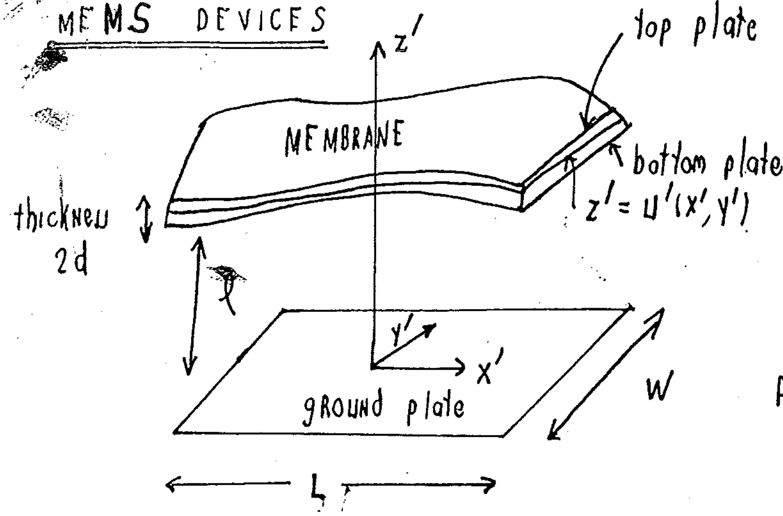
MEMS DEVICES



A MEMS CAPACITOR

"MICRO-ELECTRICAL-MECHANICAL-SYSTEMS"

WE WILL DERIVE A NONLINEAR PDE FOR THE DEFLECTION OF THE TOP MEMBRANE DUE TO AN IMPOSED VOLTAGE BIAS ACROSS THE THIN GAP OF WIDTH l.

NOW THE EQUATIONS ARE

POTENTIAL BETWEEN MEMBRANE AND BOTTOM PLATE

$$\left\{\begin{array}{ll} \Delta' \Psi_1 = 0 & |x'| \le L/2, \quad |y'| \le W/2 \\ \Psi_1 = 0 & \text{ON} \quad z' = 0 \end{array}\right.$$

INSIDE THE MEMBRANE

$$\left\{\begin{array}{ll} \nabla' \cdot [\varepsilon_2 \nabla' \Psi_2] = 0 & |x'| \le L/2, \quad |y'| \le W/2 \\ & u'(x',y') - d < z' < u'(x',y') + d \\ \Psi_2 = V & \text{ON} \quad z' = u'(x',y') + d \end{array}\right.$$

NOW THE CONTINUITY CONDITIONS ARE

$$\Psi_1 \equiv \Psi_2 \qquad \text{AT} \quad z' = u'(x',y') - d$$

$$\varepsilon_2 \nabla' \Psi_2 \cdot \hat{n} = \varepsilon_0 \nabla' \Psi_1 \cdot \hat{n} \qquad \text{AT} \quad z' = u'(x',y') - d.$$

HERE $\varepsilon_2 = \varepsilon_2(x', y')$ IS THE DIELECTRIC PERMITTIVITY OF THE MEMBRANE, AND ε_0 IS THE PERMITTIVITY OF AIR.

HERE $\hat{n}$ IS THE UNIT NORMAL TO THE SURFACE $z' - u'(x',y') + d = 0$

AND SO
$$\hat{n} = \frac{(-u'_{x'}, -u'_{y'}, 1)}{[1 + (u'_{x'})^2 + (u'_{y'})^2]^{1/2}}$$

NOW WE ASSUME THAT $W, L \gg \ell$ where ℓ is the gap width BETWEEN THE MEMBRANE AND THE BOTTOM PLATE

NOW LET $X = x'/L$, $Y = y'/L$, $Z = z'/\ell$ .. $U = u'/\ell$

THE EQUATIONS BECOME

IN GAP $\left\{ \begin{array}{l} \Psi_{1ZZ} + \dfrac{\ell^2}{L^2} (\Psi_{1XX} + \Psi_{1YY}) = 0 \qquad |X| < 1/2,\ |Y| < W/2L \\ \Psi_1 = 0 \qquad \text{ON} \quad Z = 0 \end{array} \right.$

LET $\mathcal{E}_2(X,Y) = \varepsilon_2(LX, LY)$. THEN,

$$\left\{ \begin{array}{l} \mathcal{E}_2 \Psi_{2ZZ} + \dfrac{\ell^2}{L^2} \left(\dfrac{\partial}{\partial X}, \dfrac{\partial}{\partial Y} \right) \cdot \left[\mathcal{E}_2 \left(\partial \Psi_2 / \partial X, \partial \Psi_2 / \partial Y \right) \right] = 0 \\ \qquad\qquad \text{IN} \quad |Z - U| \leq d/\ell,\ |X| \leq \frac{1}{2},\ |Y| \leq \frac{1}{2} \end{array} \right.$$

$$\mathcal{E}_2 \left(\frac{1}{L} \frac{\partial}{\partial X}, \frac{1}{L} \frac{\partial}{\partial Y}, \frac{1}{\ell} \frac{\partial}{\partial Z} \right) \cdot \left(-\frac{\ell}{L} U_X, -\frac{\ell}{L} U_Y, 1 \right) \Psi_2$$

$$= \mathcal{E}_0 \left(\frac{1}{L} \frac{\partial}{\partial X}, \frac{1}{L} \frac{\partial}{\partial Y}, \frac{1}{\ell} \frac{\partial}{\partial Z} \right) \cdot \left(-\frac{\ell}{L} U_X, -\frac{\ell}{L} U_Y, 1 \right) \Psi_1$$

ON $Z = U - d/\ell$

$\Psi_1 = \Psi_2$ ON $Z = U - d/\ell$

$\Psi_2 = V$ ON $Z = U + d/\ell$

NOW IN THE LIMIT $\ell/L \ll 1$, (thin gap), WE OBTAIN

membrane $\left\{ \begin{array}{ll} \Psi_{2ZZ} = 0 & U - d/\ell \leq Z \leq U + d/\ell \\ \Psi_2 = V & Z = U + d/\ell \end{array} \right.$

gap $\left\{ \begin{array}{ll} \Psi_{1ZZ} = 0 & 0 \leq Z \leq U - d/\ell \\ \Psi_1 = 0 & Z = 0 \end{array} \right.$

THE CONTINUITY CONDITIONS ARE

$$\varepsilon_2 \Psi_{2z} = \varepsilon_0 \Psi_{1z} \quad \text{ON} \quad z = u - d/\ell$$
$$\Psi_1 = \Psi_2 \quad \text{ON} \quad z = u - d/\ell .$$

DEFINE $Z_\ell = u - d/\ell$, $Z_u = u + d/\ell$.

WE SOLVE TO GET
$$\Psi = \begin{cases} A_1 z & \text{ON } 0 \le Z \le Z_L \\ A_2 (z - Z_u) + V & \text{ON } Z_L \le Z \le Z_u . \end{cases}$$

NOW CONTINUITY GIVES:
$$A_1 Z_L = A_2 (Z_L - Z_u) + V$$
$$\varepsilon_2 A_2 = \varepsilon_0 A_1$$

THIS YIELDS THAT
$$A_1 = \frac{\varepsilon_2/\varepsilon_0 \; V}{(\varepsilon_2/\varepsilon_0) Z_L + 2d/\ell} , \qquad A_2 = \frac{V}{[(\varepsilon_1/\varepsilon_0) Z_L + 2d/\ell]}$$

THIS YIELDS THE THIN GAP RESULT:
$$\psi = \begin{cases} \dfrac{(\varepsilon_2/\varepsilon_0) V}{(\varepsilon_1/\varepsilon_0) Z_L + 2d/\ell} \, z & 0 \le z \le Z_L \\[2ex] \dfrac{V}{[(\varepsilon_1/\varepsilon_0) Z_L + 2d/\ell]} (z - Z_u) + V & Z_L \le Z \le Z_u . \end{cases}$$

NOW A DEFORMABLE ELASTIC MEMBRANE SATISFIES
$$T \, \Delta' u'(x', y') = \frac{\varepsilon_2}{2} |\nabla' \psi_2|^2 \quad \text{ON} \quad z' = u'(x', y') - d .$$

where T is the tension in the membrane.

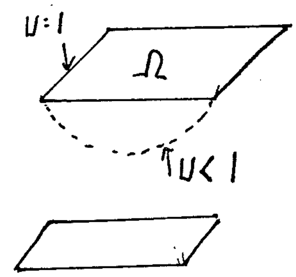
NEXT, WE WILL ASSUME EVEN FURTHER THAT $d/\ell \ll 1$. THEN ON THE SURFACE $z = u$ WE HAVE: $\psi_2 \sim V \left(\frac{\varepsilon_0}{\varepsilon_2} \right) \frac{(z - u)}{u} + V$.

NOW WRITING $x = x'/L$, $y = y'/L$, $u = u'/\ell$ WE OBTAIN THAT

$$T\left[\frac{\ell}{L^2}\right]\left(u_{xx}+u_{yy}\right)=\frac{\varepsilon_2}{2\ell^2}\left(\psi_{2z}\right)^2+\cdots$$

THIS YIELDS WITH $\psi_{2z}=V\,\dfrac{\varepsilon_0}{\varepsilon_2}\,\dfrac{1}{u}$ THAT

$$\begin{cases} u_{xx}+u_{yy}=\dfrac{\lambda f(x,y)}{u^2} & \text{IN } \Omega \\[2ex] u=1 & \text{ON } \partial\Omega \end{cases}$$



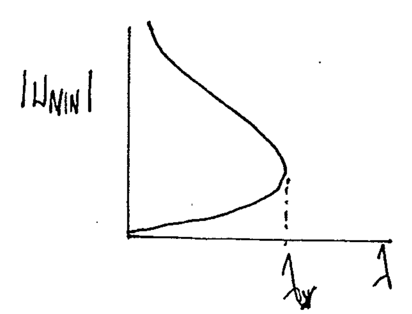
WITH $\lambda=\dfrac{\varepsilon_0 L^2 V^2}{2\ell^3 T}$ $\quad f(x,y)=\dfrac{\varepsilon_0}{\varepsilon_2(x,y)}$ $\quad\lambda$ bifurcation parameter.

THIS IS THE EQUATION FOR THE DEFLECTION u OF THE MEMBRANE.

NOTICE THAT $\lambda\propto V^2$ WHICH IS THE square of the applied voltage. NOTICE THAT WHEN $u=0$ WE HAVE A SINGULARITY AS THE MEMBRANE TOUCHES THE GROUND PLATE.

