FOURTH-ORDER DIFFUSION PROBLEM

SUPPOSE THAT $h(x,t)$ SATISFIES THE LINEAR PDE

$$h_t = -h_{xxxx}, \quad -\infty < x < \infty, \ t > 0 \ ; \quad h \to 0 \text{ AS } |x| \to \infty$$
$$h(x,0) = \delta(x).$$

THEN
$$h(x,t) = \frac{1}{t^{1/4}}\, g\left(x/t^{1/4} \right) \qquad g(z) = \frac{1}{\pi}\int_0^\infty e^{-k^4} \cos(kz)\, dk.$$

DERIVATION WE TAKE THE FOURIER TRANSFORM $H(k,t) = \int_{-\infty}^\infty e^{-ikx} h(x,t)\, dx$

TO OBTAIN
$$H_t = -k^4 H$$
$$H(k,0) = 1.$$

THUS $H = e^{-k^4 t}$. THE INVERSE TRANSFORM IS

$$h(x,t) = \frac{1}{2\pi}\int_{-\infty}^\infty H(k,t) e^{ikx}\, dk = \frac{1}{2\pi}\int_{-\infty}^\infty e^{-k^4 t} [\cos(kx) + i \sin(kx)]\, dk.$$

THE INTEGRAL INVOLVING SIN VANISHES SINCE IT IS ODD, AND WE OBTAIN

$$h(x,t) = \frac{1}{2\pi}\int_{-\infty}^\infty e^{-k^4 t} \cos(kx)\, dk = \frac{1}{\pi}\int_0^\infty e^{-k^4 t} \cos(kx)\, dk.$$

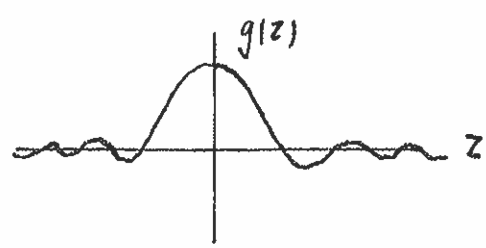
NOW LET $\sigma = k t^{1/4}$. $dk = d\sigma / t^{1/4}$ SO THAT

$$h(x,t) = \frac{1}{\pi t^{1/4}} \int_0^\infty e^{-\sigma^4} \cos\left(\sigma\, x/t^{1/4} \right) d\sigma.$$

WE WRITE THIS AS

$$h(x,t) = \frac{1}{t^{1/4}}\, g\left(x/t^{1/4} \right) \qquad \text{WITH} \quad g(z) = \frac{1}{\pi}\int_0^\infty e^{-\sigma^4} \cos(\sigma z)\, d$$

WE HAVE THAT $\int_{-\infty}^\infty g(z)\, dz = 1$ IS GUARANTEED.



SIMILARITY ANSATZ

$$h = H\bar{h}, \quad x = L\bar{x}, \quad t = T\bar{t}$$

THEN $\quad h_t = -h_{xxxx} \quad \rightarrow \quad \frac{H}{T}\,\bar{h}_{\bar{t}} = -\frac{H}{L^4}\,\bar{h}_{\bar{x}\bar{x}\bar{x}\bar{x}} \quad \Longrightarrow \quad L = T^{1/4}$

$$h(x,0) = \delta(x) \quad \rightarrow \quad H\,\bar{h}(\bar{x},0) = \frac{1}{L}\,\delta(\bar{x}) \quad \Longrightarrow \quad H = 1/L.$$

WE CALCULATE $\quad \dfrac{x}{t^{1/4}} = \dfrac{L\bar{x}}{T^{1/4}\,\bar{t}^{1/4}} = \dfrac{\bar{x}}{\bar{t}^{1/4}}.$

NOW SUBSTITUTE, $\quad h = t^{-1/4}\,g(z) \quad$ WITH $\quad z = x/t^{1/4}.$

$$h_t = -\frac{1}{4}\,t^{-5/4}\,g + t^{-1/4}\,g'(z)\left(-\frac{1}{4}\,t^{-5/4}\,x\right) = t^{-5/4}\left[-\frac{g}{4} - \frac{z}{4}\,g'\right]$$

$$h_{xxxx} = t^{-5/4}\,g''''(z)$$

SUBSTITUTE TO OBTAIN

$$4\,g''''(z) = g(z) + z\,g'(z), \quad 0 < z < \infty$$

$$g'(0) = g'''(0) = 0 \quad (g \text{ IS EVEN})$$

$$g \rightarrow 0 \text{ AS } |z| \rightarrow \infty.$$

NOW $\quad \int_{-\infty}^{\infty} u(x,t) = 1 \quad \rightarrow \quad \int_{-\infty}^{\infty} t^{-1/4}\,g(x/t^{1/4})\,dx = \int_{-\infty}^{\infty} g(\eta)\,d\eta = 1.$

BY INTEGRATING ONCE:

$$\begin{cases} g'''(z) - \dfrac{z}{4}\,g(z) = 0, \quad \text{ON} \quad 0 < z < \infty \\ g'(0) = 0, \quad g \rightarrow 0 \text{ AS } |z| \rightarrow \infty \\ \int_{-\infty}^{\infty} g(z)\,dz = 1 \end{cases}$$

NOW TAKE FOURIER TRANSFORMS.

$$\hat{g}(u) = \mathcal{F}(g) = \int_{-\infty}^{\infty} g(z)\, e^{iuz}\, dz.$$

SO $$\mathcal{F}(g''') - \frac{1}{4}\mathcal{F}(zg) = 0$$

$$\int_{-\infty}^{\infty} g''' e^{iuz}\, dz - \frac{1}{4}\int_{-\infty}^{\infty} z g\, e^{iuz}\, dz = 0.$$

USE IBP 3 TIMES ON FIRST TERM:

$$-(iu)^3\, \hat{g}(u) - \frac{1}{4}\int_{-\infty}^{\infty} g \frac{d}{du}\left(-ie^{iuz}\right) dz = 0.$$

THUS, $$i u^3 \hat{g}(u) + \frac{i}{4}\frac{d}{du}\hat{g}(u) = 0.$$

THIS MEANS $$\frac{d}{du}\hat{g} = -4u^3 \hat{g}$$

WITH $\hat{g}(0) = 1$ (WHICH YIELDS $\int_{-\infty}^{\infty} g(z)\, dz = 1$).

