O(\varepsilon^{2/3}).$$

NOW (4) IS RICATTI'S EQUATION. IT IS A <u>NORMAL FORM</u> ODE SINCE IT INVOLVES NO PARAMETERS. PUT $V = \dfrac{\phi'}{\phi}$. THEN $V' = \dfrac{\phi''}{\phi} - \dfrac{\phi'^2}{\phi^2}$ SO THAT

$$\frac{\phi''}{\phi} - \frac{\phi'^2}{\phi^2} = -\frac{\phi'^2}{\phi^2} + S$$

THIS YIELDS AIRY'S EQUATION

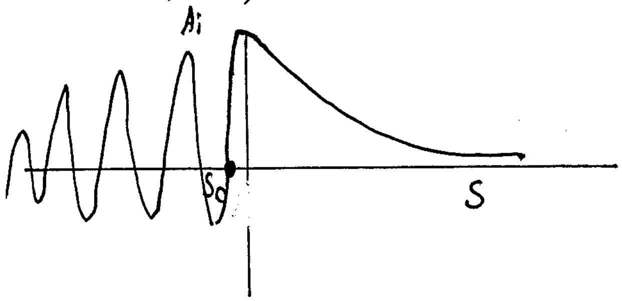
$$(5) \qquad \phi'' - S\phi = 0 \qquad \begin{array}{l} S < 0 \rightarrow \lambda > \lambda_c \\ S > 0 \rightarrow \lambda < \lambda_c . \end{array}$$

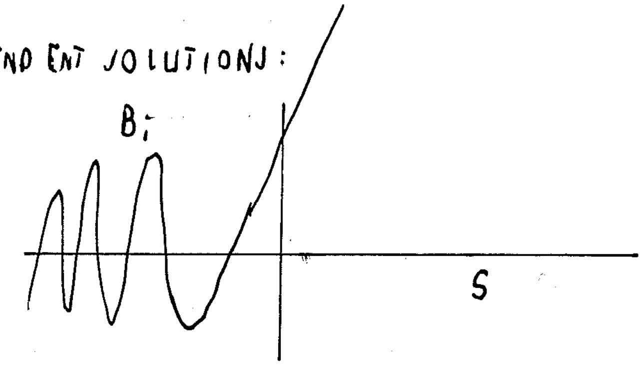
WE ARE LOOKING FOR A SOLUTION THAT CORRESPONDS TO $V$ BOUNDED AS $S \rightarrow +\infty$.

THE GENERAL SOLUTION IS

$$\phi = a_0 \, A_i(s) + a_1 \, B_i(s)$$

WHERE $A_i(s)$, $B_i(s)$ ARE TWO-INDEPENDENT SOLUTIONS:




FIRST ZERO $A_i(S_0) = 0 \longrightarrow S_0 = -2.33811$

$$\text{THUS} \quad V = \frac{a_0 A_i'(s) + a_1 B_i'(s)}{a_0 A_i(s) + a_1 B_i(s)} \quad \Big\} \qquad (6)$$

NOTICE THAT $\Phi$ DEPENDS ONLY ON 1-PARAMETER, THE RATIO $a_1/a_0$.

NOW FOR $S \to +\infty$ WE HAVE

$$A_i(s) \sim \frac{1}{2\sqrt{\pi}} s^{-1/4} e^{-\frac{2}{3} s^{3/2}}, \qquad B_i(s) \sim \frac{1}{\sqrt{\pi}} s^{-1/4} e^{2 s^{3/2}/3} \quad \Big\} \; (7)$$

THUS IF $a_1 \neq 0$ WE HAVE $V \sim \dfrac{B_i'(s)}{B_i(s)} \sim s^{1/2} + \ldots$ AS $S \to +\infty$.

THIS WOULD MEAN THAT

$$Y = Y_c + \varepsilon^{1/3} B V \sim Y_c + \varepsilon^{1/3} B \sqrt{S} \qquad \text{AS} \quad S \to +\infty$$

HOWEVER, WE MUST HAVE $Y < Y_c$ FOR $S \gg 1 \longrightarrow$ THIS IS INCORRECT. WE MUST HAVE $a_1 = 0$.