QUESTIONS

(i) WHAT IS u_{MIN} VS λ? TYPICALLY WE GET A PICTURE OF THE FORM

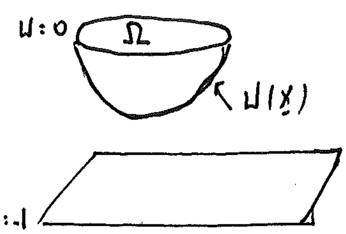


λ_* = FOLD POINT VALUE

$$V_{\text{pull-IN}}\equiv\left(\lambda_*/c\right)^{1/2}$$

(ii) WE WOULD LIKE TO ESTIMATE λ_* AND IN PARTICULAR TRY TO CHOOSE, OR OPTIMIZE ε_2, SO THAT λ_* IS LARGE. THIS WILL ALLOW STEADY-STATE SOLUTIONS FOR A LARGE INTERVAL IN V.

MEMS PDE PUT LOWER PLATE AT $u=-1 \Rightarrow u \mapsto 1+u$



THEN, WE WRITE (*) $\Delta u = \dfrac{\lambda f(\underline{x})}{(1+u)^2}$ IN Ω

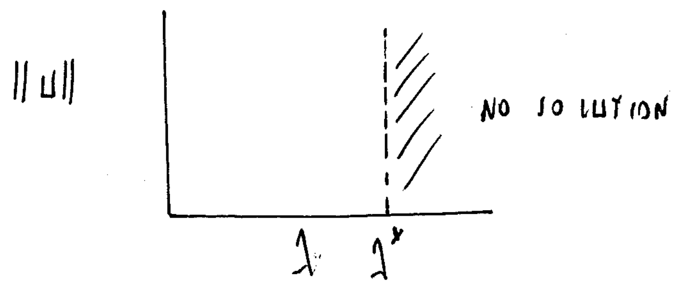
$u = 0$ ON $\partial\Omega$

WITH $f(\underline{x}) = \dfrac{\varepsilon_0}{\varepsilon_2(\underline{x})} > c$ IN Ω, WITH $c > 0$

AND $\lambda = bV^2$ IS BIFURCATION PARAMETER. WE REQUIRE $u > -1$ IN Ω.

PROPOSITION $\exists$ A $\lambda^* > 0$ SUCH THAT NO SOLUTION TO (*) EXISTS IF $\lambda > \lambda^*$.

i.e. $\exists$ A MAXIMUM voltage $V_{PULL} = \sqrt{\dfrac{\lambda^*}{b}}$ S.T. NO SOLUTION EXISTS IF $V > V_{PULL}$.



PROOF SUPPOSE THAT u EXISTS. THEN BY GREEN'S IDENTITY WITH

$\Delta \Phi_1 = -\sigma_1 \Phi_1$ WE OBTAIN

$$\int_\Omega (\Phi_1 \Delta u - u \Delta \Phi_1)\, dx = \lambda \int_\Omega \frac{\Phi_1 f}{(1+u)^2}\, dx + \int \Phi_1 u \sigma_1\, dx = 0$$

WE CONCLUDE THAT $I \equiv \displaystyle\int_\Omega \left(\frac{\lambda f}{(1+u)^2} + u\sigma_1 \right) \Phi_1\, d\underline{x} = 0$

THUS IF A SOLUTION EXISTS, WE MUST FIND THAT $I \equiv 0$.

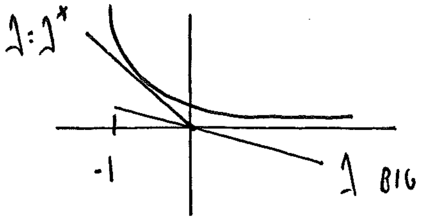
NOW $I \geq \displaystyle\int_\Omega \left[\frac{\lambda c}{(1+u)^2} + u\sigma_1 \right] \Phi_1\, dx$

SO IF $\dfrac{\lambda c}{(1+u)^2} + u\sigma_1 > 0$ $\forall\, -1 < u$ THEN $I > 0$ AND WE

HAVE A CONTRADICTION $\longrightarrow$ NO SOLUTION.

LET $f = \dfrac{1}{(1+u)^2}$ $\quad g = -\dfrac{\sigma_1}{\lambda c} u$.

IF $f > g$ ON $u > -1$ $\longrightarrow$ NO solution.



WE CONCLUDE THAT $f > g$ ON $u > -1$ IFF $\lambda > \lambda^*$ WHERE λ^*, u^* satisfy the tangency condition

$$\frac{1}{(1+u)^2} = -\frac{\sigma_1}{\lambda c} u$$

$$\frac{-2}{(1+u)^3} = \frac{-\sigma_1}{\lambda c}$$

DIVIDE $\quad -\dfrac{(1+u)}{2} = u \longrightarrow 3u = -1, \quad u = -\dfrac{1}{3}.$

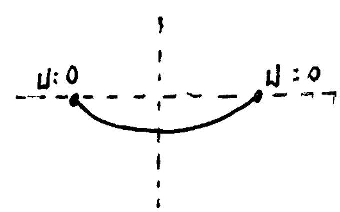
THUS $$\lambda^* = \frac{\sigma_1}{2c}(1+u)^3 = \frac{4\sigma_1}{27}.$$

EXAMPLE CONSIDER THE UNIT DISK WITH $f \equiv 1$. THEN $\Phi_1 = J_0(\sqrt{\sigma_1}\, r)$ WITH $J_0(\sqrt{\sigma_1}) = 0$ OR $\sigma_1 = (2.4048\ldots)^2 \rightarrow \lambda^* = \dfrac{4}{27}(2.4048)^2 \cong .857.$

EXAMPLE IN SLAB DOMAIN $-\dfrac{1}{2} \le x \le \dfrac{1}{2}$ THEN $\Phi_1 = \sin\left(\sqrt{\sigma_1}\,(x + \tfrac{1}{2})\right)$ SO $\sin(\sqrt{\sigma_1}) = 0 \rightarrow \sqrt{\sigma_1} = \pi$ OR $\sigma_1 = \pi^2$. WE HAVE FOR $c = 1$ THAT

$$\lambda^* = \frac{4}{27}(\pi^2) \doteq 1.46.$$

REMARK METHOD OF PROOF NEEDS $c > 0$, i.e. f BOUNDED AWAY FROM ZERO.



1-D PROBLEM $$u'' = \frac{\lambda}{(1+u)^2}, \qquad |x| \le \frac{1}{2}$$

$u(x) = u(-x)$, u CONVEX, $u(x) \le 0$ ON $|x| \le 1/2$

$u(\pm 1/2) = 0$

WE PUT $u(0) = u_0$ WITH $-1 < u(0) < 0$ AND CALCULATE:

WE MULTIPLY BY u' AND INTEGRATE

$$\frac{u'^2}{2} = -\frac{\lambda}{1+u} + C = \frac{\lambda}{1+u_0} - \frac{\lambda}{1+u} = \frac{\lambda(u-u_0)}{(1+u_0)(1+u)}$$

WE GET $$u' = \sqrt{2\lambda}\left[\frac{u-u_0}{(1+u_0)(1+u)}\right]^{1/2} \quad \text{ON } 0 < x < 1/2.$$

INTEGRATING YIELDS:

$$\int \frac{(1+u)^{1/2}}{(u-u_0)^{1/2}}\, du = \int \frac{\sqrt{2\lambda}}{(1+u_0)^{1/2}}\, dx$$

SO $$\int_{u_0}^{u} \frac{(1+\Lambda)^{1/2}}{(\Lambda-u_0)^{1/2}}\, d\Lambda = \frac{\sqrt{2\lambda}}{\sqrt{1+u_0}}\, x.$$

NOW $u(1/2) = 0$ YIELDS,

$$\sqrt{\frac{\lambda}{2(1+u_0)}} = \int_{u_0}^{0} \sqrt{\frac{1+\Lambda}{\Lambda-u_0}}\, d\Lambda = 2\int_0^{\sqrt{-u_0}} \sqrt{1+u_0+s^2}\, ds.$$

NOW LET $\Lambda - u_0 = s^2$. $\frac{d\Lambda}{\sqrt{\Lambda-u_0}} = \frac{2s}{s}\, ds = 2$

IN THIS WAY WE OBTAIN

$$\sqrt{\lambda} = 2\sqrt{2(1+u_0)} \int_0^{\sqrt{-u_0}} g(u_0,s)\, ds \qquad g(u_0,s) = \sqrt{1+u_0+s^2}.$$

NOTICE THAT $\sqrt{\lambda} = 0$ WHEN $u_0 = 0$ AND WHEN $u_0 = -1$. MOREOVER, $\sqrt{\lambda} > 0$ ON $-1 < u_0 < 0$. THUS $\exists$ A MAX. VALUE FOR λ, labelled by λ_{MAX}, AS u_0 RANGES ON $-1 < u_0 < 0$.

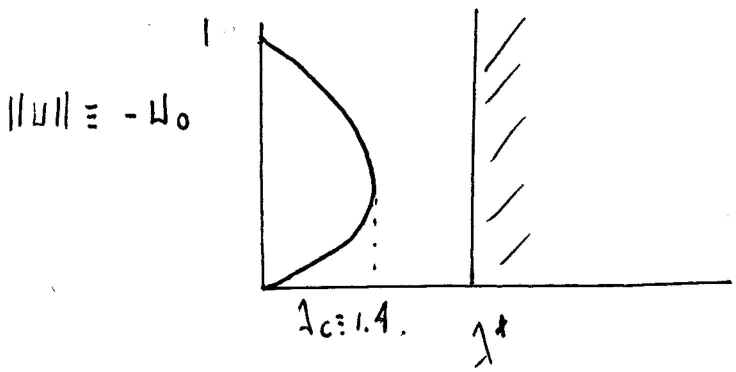
NOW WE CALCULATE AS $u_0 \to 0$ THAT

$$\sqrt{\lambda} \sim 2\sqrt{2(1+u_0)}\,\left(\sqrt{-u_0}\right) \quad \text{SO} \quad \lambda \sim 8(-u_0) + \cdots$$

NOW AS $u_0 \to -1$ WE HAVE

$$\sqrt{\lambda} \sim 2\sqrt{2}\ \sqrt{1+u_0} \int_0^1 g(-1,s)\,ds = \frac{1}{2}\sqrt{2}\ \sqrt{1+u_0}\,.$$

THUS $\lambda \sim 2(1+u_0)$ AS $u_0 \to -1$.