WE OBTAIN $$\hat{g}(u) = e^{-u^4},$$

AND BY INVERSE F.T. $$g(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-u^4}\, e^{-iuz}\, dz.$$

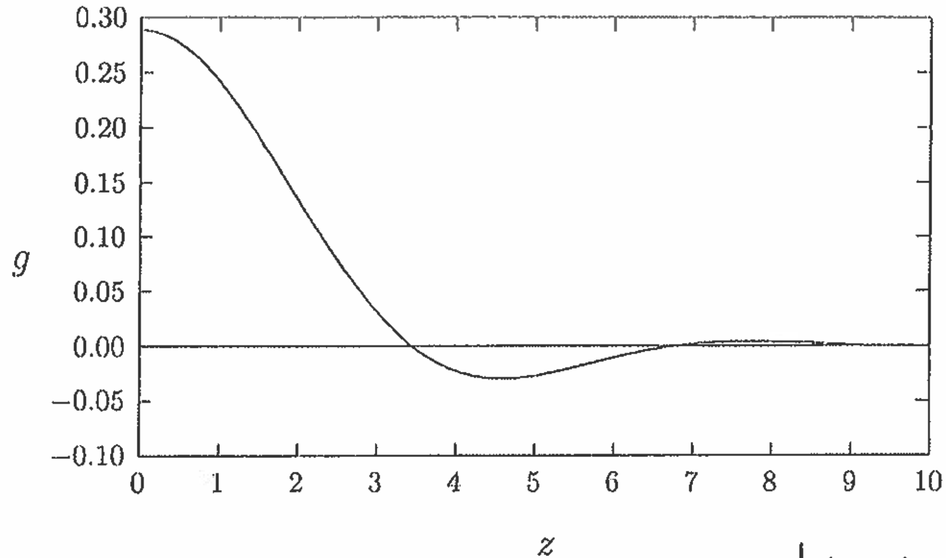
IN ORDER TO MAKE $g(z)$ EVEN WITH $g'(0) = 0$ TAKE RE(..)
AND USE $\mathrm{Re}(e^{-iuz}) = \cos(-uz) = \cos(uz)$.

THIS GIVES $$g(z) = \frac{1}{\pi}\int_{0}^{\infty} e^{-u^4} \cos(uz)\, dz \quad \text{AS BEFORE.}$$

SO $$h(x,t) = t^{-1/4}\, g(x/t^{1/4}).$$

<u>REMARK</u> $g(0) = \frac{1}{4\pi}\Gamma(1/4)$, $g''(0) = -\frac{1}{4\pi}\Gamma(3/4)$ WITH $\Gamma(s) = \int_0^{\infty} t^{s-1} e^{-t}\, dt$

NOTE: $$g(0) = \frac{1}{\pi}\int_0^{\infty} e^{-\sigma^4}\, d\sigma = \frac{1}{4\pi}\int_0^{\infty} \lambda^{-3/4} e^{-\lambda}\, d\lambda = \frac{1}{4\pi}\Gamma(1/4).$$



<u>KEY POINT:</u> g OSCILLATES AS $|z|$ INCREASES. $h(x,t)$ IS NOT STRICTLY OF ONE SIGN.

<u>2-D</u>

NOW IN 2-D FOR RADIAL SYMMETRIC δ FUNCTION INITIAL DATA WE HAVE

$$h_t = -\Delta^2 h \qquad \Delta^2 h = h_{xxxx} + 2\, h_{xx}\, h_{yy} + h_{yyyy}.$$

$$h(\underline{x},0) = \delta(\underline{x}), \quad h \to 0 \text{ AS } |\underline{x}| \to \infty \text{ WITH } \underline{x} = (x,y).$$

LET $\displaystyle H(\underline{k},t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\underline{x},t)\, e^{-i\underline{k}\cdot\underline{x}}\, d\underline{x}$. THEN

$$H_t = -|\underline{k}|^4 H, \quad H(\underline{k},0) = 1 \qquad |\underline{k}| = (k_1^2 + k_2^2)^{1/2}.$$

THIS GIVES, $H(k,t) = e^{-|\underline{k}|^4 t}$

THE INVERSE TRANSFORM IS $\displaystyle h(\underline{x},t) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-|\underline{k}|^4 t}\, e^{i\underline{k}\cdot\underline{x}}\, dk_1\, dk_2.$

NOW LET $k_1 = \rho\cos\varphi$, $k_2 = \rho\sin\varphi$, WITH $\rho = |\underline{k}|$ AND $x = |\underline{x}|\cos w$, $y = |\underline{x}|\sin$ TO GET $\underline{k}\cdot\underline{x} = \rho|x|\cos(\varphi - w)$. BY INTEGRATING IN POLAR COORDINATES

$$h(\underline{x},t) = \frac{1}{4\pi^2}\int_0^{\infty} \rho\, e^{-\rho^4 t}\left(\int_0^{2\pi} e^{i\rho|x|\cos(\varphi - w)}\, d\varphi\right) d\rho.$$

SINCE THE INNER INTEGRAL IS 2π PERIODIC WE HAVE

$$h(\underline{x},t) = \frac{1}{2\pi}\int_0^{\infty} \rho\, e^{-\rho^4 t}\left(\frac{1}{2\pi}\int_0^{2\pi} e^{i\rho|x|\cos\phi}\, d\phi\right) d\rho.$$

NEXT, WE RECOGNIZE USING THE INTEGRAL REPRESENTATION

$$J_0(s) \equiv \frac{1}{2\pi}\int_0^{2\pi} e^{is\cos\phi}\, d\phi$$

SO THAT

$$h(\underset{\sim}{x},t) = \frac{1}{2\pi}\int_0^{\infty} \rho\, J_0(\rho|\underset{\sim}{x}|)\, e^{-\rho^4 t}\, d\rho.$$

NOW LET $\sigma = \rho t^{1/4}$ TO OBTAIN

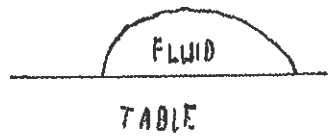
$$h(\underset{\sim}{x},t) = \frac{1}{t^{1/2}}\, g\left(|\underset{\sim}{x}|/t^{1/4}\right)$$

WHERE $$g(z) = \frac{1}{2\pi}\int_0^{\infty} \sigma\, J_0\left(\sigma z\right) e^{-\sigma^4}\, d\sigma.$$

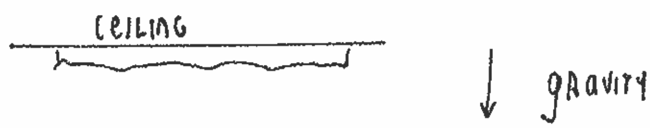
THIS FUNCTION IS ALSO SIGN-CHANGING AS $|z| \to \infty$.