. WE SET $a_1 = 0$ SO THAT $V = \dfrac{A_i'(s)}{A_i(s)}$

NOW AS $S \to +\infty$ WE USE (7) TO OBTAIN THAT $\dfrac{A_i'}{A_i} \sim -\sqrt{S}$ AS $S \to \infty$

WHICH GIVES

$$V \sim -\sqrt{S} \quad \text{AND} \quad Y \sim Y_c - \varepsilon^{1/3} B \sqrt{S} \qquad \text{FOR} \quad S \to +\infty.$$

NOW WE WRITE THIS IN TERMS OF THE ORIGINAL VARIABLES.

WE HAVE $s = \tau / \delta$ SO THAT

$$y \sim y_c - \varepsilon^{1/3} \frac{B}{\sqrt{-\delta}} \sqrt{-\tau} \qquad \text{WITH} \qquad \frac{B}{\sqrt{-\delta}} = \sqrt{\frac{2 f_\lambda^\circ}{f_{yy}^\circ}}$$

WITH $t = \varepsilon^{-1/3} \tau$ WE GET

$$y \sim y_c - \varepsilon^{1/3} \sqrt{\frac{2 f_\lambda^\circ}{f_{yy}^\circ}} \, \varepsilon^{1/6} \sqrt{-t} = y_c - \sqrt{\frac{2 f_\lambda^\circ}{f_{yy}^\circ}} \left(-\varepsilon t\right)^{1/2} \quad \text{FOR} \quad t < 0. \quad \Big\} (8)$$

THIS IS THE EXPRESSION <u>JUST BEFORE</u> THE FOLD POINT.

NOW WE HAVE, RETURNING TO (6) WITH $q_1 = 0$ THAT

$$\left. \begin{aligned} V &= \frac{A_i'(s)}{A_i(s)} \\ \text{WE THEN HAVE} \quad y &= y_c + \varepsilon^{1/3} B \, V(s) + \dots \end{aligned} \right\} (9)$$

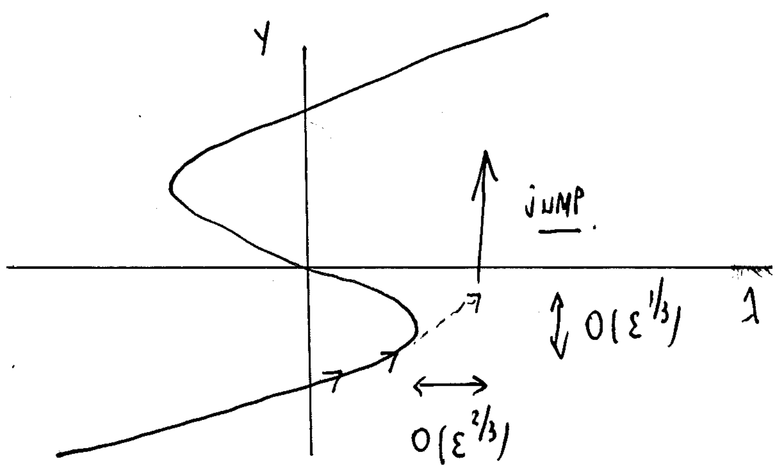
WE CONCLUDE THAT $V$ BECOMES INFINITE, i.e. $y - y_c$ IS NO LONGER SMALL WHEN $s = s_0$ WHERE $A_i(s_0) = 0$, i.e. THE FIRST ZERO OF THE AIRY FUNCTION. THIS VALUE IS $s_0 = -2.33811$, AND $V \to +\infty$ AS $s \to s_0^+$.