BY A NUMERICAL EVALUATION $d\lambda/du_0 = 0$ AT $\lambda = \lambda_c \doteq 1.4$

RECALL THE EXISTENCE THRESHOLD BOUND λ^*. WE FOUND THAT $\lambda^* \doteq 1.46$.

<u>RADIAL CASE</u> SUPPOSE $f \equiv 1$.

WE HAVE $u'' + \frac{1}{r} u' = \frac{\lambda}{(1+u)^2}$, $u(1) = 0$, $u'(0) = 0$. $J_0(z_0) = 0$, $z_0 = 2.4048$

WE WOULD LIKE TO PLOT $-u(0)$ VERSUS λ. NO SOLUTION IF $\lambda > \frac{4}{27}(z_0)^2 = .857..$

WE NOW CONVERT TO AN ODE:

SET $u = -1 + \alpha w(p)$ $\quad p = \gamma r$.

WE CAN TAKE WLOG $w(0) = 1$ AND $w'(0) = 0$.

SUBSTITUTE: $\alpha \gamma^2 w'' + \frac{\alpha \gamma^2}{p} w' = \frac{\lambda}{\alpha^2 w^2}$

NOW CHOOSE $\lambda = \alpha^3 \gamma^2$, THEN WE HAVE THE ODE IVP:

$$(1)\quad \left\{ \begin{array}{l} w'' + \frac{1}{p} w' = \frac{1}{w^2} \\ w(0) = 1,\ w'(0) = 0 \end{array} \right.$$

NOW $u(1)=0 \rightarrow -1+\alpha\, w(\gamma)=0 \rightarrow \alpha = \dfrac{1}{w(\gamma)}$.

THIS YIELDS WITH $\alpha = u(0)+1$ THAT

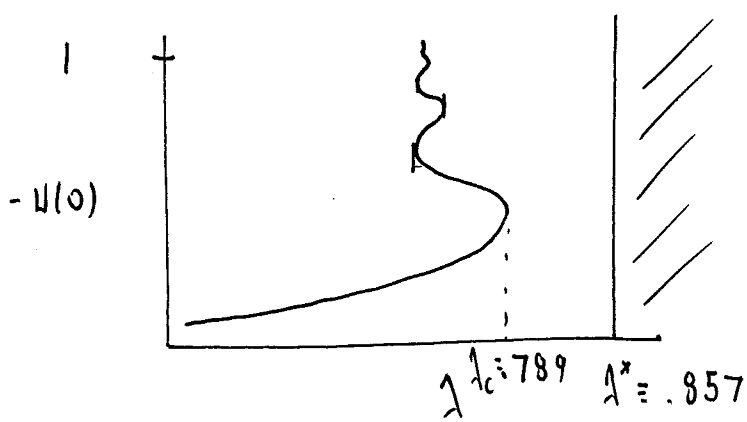
$$\lambda = \gamma^2 / [w(\gamma)]^3$$

$$u(0)+1 = \alpha = 1/w(\gamma)$$

OR

$$\left.\begin{aligned} \|u\| \equiv -u(0) &= 1 - 1/w(\gamma) \\ \lambda &= \gamma^2/(w(\gamma))^3 . \end{aligned}\right\} \qquad (2)$$

THIS YIELDS A PARAMETERIZATION OF THE RADIALLY SYMMETRIC STEADY-STATES IN THE $-u(0)$ VS λ PLANE



NUMERICALLY THIS IS NOW TRIVIAL AND WE FIND THAT $\lambda_c \doteq .789..$

<u>REMARK</u> <u>NUMERICS</u>: WE MUST HAVE $w \sim 1 + A\rho^2$ AS $\rho \to 0$

SO THAT FROM PDE $2A + 2A = 1 \rightarrow A = 1/4$.

LET $\varepsilon = .001$ $\quad w(\varepsilon) = 1 + \varepsilon^2/4, \quad w'(\varepsilon) = \varepsilon/2$.

NOW WE MUST EXPLAIN THE BIFURCATION DIAGRAM. THE KEY FEATURE OBSERVED NUMERICALLY IS THE INFINITE NUMBER OF SADDLE-NODE BIFURCATION POINTS.

WE WILL CONSIDER LONG-TIME BEHAVIOR FOR

$$w'' + \frac{1}{\rho} w' = \frac{1}{w^2}, \qquad w(0) = 1, \quad w'(0) = 0.$$

WE PUT $\Lambda = \log \rho$ SO THAT $w(\rho) \to \bar{W}(\Lambda)$.

THEN $w_\rho = \bar{W}_\Lambda \, 1/\rho \qquad w_{\rho\rho} = \bar{W}_{\Lambda\Lambda} \, 1/\rho^2 - \bar{W}_\Lambda/\rho^2$ SO

$$\bar{W}_{\Lambda\Lambda} \frac{1}{\rho^2} - \frac{1}{\rho^2}\bar{W}_\Lambda + \frac{1}{\rho^2}\bar{W}_\Lambda = \frac{1}{\bar{W}^2} \quad \to \quad \bar{W}_{\Lambda\Lambda} = \frac{e^{2\Lambda}}{\bar{W}^2}.$$

NOW LET $\bar{W} = e^{B\Lambda} V \longrightarrow \bar{W}'' = B^2 e^{B\Lambda} V + 2B e^{B\Lambda} V' + e^{B\Lambda} V''$ SO

$$V'' + 2BV' + B^2 V = \frac{e^{(2-B)\Lambda}}{e^{2B\Lambda} V^2} = \frac{e^{(2-3B)\Lambda}}{V^2}.$$

THUS $B = 2/3$ AND WE GET

$$\left\{ \begin{array}{ll} V'' + \frac{4}{3} V' + \frac{4}{9} V = \frac{1}{V^2}, & 0 < \Lambda < \infty \\ w = \rho^{2/3} V(\Lambda) & \Lambda = \log \rho. \end{array} \right.$$

NOW THIS IS A 2-DOF DYNAMICAL SYSTEM: $V_1 = V'$, $V_2 = V$ SO

$$\begin{aligned} V_2' &= V_1 \\ V_1' &= -\frac{4}{3} V_1 - \frac{4}{9} V_2 + \frac{1}{V_2^2}. \end{aligned} \quad \Longrightarrow \quad \underline{V}' = \underline{F}(\underline{V}).$$

NOW THE EQUILIBRIUM IS $\frac{4}{9} V_e = \frac{1}{V_e^2}$ SO $V_e^3 = \frac{9}{4}$ OR $V_e = \left(\frac{9}{4}\right)^{1/3}$.

NOW LINEARIZE: $V = V_e + \phi$ WITH $\phi \ll 1$

THEN $\phi'' + \frac{4}{3}\phi' + \frac{4}{9}\phi = -\frac{2}{V_e^3}\phi, \quad \to \quad \phi'' + \frac{4}{3}\phi' + \frac{4}{9}\phi + \frac{8}{9}\phi = 0$

SO $\phi'' + \frac{4}{3}\phi' + \frac{12}{9}\phi = 0.$

NOW PUT $\phi = e^{\lambda t} \longrightarrow \lambda^2 + \frac{4}{3}\lambda + \frac{12}{9} = 0$, SO $\lambda = -\frac{2}{3} \pm i\frac{2\sqrt{2}}{3}$

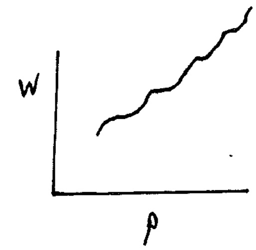
THIS YIELDS $\quad \phi = A e^{-2/3 \Lambda} \cos\left(\frac{2\sqrt{2}}{3} \Lambda + \phi_0 \right) \quad$ ϕ_0 phase shift.

NOW RECALL $\Lambda = \log p$, THEN FOR $\Lambda \gg 1$ WE HAVE

$$v \sim \left(\frac{9}{4}\right)^{1/3} + A e^{-2\Lambda/3} \cos\left(\frac{2\sqrt{2}}{3} \Lambda + \phi_0 \right) \qquad A \ll 1$$

WITH $\Lambda = \log p$, $w = p^{2/3} v$ SO $p^{2/3} e^{-2\Lambda/3} = 1$ WE GET

$$w \sim \left(\frac{9}{4}\right)^{1/3} p^{2/3} + A \cos\left(\frac{2\sqrt{2}}{3} \log p + \phi_0 \right) \quad \text{AS} \quad p \to \infty.$$

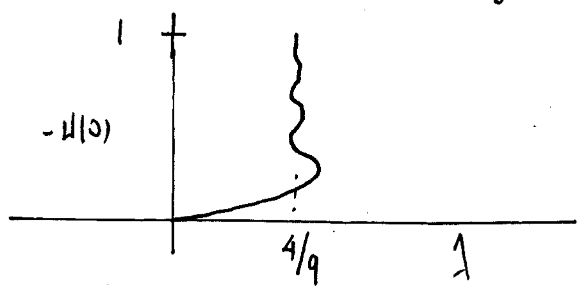


RECALL $\quad -u(0) = 1 - \frac{1}{w(\gamma)}, \quad \lambda(\gamma) = \gamma^2 / (w[\gamma])^3.$

THUS $\quad -u(0) \sim 1 - \frac{1}{\left(\frac{9}{4}\right)\gamma^{2/3} + \dots}, \quad \lambda \sim \frac{\gamma^2}{\left(\frac{9}{4}\right)\gamma^2 \left[1 + \frac{A}{\gamma^{2/3}} \left(\frac{4}{9}\right)^{1/3} \cos(\dots) \right]^3}$

THIS YIELDS FOR $\gamma \to +\infty$ THAT $-u(0) \to 1^-$

AND $$\lambda \sim \frac{4}{9}\left[1 - \frac{3A}{\gamma^{2/3}} \left(\frac{4}{9}\right)^{1/3} \cos\left(\frac{2\sqrt{2}}{3} \log \gamma + \phi_0 \right) \right]$$



DIELECTRIC PERMITTIVITY

WE LET $\frac{\varepsilon_0}{\varepsilon_2} = r^{\alpha}$ SO THAT $\varepsilon_2 = \varepsilon_0 r^{-\alpha}$ (HIGH PERMITTIVITY NEAR $r = 0$)