HIGHER ORDER PDE'S FROM LUBRICATION THEORY

WE CONSIDER THIN LIQUID FILMS OF A VISCOUS FLUID. FOR EXAMPLE, A DROPLET ON A SUBSTRATE



OR A THIN LAYER OF PAINT HANGING FROM CEILING



WE ARE INTERESTED IN DETERMINING THE SHAPE OR THICKNESS $h(\underline{x},t)$ OF THE FILM UNDER THE ACTION OF GRAVITY, SURFACE TENSION, SLIP / NO SLIP BOUNDARY CONDITIONS.

IN 1-D, THIS LEADS TO HIGH ORDER (i.e. FOURTH ORDER) DIFFUSION EQUATIONS OF THE FORM

$$(1) \qquad h_t = -[f(h)\, h_{xxx}]_x - [g(h)\, h_x]_x \qquad \text{WITH } f(h) > 0, \text{ FOR } h>0.$$

EQUATIONS OF THIS GENERAL FORM WILL BE DERIVED BY AN ASYMPTOTIC LIMIT OF THE NAVIER-STOKES EQUATIONS.

<u>CASE I</u> IF WE NEGLECT SURFACE TENSION FORCES (UNREALISTIC FOR SMALL DROPS) WE OBTAIN THE SECOND-ORDER PDE

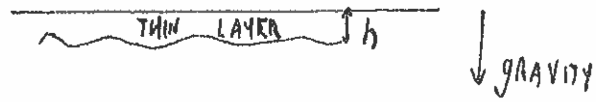
$$(2) \qquad h_t = \left[\tfrac{1}{3} h^3 h_x\right]_x$$

IN OUR ANALYSIS OF BARENBLATT'S EQUATION $u_t = (u^m u_x)_x$ WE FOUND A SIMILARITY SOLUTION WITH $u=0$ ON SOME CURVES $|x| = g(t)$.

INCLUDING THE EFFECT OF SLIPAGE, WE WILL OBTAIN IN LARGE SURFACE TENSION LIMIT

(6) $$h_t = -\frac{1}{3}\left[(\lambda_0 h^3 + b\mu h^2)\, h_{xxx} \right]_x \qquad \text{FOR SOME } b > 0.$$

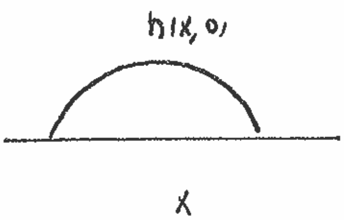
NOW FOR A THIN LAYER OF PAINT ON THE CEILING, DESTABILIZED BY GRAVITY, WE OBTAIN THAT (IN RE-SCALED TIME)



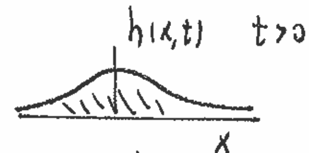
(7) $$h_t = -\left[h^3 h_{xxx} \right]_x - B\left[h^3 h_x \right]_x \qquad (B > 0)$$

FOR THIS MODEL WE ASK WHETHER $\exists$ SOLUTIONS THAT BLOW-UP IN FINITE TIME FOR SOME INITIAL CONDITIONS? i.e. PAINT DRIPS OFF THE CEILING.

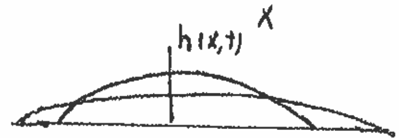
<u>MATHEMATICAL ISSUES</u> FOR $h_t = -\left[h^n h_{xxx} \right]_x$.



UNDER $h_t = h_{xx} \Longrightarrow$

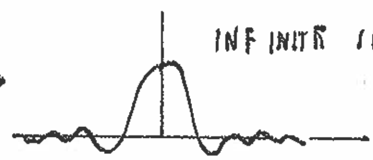


UNDER $h_t = (h^m h_x)_x$ WITH $m > 0$



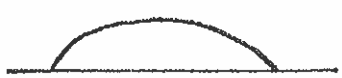
FINITE speed of propagation OF "DROPLET"

NOW FOR $h_t = -h_{xxxx} \longrightarrow$



INFINITE SPEED OF PROPAGATION BUT NO POSITIVITY

$h_t = -\left[h^n h_{xxx} \right]_x$

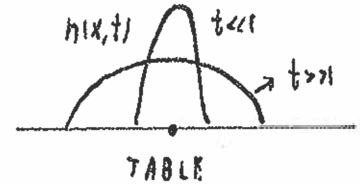


FINITE SPEED OF PROPAGATION AND POSITIVITY $0 < n < 3$

BUT $n = 3$ NO <u>DROPLET</u> SOLUTIONS!

APPLYING THAT PREVIOUS RESULT WE GET THE SPREADING DROPLET SOLUTION

$$(3) \qquad h(x,t) = (t+1)^{-1/5}\left(1 - \frac{9x^2}{10(t+1)^{2/5}}\right)_+$$



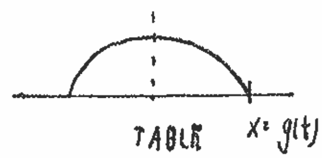
WITH $(x)_+ = \begin{cases} x & \text{IF } x>0 \\ 0 & \text{IF } x<0 \end{cases}$.