RETURNING TO THE ORIGINAL VARIABLES WE HAVE THAT THE APPROXIMATION BREAKS DOWN

$$t = t_B = \varepsilon^{-1/3} \tau_B = \varepsilon^{-1/3} \delta \, s_0 \qquad \text{WITH} \qquad \delta = -\left( \frac{2}{f_{yy}^\circ f_\lambda^\circ} \right)^{1/3}.$$

THIS YIELDS $\lambda = \lambda_c + \varepsilon t$ OR AT $t = t_B$, FOR WHICH $\lambda = \lambda_B$, given by

$$\lambda_B = \lambda_c + \varepsilon^{2/3} \tau_B = \lambda_c + \varepsilon^{2/3} (2.338..) \left( \frac{2}{f_{yy}^\circ f_\lambda^\circ} \right)^{1/3} \Big\} (10)$$




WE CONCLUDE WITH A SLOW RAMP $\lambda = \lambda_c + \varepsilon t$ THAT THE "JUMP" DOES NOT OCCUR AT $t = 0$ BUT INSTEAD AT A SCALING LAW WITH $t = O\left(\varepsilon^{-1/3}\right)$ PAST $t = 0$.

<u>EXAMPLE</u> CONSIDER $y' = y - y^3/3 + \lambda$, WHERE $f = y - y^3/3 + \lambda$.

AT THE LOWER FOLD POINT $f_{yy} = -2y$, $f_\lambda = 1$ SO WITH $y = y_c = -1$ AND $\lambda_c = 2/3$ WE GET $f_{yy}^0 = 2$, $f_\lambda^0 = 1$.

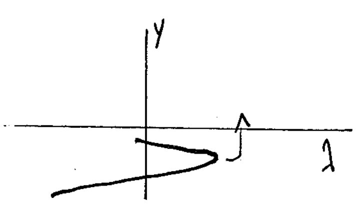
WE CALCULATE $\delta = -\left(\dfrac{2}{f_{yy}^0 f_\lambda^0}\right)^{1/3} = -1$ AND SO

$$t_B = \varepsilon^{-1/3}\,(2.33811\ldots)$$

$$\lambda = \frac{2}{3} + \varepsilon^{2/3}\,(2.3381\ldots)$$

WITH $\varepsilon = .01$ $\quad t_B = 10.85$.

<u>REMARK</u> $t_B \downarrow$ IF $f_{yy}^0 \uparrow$. THUS FOR BRANCHES WITH HIGHER CURVATURE, i.e.

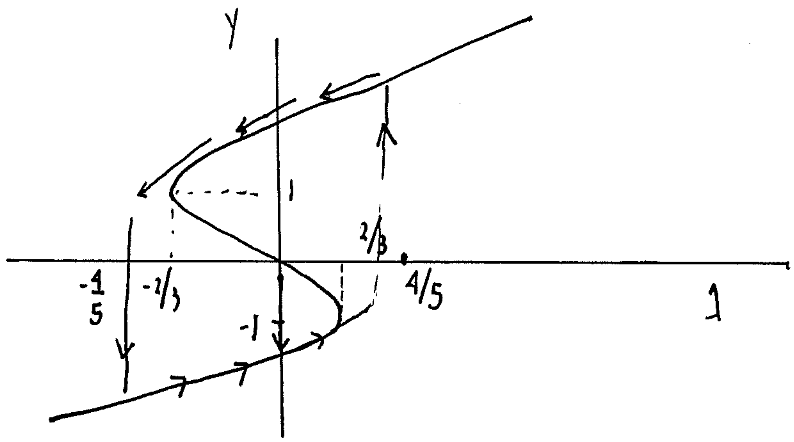



THEN BLOWUP TIME OCCURS SOONER.

## HYSTERESIS LOOP

WE CONSIDER $y' = y - \frac{1}{3} y^3 + \lambda$

WHERE $\lambda = \lambda_c (1.2) \sin(\varepsilon t)$ WITH $\varepsilon = .1$, AND $y(0) = -.1$, $\lambda_c = \frac{2}{3}$




WE GET A HYSTERESIS LOOP WITH DELAY.