THEN IN A DISK WE HAVE

$$u_{rr} + \frac{1}{r} u_r = \frac{\lambda r^{\alpha}}{(1+u)^2}, \quad u(1) = 0, \quad u'(0) = 0 \quad \text{WITH} \quad \alpha > 0.$$

NOW IF WE LET $u(r) = -1 + \alpha\, w(p)$ WITH $p = \gamma r$ WE GET

$$\gamma^2 \alpha \left(w'' + \frac{1}{p} w' \right) = \frac{\lambda p^{\alpha}}{\gamma^{\alpha} \alpha^2 w^2}$$

THIS YIELDS, $$w'' + \frac{1}{\rho} w' = \frac{\lambda \rho^{\mu}}{\gamma^{2+\mu}\alpha^{3}} \frac{1}{w^{2}}.$$

CHOOSE $w(0) = 1$ WLOG. THEN TAKE

$$\lambda = \gamma^{2+\mu}\alpha^{3}$$

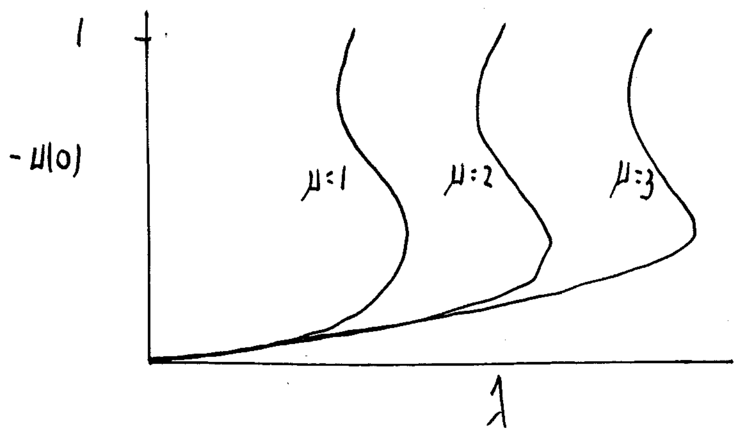
AND $w(1) = 0$ SO $-1 + \alpha w(\gamma) = 0$ OR $\alpha = 1/w(\gamma)$.

THEN WE GET

$$\lambda = \frac{\gamma^{2+\mu}}{[w(\gamma)]^{3}} \qquad \text{WHERE} \qquad w'' + \frac{1}{\rho} w' = \frac{\rho^{\mu}}{w^{2}}$$

$$-u(0) = 1 - \frac{1}{w(\gamma)} \qquad w(0) = 1, \quad w'(0) = 0.$$

WE WANT TO PLOT $-u(0)$ VERSUS λ FOR DIFFERENT μ.



NOW TO FIND THE LONG-TIME $\gamma \to +\infty$ BEHAVIOR WE LET

$\lambda = \log \rho$ AND $w = \rho^{\beta} v$, THIS YIELDS

$$\rho^{\beta-2} v'' + (2\beta-1)\rho^{\beta-2} v' + \beta(\beta-1)\rho^{\beta-2} v + \rho^{\beta-2} v' + \beta\rho^{\beta-2} v = \frac{\rho^{\mu-2\beta}}{v^{2}}$$

choose $\beta - 2 = \mu - 2\beta \rightarrow \beta = (\mu+2)/3$.

$$v'' + \frac{2(\mu+2)}{3} v' + \frac{(\mu+2)^{2}}{9} v = \frac{1}{v^{2}}$$

EQUILIBRIUM IS $v_e = \left(\frac{9}{(\mu+2)^{2}}\right)^{1/3}$.

NOW IF WE SUBSTITUTE $V = V_e + e^{\lambda\Lambda} A$ WE GET

$$\lambda^2 + \frac{2(\mu+2)}{3}\lambda + \left(\frac{(\mu+2)^2}{9} + \frac{2}{V_e^3}\right) = 0$$

OR

$$\lambda^2 + \frac{2(\mu+2)}{3}\lambda + \left(\frac{(\mu+2)^2}{9} + 2\frac{(\mu+2)^2}{9}\right) = 0$$

THUS YIELDS

$$\lambda^2 + \frac{2(\mu+2)}{3}\lambda + \frac{(\mu+2)^2}{3} = 0.$$

WE GET EITHER NEGATIVE OR COMPLEX CONJUGATE ROOTS WITH NEGATIVE REAL PART.

$$\lambda = -\frac{(\mu+2)}{3} \pm \frac{1}{2}\sqrt{\frac{4}{9}(\mu+2)^2 - \frac{4(\mu+2)^2}{3}} = -\frac{(\mu+2)}{3} \pm \frac{i\sqrt{2}}{3}(\mu+2)$$

THUS

$$V \sim \left(\frac{9}{(2+\mu)^2}\right)^{1/3} + A e^{-\frac{(\mu+2)}{3}\Lambda} \cos\left(\frac{\sqrt{2}}{3}(\mu+2)\Lambda + \phi_0\right)$$

NOW IN TERMS OF W WE HAVE

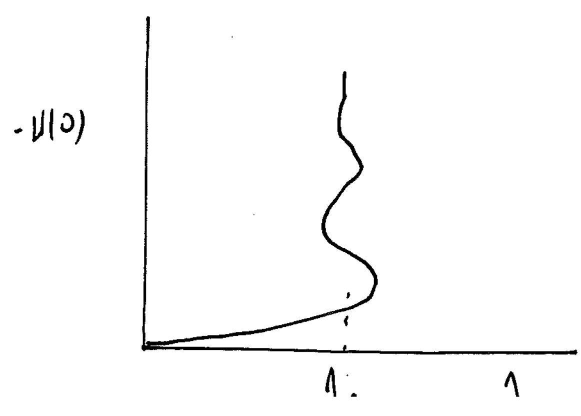
$$W \sim \rho^{(\mu+2)/3}\left(\frac{9}{(2+\mu)^2}\right)^{1/3} + A \cos\left(\frac{\sqrt{2}}{3}(\alpha+2)\log\rho + \phi_0\right)$$

NOW

$$\lambda = \frac{\gamma^{2+\mu}}{(w(\gamma))^3}, \qquad |w(0)| = 1 - \frac{1}{w(\gamma)}$$

THUS

$$\lambda \sim \lambda_* = \frac{(\mu+2)^2}{9} \quad \text{AS} \quad \gamma \to \infty.$$



BLOWUP PROFILE - SINGULARITY FORMATION

LET $\Phi_0(\underline{x}) > 0$ AND μ_0 BE FIRST EIGENPAIR OF LAPLACIAN, I.E.

$$\Delta \Phi_0 + \mu_0 \Phi_0 = 0 \quad \text{IN } \Omega$$
$$\Phi_0 = 0 \quad \text{ON } \partial\Omega$$

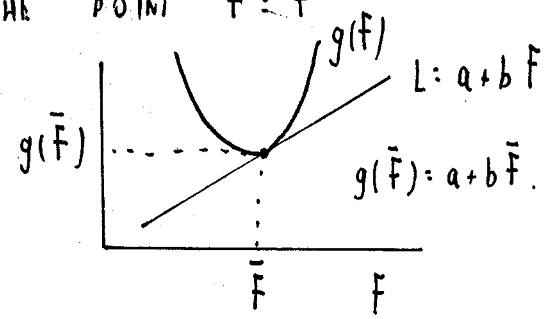
WHICH WE NORMALIZE AS $\int_\Omega \Phi_0 \, d\underline{x} = 1$.

WE FIRST STATE AND PROVE JENSEN'S INEQUALITY:

LEMMA (JENSEN'S INEQUALITY) LET $\Phi_0(\underline{x}) \geq 0$ IN Ω WITH $\int_\Omega \Phi_0 \, d\underline{x} = 1$. ASSUME THAT $g(u)$ IS CONVEX $\forall u$. DEFINE $\bar{F} \equiv \int_\Omega F \Phi_0 \, d\underline{x}$. THEN

$$g\Big(\underbrace{\int_\Omega \Phi_0 F \, d\underline{x}}_{\bar{F}}\Big) \leq \int_\Omega \Phi_0 \, g(F) \, d\underline{x}.$$

PROOF LET $L = a + bF$ BE THE STRAIGHT LINE TANGENT TO $g(F)$ AT THE POINT $F = \bar{F}$



THEN BY CONVEXITY

$$g(F) \geq a + bF, \quad \forall F.$$

NOW MULTIPLY BY $\Phi_0(\underline{x}) > 0$ IN Ω AND INTEGRATE

$$\int_\Omega g(F) \Phi_0 \, d\underline{x} \geq \int_\Omega [a + bF] \Phi_0(\underline{x}) \, d\underline{x} \geq a \underbrace{\int_\Omega \Phi_0(\underline{x}) \, d\underline{x}}_{=1} + b \underbrace{\int_\Omega F \Phi_0 \, d\underline{x}}_{\bar{F}}$$

THUS $\int_\Omega g(F) \Phi_0 \, d\underline{x} \geq a + b\bar{F} = g(\bar{F})$.

THEN WE HAVE $g(\bar{F}) \leq \int_\Omega g(F) \Phi_0 \, d\underline{x}$

$\longrightarrow$ THIS MEANS $g\left(\int_\Omega \Phi_0 F \, d\underline{x}\right) \leq \int_\Omega g(F) \Phi_0 \, d\underline{x}$ ∎.

WE WILL USE THIS AS

$$\int_\Omega g(F) \Phi_0 \, d\underline{x} \geq g\left(\int_\Omega \phi_0 F \, d\underline{x}\right)$$

NOW CONSIDER THE PDE

$$\Delta u = \frac{\lambda}{(1+u)^2}, \quad \underline{x} \in \Omega$$

$$u = 0 \text{ ON } \partial\Omega$$

AND ASSUME FOR SIMPLICITY THAT $u = 0$ AT $t = 0$.

PROPOSITION SUPPOSE THAT $\lambda > \frac{4\mu_0}{27}$. THEN u REACHES -1 AT A FINITE TIME (I.E. THE TOP PLATE COLLAPSES ONTO BOTTOM PLATE IN FINITE TIME).

PROOF DEFINE $E(t) \equiv \int_\Omega u \Phi_0 \, d\underline{x}$. NOTICE THAT $E(0) = 0$.