HOWEVER, IF WE INCLUDE THE EFFECT OF SURFACE TENSION THEN FOR A SPREADING DROP ON A TABLE WE OBTAIN

$$(4) \qquad h_t = -\frac{\lambda_0}{3}\left[h^3 h_{xxx}\right]_x + \frac{1}{3}\left(h^3 h_x\right)_x$$

WHERE $\lambda_0 > 0$ IS A DIMENSIONLESS CONSTANT related to the strength OF THE SURFACE TENSION. IN THE LIMIT $\lambda_0 >> 1$ AND UNDER A RE-SCALING IN TIME WE GET THE APPROXIMATE PROBLEM.

$$(5) \qquad H_t = -\left[H^3 H_{xxx}\right]_x$$



A DROPLET SOLUTION TO (5) IS A SOLUTION WITH

$$H_x = H_{xxx} = 0 \quad \text{AT} \quad x=0 \qquad \text{(EVEN-NESS)}$$

WITH $H = 0$ ON $x = g(t)$ (CONTACT CONDITION)

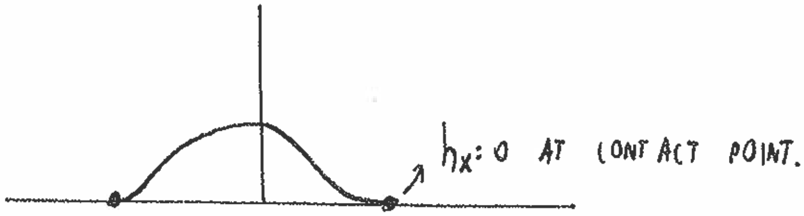
AND SOME OTHER CONDITION TO BE IMPOSED AT $x = g(t)$

THE QUESTION IS WHETHER (5) HAS A SIMILARITY SOLUTION DESCRIBING THE SHAPE OF THE DROPLET.

IN THE DERIVATION OF (5) IT IS ASSUMED THAT THERE IS NO SLIP OF THE FLUID ALONG THE SUBSTRATE. YET FOR FLUID DROP TO SLIDE, THERE MUST BE SOME SLIP AT THE CONTACT POINT. PARADOX?

<u>EXAMPLE</u> CONSIDER $h_t = -[h^3 h_{xxx}]_x$, WITH $\int_{-\infty}^{\infty} h\, dx = 1$

WE WANT TO TRY TO CONSTRUCT A STATIONARY DROPLET SOLUTION OF THE SHAPE



WHERE WE ASSUME A ZERO CONTACT ANGLE AT THE CONTACT POINT. THIS IS APPROPRIATE FOR A SLIGHTLY WETTED SUBSTRATE.

NOW PUT $h = H\bar{h}$ $t = T\bar{t}$, $x = L\bar{x}$ TO OBTAIN

$$\frac{H}{T}\bar{h}_{\bar{t}} = -\frac{H^4}{L^4}[\bar{h}^3 \bar{h}_{\bar{x}\bar{x}\bar{x}}]_{\bar{x}} \quad \text{SO} \quad \bar{h}_{\bar{t}} = -\frac{TH^3}{L^4}[\bar{h}^3\bar{h}_{\bar{x}\bar{x}\bar{x}}]_{\bar{x}}$$

WHILE $\int_{-\infty}^{\infty} h\, dx = 1$ YIELDS $HL\int_{-\infty}^{\infty}\bar{h}\, d\bar{x} = 1$ SO $H = 1/L$, $TH^3/L^4 = 1$

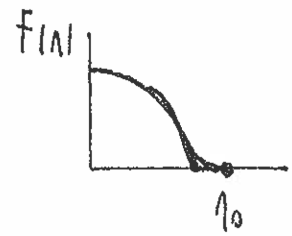
THUS $T/L^7 = 1$, $L = T^{1/7}$ AND $H = 1/L = 1/T^{1/7}$.

$$\frac{\bar{x}}{\bar{t}^{1/7}} = \frac{(1/L)x}{(t/T)^{1/7}} = \frac{T^{1/7}}{L}\frac{x}{t^{1/7}} = x/t^{1/7}$$

THIS SUGGESTS THAT WE PUT

$$h(x,t) = t^{-1/7} F(x/t^{1/7}) \quad \text{WITH} \quad \int_{-\infty}^{\infty} F(\eta)\, d\eta = 1$$

WHERE $\eta = x/t^{1/7}$. WE SUBSTITUTE TO OBTAIN



$$(1) \qquad F^3 F''' = \frac{\eta F}{7}, \quad 0 < \eta < \infty$$

WITH $F'(0) = 0$ (SO $F(\eta)$ EVEN. NOTICE $F'''(0) = 0$ AUTOMATICALLY SATISFIED)

$$(2)\left\{\begin{array}{ll} F(\eta_0) = 0 & \text{DRY AT EDGE OF DROPLET} \\ F'(\eta_0) = 0 & \text{ZERO CONTACT ANGLE} \quad \text{AND} \quad \int_0^{\eta_0} F(\eta)\, d\eta = \frac{1}{2}. \end{array}\right.$$

THESE ARE FOUR CONDITIONS FOR THE THIRD ORDER EQUATION AND UNKNOWN $\eta_0 > 0$.

REMARK TO DERIVE (1) WE PUT

$$h_t = -\frac{1}{7} t^{-8/7} F + t^{-1/7}\left[-\frac{1}{7} x t^{-8/7} F'\right] = -t^{-4/7}\, t^{-4/7}\left[F^3 F'''\right]'.$$

THUS

$$-\left(F^3 F'''\right)' = -\frac{1}{7}\left(F + \eta F'\right) = -\frac{1}{7}\left(\eta F\right)'$$

INTEGRATE ONCE

$$F^3 F''' = \frac{1}{7}\eta F + C.$$

BUT $F'''(0) = 0$ YIELDS

$$F^3 F''' = \frac{1}{7}\eta F.$$

HOWEVER, BERNIS IN 1991 PROVED THAT NO SOLUTION TO (1) AND (2) EXISTS! ONE SIMPLE WAY TO ARGUE THE NON-EXISTENCE OF SUCH A SOLUTION IS TO LOOK FOR A LOCAL SOLUTION NEAR THE CONTACT POINT η_0. WE TRY $F(\eta) \sim A(\eta_0 - \eta)^B$ AS $\eta \to \eta_0^-$, FOR SOME $B > 0$. WE MUST HAVE $B > 1$ TO ENFORCE $F(\eta_0) = F'(\eta_0) = 0$. HOWEVER FROM THE EQUATION

$$-\left[A^3(\eta_0-\eta)^{3B}\right] A B(B-1)(B-2)(\eta_0-\eta)^{B-3} \sim \frac{1}{7}\eta_0 A(\eta_0-\eta)^B.$$

BALANCING POWERS, WE HAVE $3B + (B-3) = B \to B = 1$.

THIS IS INCONSISTENT WITH REQUIRING $B > 1$.