THEN, MULTIPLY PDE BY Φ_0 AND INTEGRATE:

$$\Phi_0 u_t = \Phi_0 \Delta u - \frac{\lambda \Phi_0}{(1+u)^2}.$$ DEFINE $g(u) = \frac{1}{(1+u)^2}$ SO $g''(u) = \frac{6}{(1+u)^4} > 0$. CONVEX

WE GET
$$\frac{d}{dt}\int_\Omega \Phi_0 u \, d\underline{x} = \int_\Omega \Phi_0 \Delta u \, d\underline{x} - \lambda \int_\Omega g(u) \Phi_0 \, d\underline{x}$$

$$= +\int_\Omega u \Delta \Phi_0 \, d\underline{x} - \lambda \int_\Omega g(u) \Phi_0 \, d\underline{x}$$

$$= +\int_\Omega u \mu_0 \Phi_0 \, dx - \lambda \int_\Omega g(u) \Phi_0 \, d\underline{x}.$$

THEREFORE, $E(t) \equiv \int_\Omega \Phi_0 u \, d\underline{x}$. THEN, BY JENSEN,

$$(*) \begin{cases} \dfrac{dE}{dt} + \mu_0 E = -\lambda \displaystyle\int_\Omega g(u) \Phi_0 \, d\underline{x} \leq -\lambda g(E) = -\dfrac{\lambda}{(1+E)^2}. \\ E(0) = 0. \end{cases}$$

NOW $E(t) = \int_\Omega \phi_0 u \, dx \geq (\inf_\Omega u) \int_\Omega \Phi_0 \, d\underline{x} = (\inf_\Omega u)$.

NOW WE HAVE THE PROBLEM

$$(*)\begin{cases} \dfrac{dE}{dt} + \mu_0 E \leq -\dfrac{\lambda}{(1+E)^2} \\ \text{WITH} \quad E(0) = 0 \quad \text{AND} \quad E(t) \geq \inf_\Omega u . \end{cases}$$

NOW DEFINE $f(t)$ TO BE THE SOLUTION TO THE ODE

$$(+)\begin{cases} \dfrac{df}{dt} + \mu_0 f = -\dfrac{\lambda}{(1+f)^2} \\ \text{WITH} \quad f(0) = 0 \end{cases}$$

BY STANDARD COMPARISON PRINCIPLE $E(t) \leq f(t)$ FOR ALL t.

(PROOF DEFINE $v = f - E$. THEN

$$\frac{dv}{dt} + \mu_0 v \geq -\frac{\lambda}{(1+f)^2} + \frac{\lambda}{(1+E)^2} = \frac{\lambda[(1+f)^2 - (1+E)^2]}{(1+E)^2(1+f)^2} = \frac{\lambda[2(f-E) + (f-E)(f+E)]}{(1+E)^2(1+f)^2}$$

OR $$\frac{dv}{dt} + \mu_0 v \geq \lambda v\, h(t) \quad \text{WITH} \quad h(t) = \frac{\lambda[2 + (f+E)]}{(1+E)^2(1+f)^2} \geq 0 .$$

NOW $$(e^{\mu_0 t} v)' \geq \lambda v e^{\mu_0 t} h(t) \quad \text{WITH} \quad v(0) = 0 .$$

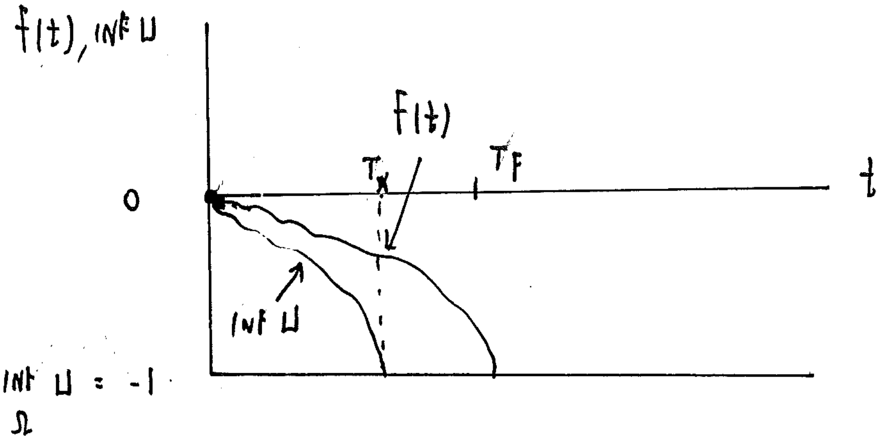
THUS $$e^{\mu_0 t} v \geq \lambda \int_0^t v(\tau) h(\tau) e^{\mu_0 \tau} d\tau > 0 .)$$

THIS SHOWS THAT WITH $E(t) \geq \inf_\Omega u$ AND $E(t) \leq f(t)$ THAT

$$f(t) \geq \inf_\Omega u .$$

NOW IF WE CAN SHOW THAT $f(t)$ REACHES -1 IN FINITE TIME

THEN SO WILL $\inf u$



HENCE IF f HAS A "TOUCHDOWN" TIME T_f, THEN THE TOUCHDOWN TIME FOR u IS AT T_* WHERE $T_* \leq T_f$.

NOW WE FIND RANGE OF λ FOR WHICH $\exists\, T_f$.

WE SEPARATE VARIABLES IN (+) TO GET

$$\frac{df}{\mu_0 f + \lambda/(1+f)^2} = -\,dt. \quad \longrightarrow \quad t = -\int_0^f \frac{ds}{(\mu_0 s + \lambda/(1+s)^2)}$$

INTEGRATING WITH $f(0)=0$ WE GET SETTING $f(T_f)=-1$ THAT

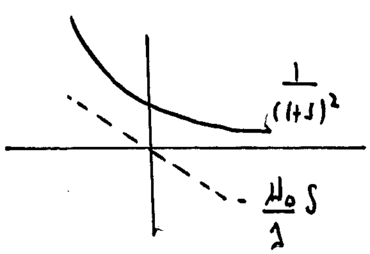
$$T_f = \int_{-1}^{0} \frac{ds}{(\mu_0 s + \lambda/(1+s)^2)}$$

IF THIS INTEGRAL IS FINITE, THEN THE TOUCHDOWN TIME T_* FOR u SATISFIES

$$T_* \leq \int_{-1}^{0} \frac{ds}{(\mu_0 s + \lambda/(1+s)^2)}$$

THIS INTEGRAL IS FINITE IFF THE DENOMINATOR NEVER VANISHES.

NOW



THE INTEGRAL IS FINITE IFF $\frac{1}{(1+s)^2} \geq -\frac{\mu_0 s}{\lambda}$ ON $s > -1$

THIS IS THE TANGENCY ARGUMENT BEFORE, SO

$$\lambda > \lambda_* = \frac{4}{27}\mu_0 \quad \text{AND} \quad T_* \leq \int_{-1}^{0} \frac{ds}{\mu_0 s + \lambda/(1+s)^2}$$

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IN CONCLUSION, WHEN $\lambda > \lambda_*$ WHERE WE CAN GUARANTEE NO STEADY-STATE SOLUTIONS, IT FOLLOWS THAT THE SOLUTION TO

$$u_t = \Delta u - \frac{\lambda}{(1+u)^2} \quad \text{IN } \Omega$$

$$u = 0 \quad \text{ON } \partial\Omega; \qquad u = 0 \quad \text{AT } t = 0$$

IS SUCH THAT $\inf_{\Omega} u = -1 \longrightarrow$ SINGULARITY IN FINITE-TIME (QUENCHING!).

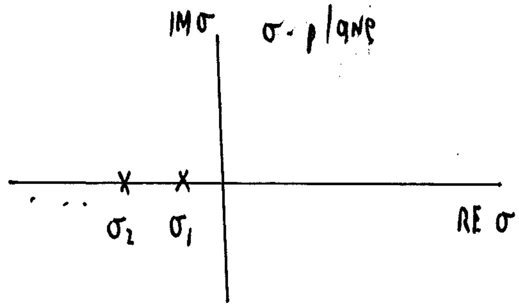
LINEAR STABILITY ANALYSIS

WE BEGIN BY WRITING $u_t = \Delta u - \lambda f(u)$ IN Ω. LET $\bar{u}$ BE steady-state solution so that $\Delta \bar{u} = \lambda f(\bar{u})$, $x \in \Omega$, $\quad f(\bar{u}) = \dfrac{1}{(1+\bar{u})^2}$

WITH $\bar{u} = 0$ ON $\partial\Omega$.

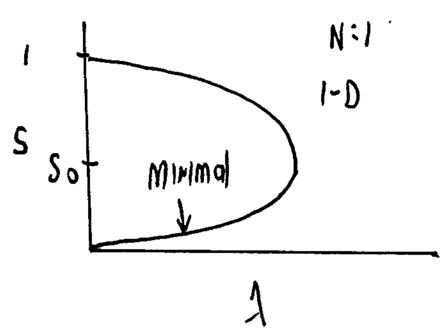
THE LINEARIZATION YIELDS
$$\Delta \Phi = \lambda f'(\bar{u})\,\Phi + \sigma \Phi \quad \text{IN } \Omega$$
$$\Phi = 0 \quad \text{ON } \partial\Omega.$$

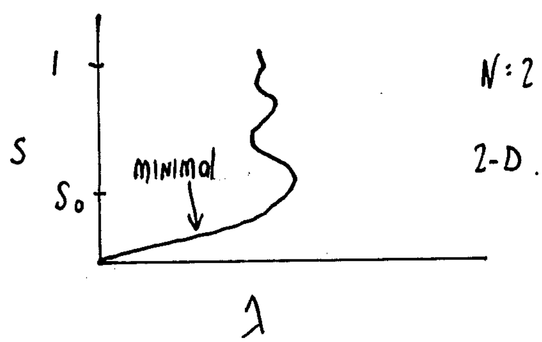
WE HAVE THAT σ_j, Φ_j FOR $j = 1, 2, \ldots$ ARE REAL WITH $j = 1, 2, \ldots$. FOR LINEAR STABILITY WE MUST SHOW THAT $\sigma_1 < 0$ SO THAT



NOW CONSIDER FOR SIMPLICITY A BALL DOMAIN IN $\mathbb{R}^N$ SO THAT
$$\Delta \bar{u} = \bar{u}_{rr} + \frac{(N-1)}{r}\bar{u}_r \quad \text{WITH EITHER } N=1 \text{ OR } N=2.$$

LABEL $\bar{u}(0) = -S$ SO THAT $0 < S < 1$. WE HAVE SHOWN THAT





THE MINIMAL BRANCH IS THE ONE FOR WHICH $d\lambda/ds > 0$ ON $0 \le S < S_0$.

NOTICE THAT AT THE SIMPLE FOLD POINT WE HAVE $\lambda'(S_0) = 0$ WITH $\lambda'(S) < 0$ FOR $S > S_0$ (SMALL) AND $\lambda'(S) > 0$ FOR $S < S_0$ (SMALL).

THE PROBLEM FOR $\bar{u}(r, S)$ IS
$$(*)\left\{\begin{array}{l} \bar{u}_{rr} + \dfrac{(N-1)}{r}\bar{u}_r = \dfrac{\lambda}{(1+\bar{u})^2} \quad \text{IN } 0 < r < 1 \\ \bar{u}(0,S) = -S, \quad \bar{u}_r(0,S) = 0, \quad \bar{u}(1,S) = 0. \end{array}\right.$$

THIS YIELDS THE SOLUTION $\bar{u}(r,s)$ AND $\lambda(s)$.

WE NOW LABEL $\Delta_N \bar{u} = \bar{u}_{rr} + \frac{(N-1)}{r}\bar{u}_r = r^{1-N}\left[r^{N-1}\bar{u}_r \right]_r = \frac{\lambda}{(1+\bar{u})^2}$

WE DIFFERENTIATE WRT s TO OBTAIN

$$\Delta_N \bar{u}_s - \lambda f'(\bar{u})\bar{u}_s = \lambda_s f(\bar{u}) \quad \text{WITH} \quad f(\bar{u}) = \frac{1}{(1+\bar{u})^2}, \quad f'(\bar{u}) = -\frac{2}{(1+\bar{u})^3}.$$

$$\bar{u}_s(0;s) = -1, \quad \bar{u}_{sr}(0;s) = 0, \quad \bar{u}_s(1;s) = 0.$$

NOW WE CONSIDER THE EIGENVALUE PROBLEM

$$(+) \left\{ \begin{array}{l} \Delta_N \Phi - \lambda f'(\bar{u})\Phi = \sigma \Phi. \\ \Phi_r = 0 \text{ AT } r = 0; \ \Phi = 0 \text{ at } r = 1. \end{array} \right.$$

WE THEN USE LAGRANGE TO OBTAIN

$$\int_\Omega \Phi\left(\Delta_N \bar{u}_s - \lambda f'(\bar{u})\bar{u}_s\right) - \bar{u}_s\left(\Delta_N \Phi - \lambda f'(\bar{u})\Phi\right) \, d\underline{x} = \int_{\partial\Omega} \left(\Phi \, \partial_n \bar{u}_s - \bar{u}_s \, \partial_n \Phi\right) d\underline{x} = 0.$$

THIS YIELDS THAT
$$\int_\Omega \Phi \lambda_s f(\bar{u}) \, d\underline{x} = \sigma \int_\Omega \bar{u}_s \Phi \, d\underline{x}$$

WE CONCLUDE THAT $\boxed{\sigma \int_\Omega \bar{u}_s \Phi \, d\underline{x} = \lambda_s \int_\Omega \Phi f(\bar{u}) \, d\underline{x}}$ (*).

WHERE $f(\bar{u}) = \frac{1}{(1+\bar{u})^2}$. THEREFORE, AT A FOLD POINT $\lambda_s = 0$ WE HAVE THAT THE LINEARIZED PROBLEM (+) HAS A ZERO EIGENVALUE. WE OBSERVE THE FOLLOWING:

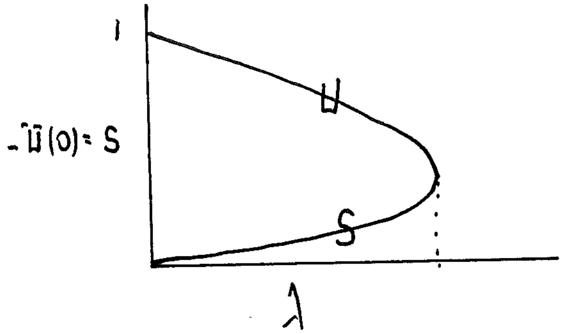
(i) SUPPOSE THAT $\bar{u}_s < 0$ ON $0 \leq r \leq 1$. THEN APPLYING (*) WITH $\Phi = \Phi_1 > 0$ WE CONCLUDE THAT $\sigma_1 < 0$ IF $\lambda_s > 0 \rightarrow$ stable

$\sigma_1 > 0$ IF $\lambda_s < 0. \rightarrow$ unstable.

(ii) FROM THE PDE CAN WE INFER ANYTHING ABOUT $\bar{u}_s$?

(iii) IF $\bar{u}_s < 0$ ON $0 \leq r \leq 1$ THEN $\Phi_1 = -\bar{u}_s > 0$ ON $0 < r < 1$ AND THE FOLD POINT IS DUE TO EXCHANGE OF STABILITY.

IN 1-D IF $\bar U_s < 0$ ON $0 \le r \le 1$ WE HAVE



• JUST BECAUSE $\lambda_s = 0$ DOES NOT MEAN THAT THE PRINCIPAL EIGENVALUE HAS CROSSED THROUGH ZERO, SINCE $\bar U_s$ MAY CHANGE SIGN. IT IS HOWEVER DIFFICULT TO FIND WHETHER $\bar U_s(r)$ IS OF ONE SIGN IN GENERAL.

1-D PROBLEM (*) BECOMES ON $0 < x < 1/2$

$$\sigma \int_0^{1/2} \bar U_s \, \Phi \, dx = \lambda_s \int_0^{1/2} \Phi \, f(\bar U) \, dx$$

THE PROBLEM FOR $\bar U$ IS $\bar U_{xx} = \dfrac{\lambda}{(1+\bar U)^2}$ $\quad \bar U(0) = -S, \; \bar U_x(0) = 0, \; \bar U(\tfrac{1}{2}) = 0.$

INTEGRATING ONCE $\quad \dfrac{\bar U_x^2}{2} = -\dfrac{\lambda}{1+\bar U} + \dfrac{\lambda}{1-S} = \dfrac{(\bar U + S)\lambda}{(1+\bar U)(1-S)}$

NOW $\quad \bar U_x = \sqrt{2\lambda} \left(\dfrac{\bar U + S}{(1+\bar U)(1-S)} \right)^{1/2}$

$$\rightarrow \quad \frac{(1+\bar U)^{1/2}}{(\bar U + S)^{1/2}} \, d\bar U = \frac{\sqrt{2\lambda}}{\sqrt{1-S}} \, dx.$$

NOW INTEGRATING $\quad \displaystyle\int_{-S}^{\bar U} \frac{(1+\Lambda)^{1/2}}{(\Lambda + S)^{1/2}} \, d\Lambda = \sqrt{\frac{2\lambda}{1-S}} \; x$

LET $\Lambda + S = t^2$. THEN $(\Lambda + S)^{1/2} = t$ AND $d\Lambda = 2t \, dt$.

THEN $\dfrac{d\Lambda}{(\Lambda+S)^{1/2}} = 2$ AND $\displaystyle 2\int_0^{\sqrt{U+S}} \sqrt{1-S+t^2} \, dt = \frac{\sqrt{2\lambda}}{\sqrt{1-S}} \, x.$

NOW THIS YIELDS

$$x\sqrt{\lambda} = \sqrt{2}\,\sqrt{1-S} \int_0^{\sqrt{U+S}} \sqrt{1-S+t^2} \, dt.$$

NOW LET $t = (1-S)^{1/2}\eta$. $dt = (1-S)^{1/2} d\eta$.

THEN
$$x\sqrt{\lambda} = \sqrt{2(1-S)} \int_0^{\sqrt{\frac{U+S}{1-S}}} (1-S)\sqrt{1+\eta^2}\, d\eta.$$

WE CONCLUDE THAT
$$x\sqrt{\lambda} = \sqrt{2}\,(1-S)^{3/2} \int_0^{\eta(U,S)} \sqrt{1+\eta^2}\, d\eta$$

WITH
$$\eta(U,S) = \left(\frac{U+S}{1-S}\right)^{1/2} = \left(-1 + \frac{(U+1)}{(1-S)}\right)^{1/2}$$

NOW WHEN $x = \frac{1}{2}$, $U = 0$ SO THAT
$$\sqrt{\lambda} = 2\sqrt{2}\,(1-S)^{3/2} \int_0^{\sqrt{\frac{S}{1-S}}} \sqrt{1+\eta^2}\, d\eta.$$

DEFINES λ AS A FUNCTION OF S.

NOW WE WOULD LIKE TO ESTABLISH THE SIGN OF U_S ON $0 < x < \frac{1}{2}$.

WE CALCULATE $\frac{1}{2} x \lambda^{-1/2} \lambda_S = -\frac{3}{2}\sqrt{2}\,(1-S)^{1/2} \int_0^{\eta} \sqrt{1+\eta^2}\, d\eta + \sqrt{2}\,(1-S)^{3/2}\sqrt{1+\eta^2}\,\eta_S.$

THUS
$$x \lambda^{-1/2}\lambda_S + \sqrt{2}(1-S)^{1/2}\int_0^{\eta}\sqrt{1+\eta^2}\, d\eta = \sqrt{2}\,(1-S)^{3/2}\sqrt{1+\eta^2}\,\eta_S.$$

THIS YIELDS
$$\eta_S = \frac{1}{\sqrt{2}\,(1-S)^{3/2}\sqrt{1+\eta^2}}\left[x\frac{\lambda_S}{\sqrt{\lambda}} + \sqrt{2}\,(1-S)^{1/2}\int_0^{\eta}\sqrt{1+\eta^2}\, d\eta\right].$$

NOW
$$2\eta\eta_S = U_S(1-S)^{-1} + (U+1)(1-S)^{-2}$$

SO
$$U_S(1-S)^{-1} + (U+1)(1-S)^{-2} = \frac{1}{2\sqrt{2}\,\eta\,(1-S)^{3/2}\sqrt{1+\eta^2}}\left[x\frac{\lambda_S}{\sqrt{\lambda}} + \sqrt{2}\,(1-S)^{1/2}\int_0^{\eta}\sqrt{1+\eta^2}\, d\eta\right].$$

WE WOULD LIKE TO CLAIM THAT $U_S < 0$ ON THIS BRANCH.