REMARK (i) THERE IS A DROPLET SOLUTION FOR THE MODIFIED PROBLEM

$$h_t = -\left[h^m h_{xxx} \right]_x \quad \text{WITH} \quad 0 < m < 3.$$

THE SIMILARITY PROBLEM IS $h = t^{-1/(m+4)} F\left(x / t^{1/m+4} \right)$ AND

$$(+) \quad \begin{cases} F^m F''' = \dfrac{\eta F}{m+4} \\ F'(0) = 0, \quad F(\eta_0) = F'(\eta_0) = 0 \\ \int_0^{\eta_0} F(\eta)\, d\eta = 1/2 \end{cases}$$

THE LOCAL ANALYSIS GIVES $F(\eta) \sim A(\eta_0 - \eta)^\beta$ WITH $\beta = 3/m$.

FOR $0 < m < 3$, $\beta > 1$ WHICH IS CONSISTENT.

HOWEVER, $m = 3$ IS THE CORRECT EXPONENT WHEN SURFACE TENSION IS INCLUDED. REMEDY : WE MUST ALLOW FOR SMALL SPLIPPAGE ALONG THE SUBSTRATE.

REMARK (ii) FROM THE SIMILARITY EQUATION WE MUST HAVE THAT AS $\eta \to \eta_0$ THAT $F(\eta) \to 0$ BUT $F^3 F''' \to 0$. THIS IMPLIES THAT WE CAN LOOSE REGULARITY IN THAT F''' CAN BLOW UP AS $\eta \to \eta_0$.

REMARK (iii) IN THE HW YOU WILL DO A SHOOTING TYPE NUMERICAL METHOD TO COMPUTE SOLUTIONS TO (+) FOR $0 < m < 3$ AND INVESTIGATE WHAT HAPPENS AS $m \to 3^-$.

NOW TO SOLVE (+) WE CONSIDER AN AUXILLIARY PROBLEM FOR $g(y)$ GIVEN BY

$$g^m(y)\, g'''(y) = \frac{y\, g(y)}{m+4}, \quad y \geq 0$$

WITH $g(0) = 1,\ g'(0) = 0,\ g''(0) = -\mu < 0$

WITH $\mu > 0$ A PARAMETER. WE USE μ AS A SHOOTING PARAMETER AND TRY TO FIND A VALUE $y = a$ FOR WHICH

$$g(a) = g'(a) = 0$$

THE RESULTS ARE SHOWN ON NEXT PAGE AND WE FIND THAT $\exists$ SUCH A SOLUTION WHEN $m = 1, 2$ BUT NOT $m = 3$. WHEN THIS SOLUTION EXISTS WE CALCULATE B BY

$$\int_0^a g(y)\, dy = B.$$

TO RECOVER OUR ORIGINAL FUNCTION $F(\Lambda)$ WE PUT

$$y = S\Lambda, \quad g = AF(\Lambda) \rightarrow g^m g''' = \frac{y g}{m+4} \rightarrow \frac{A^{m+1}}{S^3} F^m F''' = SA \frac{\Lambda F}{m+4}$$

THIS MEANS THAT $A^m = S^4$. NOW

$$\int_0^a g(y)\, dy = B \rightarrow SA \underbrace{\int_0^{\Lambda_0} F(\Lambda)\, d\Lambda}_{= 1/2} = B \qquad \text{SO} \quad SA = 2B.$$

$$\rightarrow \qquad \text{AND} \quad \Lambda_0 = a/S$$

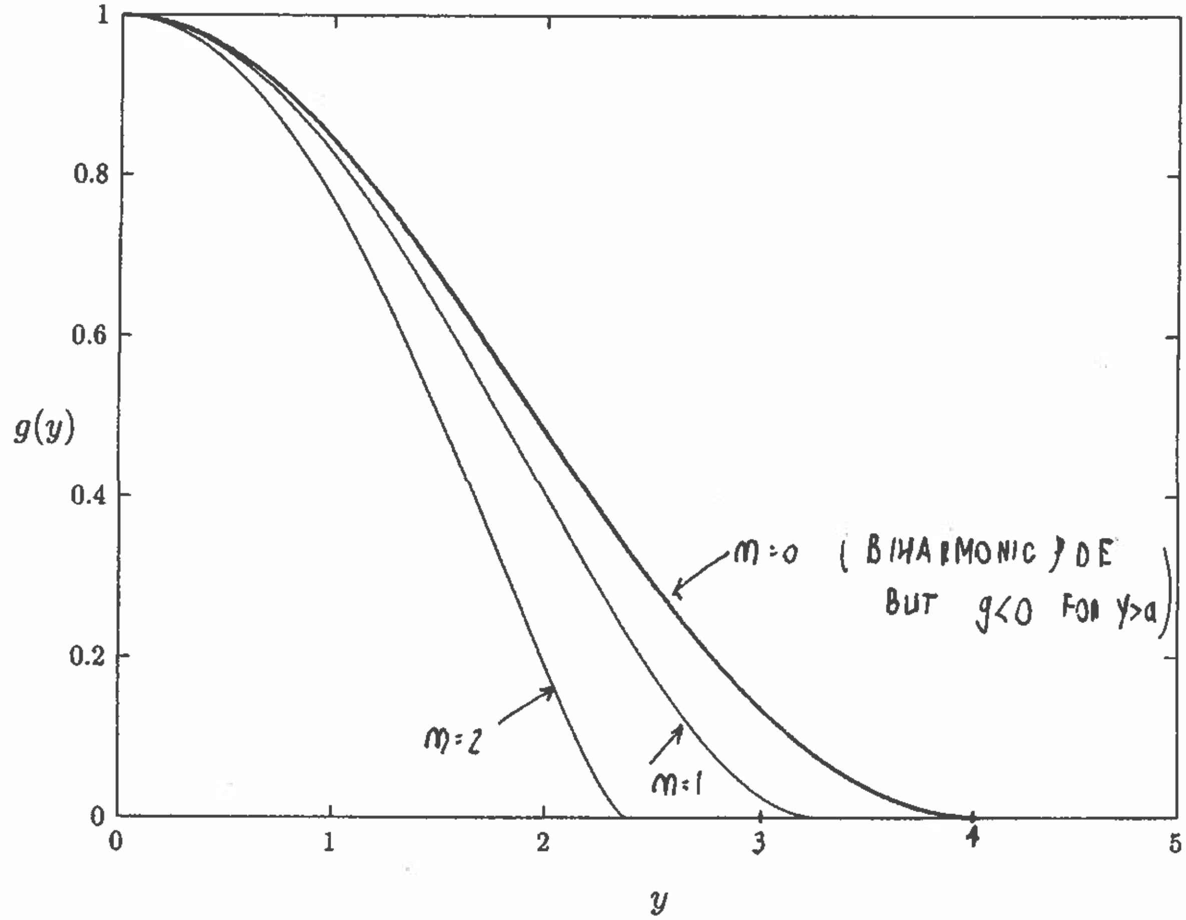
WE SOLVE FOR A AND S:

$$A = (2B)^{4/m+4}, \quad S = (2B)^{m/m+4}, \quad \Lambda_0 = a/S$$

SO THAT IF $F(\Lambda) = \frac{1}{A} g(S\Lambda)$ SO WE HAVE

$$F^m F''' = \frac{\Lambda F}{m+4}, \qquad F'(0) = 0, \quad F(\Lambda_0) = F'(\Lambda_0) = 0$$

$$\text{AND} \quad \int_0^{\Lambda_0} F\, d\Lambda = 1/2.$$



$$g^m g''' = \frac{y\, g}{m+4} \qquad g(0)=1,\ g'(0)=0,\ g''(0)=-\mu$$
$$g(a) = g'(a) = 0.$$

WE USE A 2-PARAMETER NEWTON'S METHOD TO COMPUTE μ, a.

THIS YIELDS,

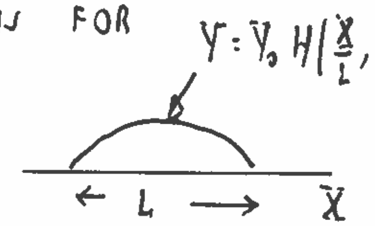
m	μ	a
0	.332	4.08
1	.365	3.31
2	.477	2.39

THE PROGRAM DOES NOT CONVERGE FOR $m=3$.

THIS SUGGESTS NON-EXISTENCE OF SUCH A SIMILARITY SOLUTION

THIN FILMS AND FREE SURFACES

WE BEGIN WITH THE 2-D NAVIER STOKES EQUATIONS FOR A VISCOUS INCOMPRESSIBLE FLUID OF DENSITY ρ AND VISCOSITY μ, GRAVITY IS ACTING DOWNWARDS.



$$(1)\left\{\begin{aligned} \rho\left(U_T + U U_X + V U_Y\right) &= -\frac{\partial P}{\partial X} + \mu\left(U_{XX} + U_{YY}\right) \\ \rho\left(V_T + U V_X + V V_Y\right) &= -\frac{\partial P}{\partial Y} + \mu\left(V_{XX} + V_{YY}\right) - \rho g. \end{aligned}\right.$$

$$U_X + V_Y = 0.$$

INCOMPRESSIB

WHERE X AND Y COMPONENTS OF FLUID VELOCITY ARE U AND V:

NOW LET

$$\left.\begin{aligned} \underline{X} &= (X, Y) = L\,\underline{x} \\ \underline{U} &= U_0\,\underline{u} \qquad \underline{U} = (U, V) \\ T &= \frac{L}{U_0}\,t \\ P &= \rho U_0^2\,p_0 \end{aligned}\right\} (2)$$

WE OBTAIN

$$\frac{\rho U_0^2}{L}\frac{D}{Dt}\underline{u} = -\frac{\rho U_0^2}{L}\nabla_{\underline{x}} p_0 - \rho g\begin{pmatrix}0\\1\end{pmatrix} + \frac{\mu}{L^2}U_0\,\Delta\underline{u}$$

WHERE $\frac{D}{Dt}\underline{u} = \underline{u}_t + (\underline{u}\cdot\nabla)\underline{u}$ "MATERIAL DERIVATIVE" AND $\Delta\underline{u} = \begin{pmatrix} u_{xx} + u_{yy} \\ v_{xx} + v_{yy}\end{pmatrix}$

THE INCOMPRESSIBILITY CONDITION GIVES $\nabla_{\underline{x}}\cdot\underline{u} = 0$.

THIS YIELDS:

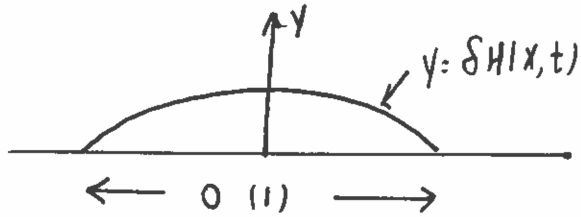
$$\frac{D\underline{u}}{Dt} = -\frac{gL}{U_0^2}\begin{pmatrix}0\\1\end{pmatrix} - \nabla_{\underline{x}} p_0 + \frac{\mu/\rho}{L U_0}\Delta\underline{u}.$$

$$\nabla_{\underline{x}}\cdot\underline{u} = 0.$$

BUT $\nu = \mu/\rho$ KINEMATIC VISCOSITY. DEFINE $Re = L U_0/\nu$. THEN

$$(*)\left\{\ \frac{D\underline{u}}{Dt} = -\frac{gL}{U_0^2}\begin{pmatrix}0\\1\end{pmatrix} - \nabla_{\underline{x}} p_0 + \frac{1}{Re}\Delta\underline{u}, \qquad \nabla_{\underline{x}}\cdot\underline{u} = 0.\right.$$

NOW WE WILL CONSIDER THE SCALING FOR A THIN DROP



$(\delta \ll 1)$ LENGTH OF DROP ≫ HEIGHT

$\delta = Y_0/L$ $\qquad$ $RE = L U_0/\nu$.

NOW THE BOUNDARY CONDITION ARE (EXPLAINED LATER)

- $p = -\lambda \kappa$ ON $y = \delta H$ $\qquad \kappa = \dfrac{\delta H_{xx}}{[1+\delta^2 H_x^2]^{3/2}} \sim \delta H_{xx} \quad (\delta \ll 1)$

  (NORMAL FORCE THAT IS PROPORTIONAL TO CURVATURE OF THE DROP. THIS IS THE CAPILLARY OR SURFACE TENSION FORCE - LAPLACE-YOUNG CONDITION).

- $U_y = 0$ ON $y = \delta H$ (COMES FROM IMPOSING NO SHEAR STRESS ON DROPLET)

- $V = \delta(H_t + U H_x)$ ON $y = \delta H$ (PARTICLE THAT IS ON SURFACE REMAINS ON SURFACE)

  <u>NOTE</u>: WRITE $F(x,y,t) = 0$ FOR $y - \delta H(x,t) = F = 0$.

  IMPOSE $\dfrac{D}{Dt} F = 0 \rightarrow F_t + \underline{u} \cdot \nabla F = 0$ ON $y = H$.