NOW FROM (*) WE DEVELOP A FORMULA TO HOLD AT ANY FOLD POINT WHERE $\lambda_s = 0$ BUT $\lambda_{ss} \neq 0$. WE DIFFERENTIATE (*) WRT S AND EVALUATE AT $S = S_0$ WHERE $\sigma = 0$ AND $\lambda_s = 0$. AT FOLD POINT WE HAVE $\Phi = \bar{u}_s$

DIFFERENTIATING USING PRODUCT RULE

$$\sigma' \int_\Omega \bar{u}_s \Phi \, dx + \sigma \frac{d}{ds} \int_\Omega \bar{u}_s \Phi \, d\underline{x} = \lambda_{ss} \int_\Omega \Phi f(\bar{u}) \, dx + \lambda_s \frac{d}{ds}\left(\int_\Omega \Phi f(\bar{u}) \, d\underline{x} \right)$$

EVALUATE AT $S = S_0$ WHERE $\Phi = \bar{u}_s$ AND $\lambda_s = \sigma = 0$ WE GET

$$\boxed{\sigma'(s_0) \int_\Omega (\bar{u}_s)^2 \, d\underline{x} = \lambda_{ss}(s_0) \int_\Omega \bar{u}_s f(\bar{u}) \, d\underline{x}} \qquad (+)$$

REMARK WE CAN OBTAIN (+) BY USING SOLVABILITY CONDITIONS.

WE HAVE $\Delta \Phi_s - \lambda f'(\bar{u}) \Phi_s = \lambda_s f'(\bar{u}) \Phi + \lambda f''(\bar{u}) \bar{u}_s \Phi + \sigma \Phi_s + \sigma' \Phi$

NOW AT $s = s_0 \longrightarrow \Delta \Phi_s - \lambda F'(\bar{u}) \Phi_s = \lambda F''(\bar{u}) \bar{u}_s^2 + \sigma' \bar{u}_s$ SINCE $\Phi = \bar{u}_s = \psi$.

SOLVABILITY YIELDS $\lambda \int_\Omega f''(\bar{u}) \bar{u}_s^3 \, d\underline{x} = -\sigma' \int_\Omega \bar{u}_s^2 \, dx$. (1)

NOW $\Delta \psi - \lambda f'(\bar{u}) \psi = \lambda_s f(\bar{u})$ WITH $\psi = u_s$.

DIFFERENTIATE WRT S: $\longrightarrow \Delta \psi_s - \lambda f'(\bar{u}) \psi_s = \lambda_s f'(\bar{u}) \psi + \lambda f''(\bar{u}) \bar{u}_s \psi + \lambda_{ss} f(\bar{u}) + \lambda_s f'(\bar{u}) \bar{u}_s$.

AT $S = S_0$, WE HAVE

$$\Delta \psi_s - \lambda f'(\bar{u}) \psi_s = \lambda f''(\bar{u}) \bar{u}_s^2 + \lambda_{ss} f(\bar{u})$$

SOLVABILITY YIELDS $\lambda \int_\Omega f''(\bar{u}) \bar{u}_s^3 \, d\underline{x} = -\lambda_{ss} \int_\Omega f(\bar{u}) \bar{u}_s \, d\underline{x}$ (2)

NOW COMBINING, WE OBTAIN FROM (1) AND (2), THAT

$$\lambda_{ss} \int_\Omega f(\bar{u}) \bar{u}_s \, d\underline{x} = \sigma' \int_\Omega (\bar{u}_s)^2 \, d\underline{x}.$$

WE WANT TO FIND THE SIGN OF σ'. THIS IS SAME AS (+).

NOW WE WRITE $$\lambda_{ss} \frac{d}{ds}\left(\int_\Omega \hat{f}(\bar{u})\, d\underline{x} \right) = -\sigma' \int_\Omega (\bar{u}_s)^2\, d\underline{x}$$

WHERE $$\hat{f}(u) = -\int^{\bar{u}} f(\eta)\, d\eta = -\int^{\bar{u}} \left(\frac{1}{(1+\eta)^2} \right) d\eta = \frac{1}{1+\bar{u}}. \qquad (3)$$

THUS, $$\lambda_{ss} \frac{d}{ds}\left(\int_\Omega \frac{1}{(1+\bar{u})}\, d\underline{x} \right) = -\sigma' \int_\Omega (\bar{u}_s)^2\, d\underline{x} \quad \text{AT } s = s_0 \qquad (4).$$

THIS IS A KEY FORMULA WHICH WE USE AS FOLLOWS:

NOW OUR PDE IS THE EULER-LAGRANGE EQUATION OF THE FUNCTIONAL $I(u,\lambda)$ DEFINED BY

$$I(u,\lambda) \equiv \int_\Omega \underbrace{\left(\frac{1}{2} |\nabla u|^2 - \frac{\lambda}{1+u} \right)}_{G(u,\nabla u)} d\underline{x}.$$

THE E-L EQUATION IS $$G_u - \frac{d}{dx} G_{u_x} - \frac{d}{dy} G_{u_y} = 0. \qquad G_{u_x} = u_x,\; G_{u_y} = u_y, \quad G_u = \frac{\lambda}{(1+u)^2}$$

SO WE OBTAIN THAT $\bar{u}$ SATISFIES
$$\frac{\lambda}{(1+\bar{u})^2} = u_{xx} + u_{yy}.$$

NOW $$-I_\lambda = \int_\Omega \frac{1}{(1+\bar{u})}\, d\underline{x}$$ EVALUATED ON SOLUTION BRANCH.

THUS INTRODUCE A VARIABLE S BY $$S = \int_\Omega \frac{1}{(1+\bar{u})}\, d\underline{x}.$$

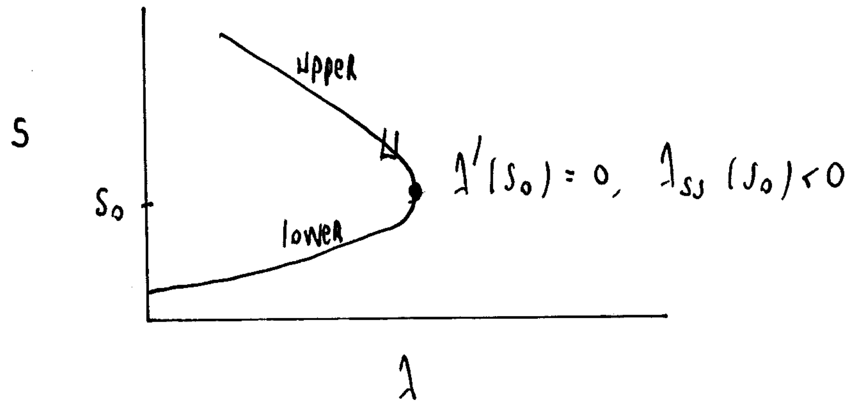
WHERE $$\Delta \bar{u} = \frac{\lambda}{(1+\bar{u})^2}, \qquad \bar{u} = 0 \text{ ON } \partial\Omega.$$

THEN (4) BECOMES $$\lambda_{ss} = -\sigma' \int_\Omega (\bar{u}_s)^2\, d\underline{x}. \qquad (5)$$

THE IDEA THEN IS TO PLOT THE "PREFFERED" - BIFURCATION DIAGRAM (M21)

OF $S \equiv \int_\Omega \frac{1}{(1+u)}\, dx$ VERSUS λ.

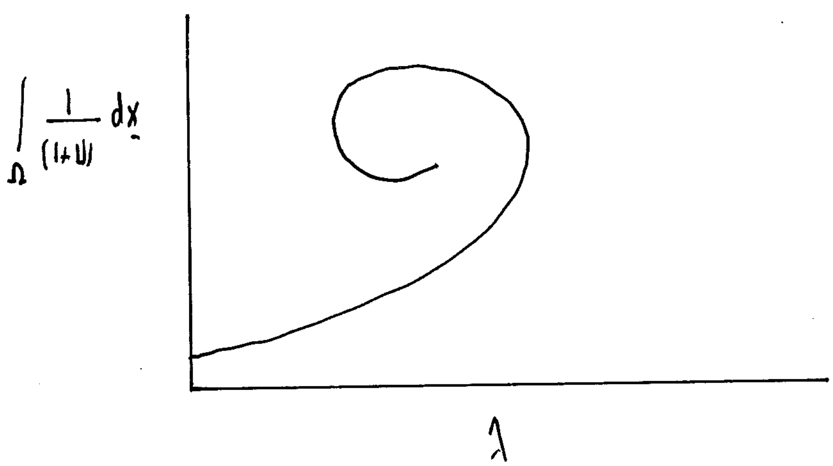
SUPPOSE WE GET A PICTURE AS SHOWN



USING (5) WE CAN OBTAIN THAT $\sigma'(S_0) > 0$ SO $\sigma > 0$ IF $S > S_0$ (locally), $\sigma < 0$ IF $S < S_0$ locally.

CONCLUSION ON UPPER BRANCH WE HAVE THAT LINEARIZATION HAS AN UNSTABLE EIGENVALUE SINCE $\sigma'(S_0) > 0 \rightarrow \sigma > 0$ FOR $0 < S - S_0 \ll 1$.

IN OTHER CASES WE PERHAPS HAVE TO TAKE A DIFFERENT PARAMETRIZATION, AND NOT $S = \int_\Omega \frac{1}{(1+u)}\, d\underline{x}$ AS THE CURVE CAN BE AS SHOWN



WE WILL NOW IMPLEMENT THIS FOR THE RADIALLY SYMMETRIC MEMS PDE WHERE WE CAN USE THE SCALE INVARIANCE TECHNIQUE TO COMPUTE $\int_\Omega \frac{1}{(1+u)}\, d\underline{x} = \int^1 \frac{r}{(1+u)}\, dr$ VERSUS λ.