  $\rightarrow -\delta H_t - \delta U H_x + V = 0$, ON $y = H$

- ON BOTTOM WE IMPOSE

  $V = 0$ ON $y = 0$

  $U = b U_y$ ON $y = 0$ $\qquad b$ IS "SLIPPAGE LENGTH".

  IF $b = 0$ NO SLIP CONDITIONS.

  WE WILL ASSUME THAT $b = O(\delta) \ll 1$.

NOW PUT $p_0 = \frac{1}{Re} p$. THEN

$$R_e \frac{D}{Dt} \underline{u} = -\frac{g L Re}{U_0^2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \nabla_{\underline{x}} p + \Delta \underline{u}$$

$$\nabla_{\underline{x}} \cdot \underline{u} = 0$$

HOWEVER, $\frac{gL}{U_0^2} RE = \frac{gL}{U_0^2}\left(\frac{L U_0}{\nu}\right) = \frac{g L^2}{U_0 \nu}$. THIS YIELDS THAT

$$\left.\begin{aligned} RE \frac{D}{Dt} \underline{u} &= -\frac{g L^2}{U_0 \nu} \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \nabla_x p + \Delta \underline{u} \\ \nabla_{\underline{x}} \cdot \underline{u} &= 0 \end{aligned}\right\} \quad (3)$$

NOW WE INTRODUCE THE MEASURE $\delta = Y_0/L$ OF HEIGHT OF DROP.

WE INTRODUCE THE RESCALING:

$$\hat{y} = y/\delta, \quad v = \delta \hat{v}, \quad p = \hat{P}/\delta^2. \quad \Big\} \quad (3.5)$$

THEN $v_y = \hat{v}_{\hat{y}}$ AND WE OBTAIN:

$$\left.\begin{aligned} RE\,[u_t + u u_x + \hat{v} u_{\hat{y}}] &= -\frac{1}{\delta^2} \hat{P}_x + u_{xx} + \frac{1}{\delta^2} u_{\hat{y}\hat{y}} \\ \delta\, RE\,[\hat{v}_t + u \hat{v}_x + \hat{v} \hat{v}_{\hat{y}}] &= -\frac{1}{\delta^3} \hat{P}_{\hat{y}} + \delta \hat{v}_{xx} + \frac{1}{\delta} \hat{v}_{\hat{y}\hat{y}} - \frac{g L^2}{U_0 \nu} \\ u_x + \hat{v}_{\hat{y}} &= 0. \end{aligned}\right\} \quad (4)$$

BALANCE

WE REMARK THAT SCALE U_0 FOR VELOCITY OF THE DROP IS ARBITRARY AT THIS STAGE. TO BALANCE TERMS (HYDROSTATIC PRESSURE) CHOOSE

$$U_0 = \frac{g L^2}{\nu} \delta^3. \quad (5)$$

WE WILL THEN ASSUME THAT WITH $RE = L U_0/\nu$ WE HAVE.

$$\delta^2 \, RE \ll 1.$$

NOW IN TERMS OF THE NEW VARIABLES IN (3.5) WE GET:

- $V = 0$ ON $\hat{y} = 0$, $\quad U = b_0 U_{\tilde{y}}$ ON $\hat{y} = 0$

  WHERE $\quad b_0 = b/\delta = O(1)$.

- $\hat{v} = H_t + U H_x$ ON $\hat{y} = H$

- $\hat{p} = -\lambda_0 H_{xx}$ ON $\tilde{y} = H$ WITH $\lambda_0 \equiv \lambda \delta^3$.

  TYPICALLY λ_0 IS A SMALL PARAMETER BUT WE KEEP THIS TERM

- $U_{\tilde{y}} = 0$ ON $\hat{y} = H$

NOW WITH THIS CHOICE WE OBTAIN

$$\tilde{p}_{\tilde{y}} = -1 \quad \text{WITH} \quad \tilde{p} = -\lambda_0 H_{xx} \text{ ON } \hat{y} = H.$$

AND $\quad \tilde{p}_x = U_{\tilde{y}\hat{y}}$.

WE OBTAIN THAT

$$\tilde{p} = (H - \hat{y}) - \lambda_0 H_{xx}$$

SO THAT BY CALCULATING $\tilde{p}_x$ WE GET

$$(6) \quad \begin{cases} U_{\tilde{y}\hat{y}} = H_x - \lambda_0 H_{xxx} & \text{ON } 0 < \hat{y} < H \\ U_{\tilde{y}} = 0 & \text{ON } \tilde{y} = H \\ U = b_0 U_{\tilde{y}} & \text{ON } \hat{y} = 0. \end{cases}$$

THE SOLUTION IS

$$U = -\frac{1}{2}\left[H_x - \lambda_0 H_{xxx} \right] \hat{y} (2H - \hat{y}) + B$$

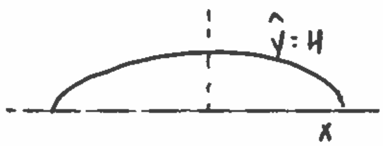
WHICH SATISFIES THE EQUATION AND $U_{\hat{y}} = 0$ ON $\hat{y} = H$.

THIS YIELDS THAT ← TERM FROM SURFACE TENSION →

$$\left.\begin{aligned} H_t = &-\frac{\partial}{\partial x}\left[\gamma_0\left(\frac{H^3}{3}+\frac{b_0 H^2}{2}\right)H_{xxx}\right] \\ &+\frac{\partial}{\partial x}\left[H_x\left(\frac{H^3}{3}+\frac{b_0 H^2}{2}\right)\right]. \end{aligned}\right\}$$

← TERM FROM GRAVITY →

THIS IS THE PDE (FOURTH ORDER) FOR EVOLUTION OF DROP



- IF NO SLIP IS ASSUMED THEN $b_0 = 0$, AND

$$H_t = -\frac{\partial}{\partial x}\left[\frac{\gamma_0 H^3}{3}H_{xxx}\right] + \frac{1}{3}\frac{\partial}{\partial x}\left(H^3 H_x\right)$$

IF WE FURTHER IGNORE SURFACE TENSION THEN $\gamma_0 = 0$

AND WE GET $H_t = \frac{1}{3}\frac{\partial}{\partial x}\left(H^3 H_x\right)$,

WHICH HAS THE BARENBLATT SIMILARITY PROFILE.

- RECALL, IF WE NEGLECT GRAVITY AND IMPOSE NO SLIP CONDITION ($b_0 = 0$) THEN <u>NO</u> DROPLET SOLUTION TO

$$H_t = -\frac{\partial}{\partial x}\left[\frac{\gamma_0 H^3}{3}H_{xxx}\right]$$

AS WE SAW EARLIER.

FORCE BALANCE ON DROPLETS

TO DERIVE THE BOUNDARY CONDITIONS, WE RECALL THE STRESS TENSOR IN THE ORIGINAL VARIABLES:

$$\sigma = \begin{pmatrix} -P + 2\mu U_X & \mu(U_Y + V_X) \\ \mu(U_Y + V_X) & -P + 2\mu V_Y \end{pmatrix}$$

NOW WE RECALL THE SCALING USED

$$U = U_0 u, \quad X = x L, \qquad P = \rho U_0^2 p_0 = \frac{\rho U_0^2}{R_E} p = \frac{\rho U_0^2}{R_E} \frac{\hat{p}}{\delta^2} \qquad R_E = \frac{UL}{\nu}$$

$$V = U_0 \delta \tilde{v} \quad \text{WITH } \tilde{y} = y/\delta \text{ AND } Y = yL$$

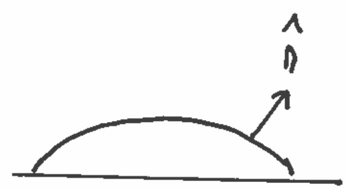
BUT $U_0 = \dfrac{g L^2}{\nu} \delta^3$ SO $P = \dfrac{\rho \nu U_0}{L} \dfrac{\hat{p}}{\delta^2}$, AND $\rho\nu = \mu$

NOW SUBSTITUTE INTO STRESS TENSOR:

$$\sigma = \frac{\mu U_0}{L} \begin{pmatrix} -\hat{p}/\delta^2 + 2u_x & \frac{1}{\delta} u_{\tilde{y}} + \delta \tilde{v}_x \\ \frac{1}{\delta} u_{\tilde{y}} + \delta v_x & -\hat{p}/\delta^2 + 2 v_{\tilde{y}} \end{pmatrix}$$

NOW THE UPPER SURFACE NORMAL ON THE SURFACE OF THE DROP IS TO CALCULATE $\nabla[-\tilde{y} + H(x,t)]$. THIS YIELDS THAT

$$\hat{n} = \frac{(-H_x, 1)^T}{(1 + \delta^2 H_x^2)^{1/2}}$$ FOR UNIT OUTWARD NORMAL



NOW THE FORCE ON THE SURFACE OF THE DROP IS

$$T = \sigma \hat{n}$$

WHICH IS A 2-VECTOR.

NOW $\quad T = \sigma \hat{n} = \frac{\mu U_0}{L}\begin{pmatrix} -\hat{P}/\delta^2 + 2U_x & \frac{1}{\delta}\hat{U}_y + \delta \tilde{V}_x \\ \frac{1}{\delta}U_{\tilde{y}} + \delta V_x & -\frac{\hat{P}}{\delta^2} + 2\tilde{V}_y \end{pmatrix}\begin{pmatrix} -H_x \\ 1 \end{pmatrix}\frac{1}{\sqrt{1+\delta^2 H_x^2}}$.

NOW THIS YIELDS THAT ON $\hat{y} = H(X)$:

$$T = \sigma \hat{n} = \frac{\mu U_0}{L}\begin{pmatrix} \frac{\hat{P} H_x}{\delta^2} + \frac{\tilde{U}_y}{\delta} + O(1) \\ -\frac{\hat{P}}{\delta^2} - \frac{H_x U_{\tilde{y}}}{\delta} + O(1) \end{pmatrix}$$

WE EARLIER IMPOSED THAT

$$\hat{P} = -\lambda \delta^3 H_{xx} \quad \text{ON BOUNDARY.}$$

THIS IMPLIES THAT TO ELIMINATE FIRST COMPONENT OF T, WHICH IS THE SHEAR STRESS WE NEED

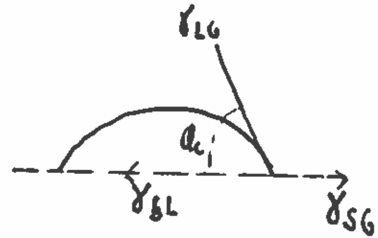
$$\tilde{U}_y = 0 \quad \text{ON} \quad \hat{y} = H(X)$$

THIS GIVES US OUR BOUNDARY CONDITION $U_{\tilde{y}} = 0$ ON DROPLET.

THE BEHAVIOR AT A CONTACT LINE WHERE THE DROP HITS THE SURFACE IS EXTREMELY COMPLICATED TO MODEL PHYSICALLY AND TO ANALYZE MATHEMATICALLY. THERE ARE MANY CONSIDERATIONS.

- INCLUDE A CONTACT ANGLE (WHICH INCLUDES CAPILLARY EFFECTS)

  YOUNG EQUATION: $\gamma_{SG} - \gamma_{SL} - \gamma_{LG}\cos\varphi_c = 0$

  γ_{SG} INTERFACIAL ENERGY SOLID-GAS

  γ_{SL} " " SOLID-LIQUID

  γ_{LG} " " LIQUID-GAS

  → THERMAL EQUILIBRIUM BETWEEN PHASES DESCRIBES "WETTABILITY" OF THE SOLID.

  WHAT ABOUT DYNAMICS AND NOT EQUILIBRIUM?

- IS THERE A PRECURSOR FILM OVER WHICH DROP SPREADS?

IN RECENT ARTICLE, W. REN, P. TRINH, W. E. THEY ANALYZE THE THIN FILM EQUATION WITH NO GRAVITY AND WITH A SLIP BC. (JFM, 772, p. 107-126, 2016)

$$h_t = -\left[\left(\frac{h^3}{3} + b h^2\right) h_{xxx}\right]_x \quad \text{IN} \quad 0 \le x \le s(t)$$

$$h_x = h_{xxx} = 0 \quad \text{AT} \quad x = 0$$

$$h = 0 \quad \text{AT} \quad x = s(t)$$

$$\beta \dot{s} = \frac{1}{2}\left[h_x^2 - \varphi_c^2\right] \quad \text{AT} \quad x = s(t)$$

THIS IS A MOVING BOUNDARY PROBLEM TO BE SOLVED FOR $h(x,t)$, $s(t)$.