LINEAR STABILITY: RADIALLY SYMMETRIC

WE NOW CONSIDER THE STEADY-STATE PROBLEM

$$u'' + \frac{1}{r}u' = \frac{\lambda}{(1+u)^2}, \quad u'(0)=0, \quad u(1)=1.$$

WE WANT TO PLOT $I = \int_0^1 \frac{r}{(1+u)}\,dr$ VERSUS λ. THIS "PREFERRED" DIAGRAM HAS STABILITY INFORMATION. RECALL OUR SCALE INVARIANCE TECHNIQUE:

$$u(r) = -1 + \alpha w(\gamma r).$$

THEN WITH $\alpha = 1/w(\gamma)$ WE GET

$$\lambda = \gamma^2/[w(\gamma)]^3, \quad -u(0) = 1 - 1/w(\gamma), \quad \text{WHERE} \quad w'' + \frac{1}{\rho}w' = \frac{1}{w^2} \text{ WITH } w(0)=1,\ w'(0)=0.$$

NOW WE CALCULATE $I = \int_0^{\gamma} \frac{(\rho/\gamma)(1/\gamma)\,d\rho}{\alpha[w(\rho)]^2} = \frac{w(\gamma)}{\gamma^2}\int_0^{\gamma}\frac{\rho}{w(\rho)}\,d\rho \equiv \frac{w(\gamma)}{\gamma^2}J[\gamma].$

NUMERICAL IMPLEMENTATION

DEFINE $w_1 = w$, $w_2 = w'$, $J' = \rho/w_1$

THEN WE HAVE THE FIRST ORDER SYSTEM

$$w_1' = w_2 \qquad \text{WITH} \qquad w_1(0) = 1$$

$$w_2' = -\frac{1}{\rho}w_2 + \frac{1}{w_1^2} \qquad w_2(0) = 0$$

$$J' = \rho/w_1(\rho) \qquad J(0) = 0$$

THEN WE OUTPUT AT "TIME" $\rho = \gamma$

$$\lambda = \frac{\gamma^2}{[w(\gamma)]^3}, \quad I = \frac{w(\gamma)}{\gamma^2}J(\gamma).$$

REMARK TO AVOID THE SINGULARITY NEAR $\rho = 0$ WE USE $w \sim 1 + \rho^2/4$, $w' \sim \rho/2$, $J \sim \rho^2/2$. FOR $\rho \ll 1$. SET $\rho = \varepsilon = 10^{-5}$ AND PUT

$$w_1(\varepsilon) = 1 + \varepsilon^2/4$$

$$w_2(\varepsilon) = \varepsilon/2$$

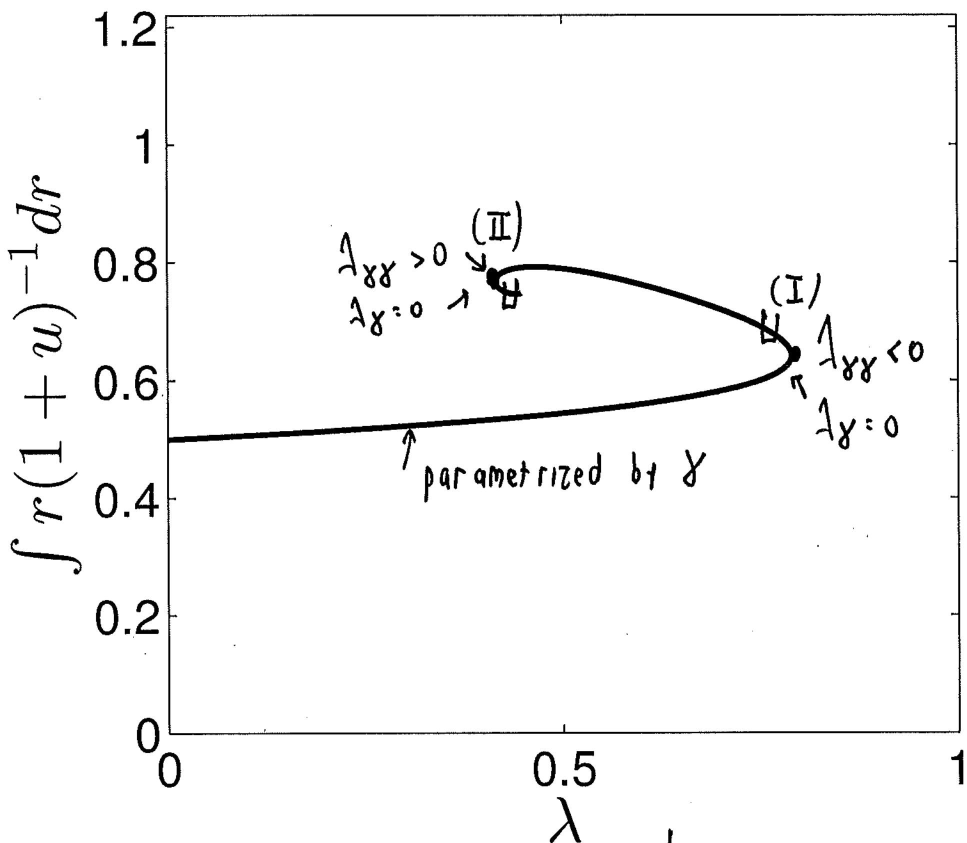
$$J(\varepsilon) = \varepsilon^2/2$$

AS THE INITIAL CONDITIONS.

WE THEN PLOT $I = \int_0^1 \frac{r}{(1+u)}\,dr$ VERSUS λ AND OBTAIN THE PLOT AS SHOWN BELOW. RECALL THAT IN TERMS OF A "PARAMETRIZATION" S, WE HAVE

$$\lambda_{ss}\,\frac{d}{ds}\left(\int_0^1 \frac{r}{(1+u)}\,dr\right) = -\sigma' \int_0^1 (u_s)^2\, r\,dr. \quad (*)$$

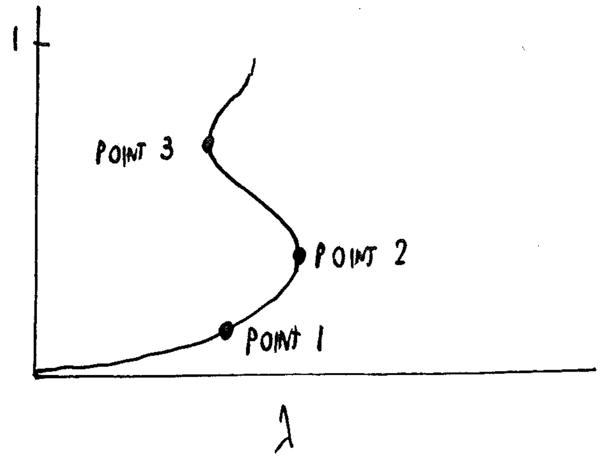
WE CAN CHOOSE $S = \gamma$ FROM THE SCALE INVARIANCE ODE SYSTEM:

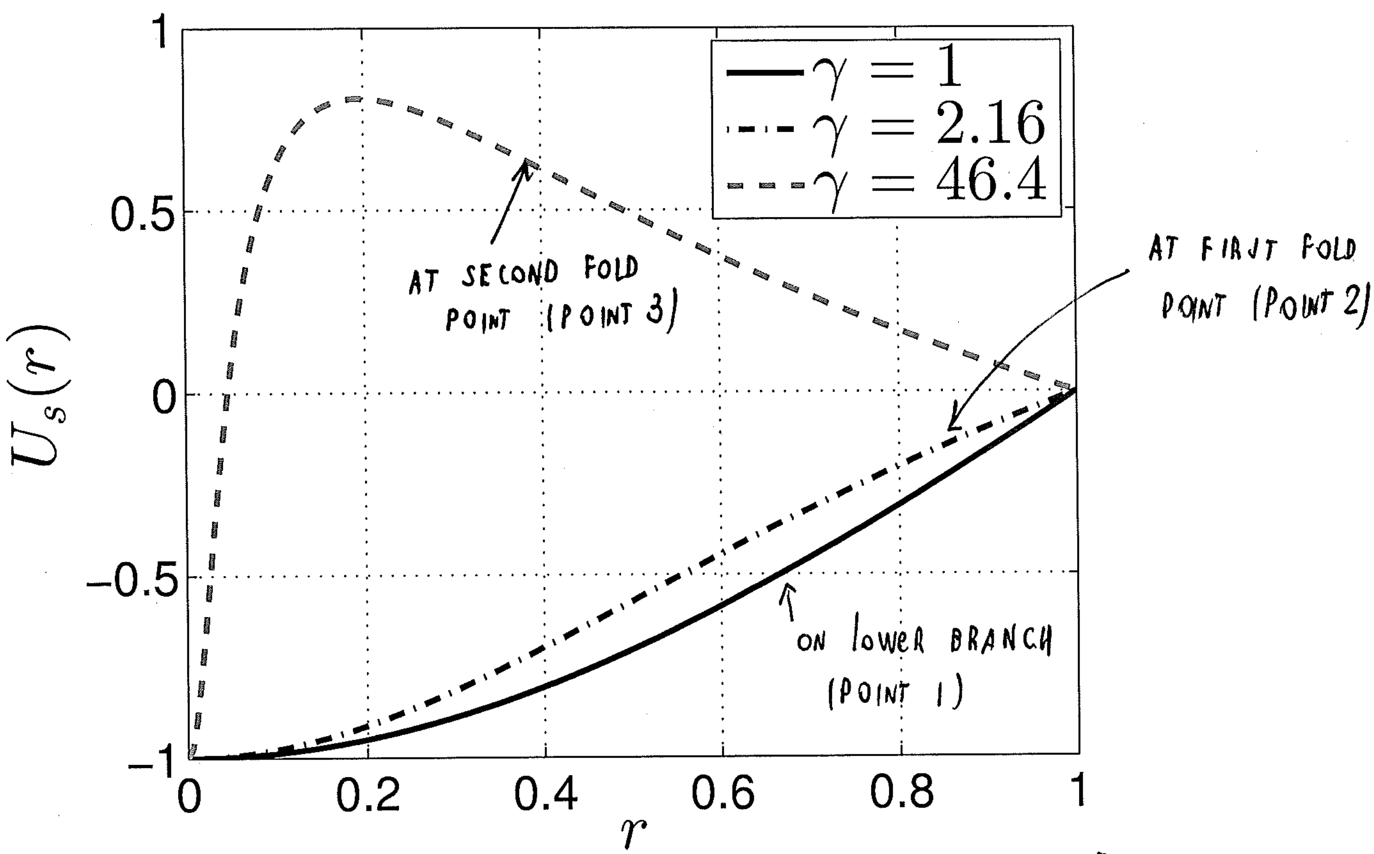


- AT POINT (I) WE HAVE $\lambda_{\gamma\gamma} < 0$ AND $\frac{d}{d\gamma}\left(\int_0^1 \frac{r\,dr}{(1+u)}\right) > 0$ SO $\sigma' > 0$ THUS SLIGHT ABOVE FOLD POINT WE HAVE INSTABILITY SINCE $\exists\ \sigma > 0$.
- AT POINT (II) WE HAVE $\lambda_{\gamma\gamma} > 0$ AND $\frac{d}{d\gamma}\left(\int_0^1 \frac{r\,dr}{(1+u)}\right) < 0$ SO $\sigma' > 0$ AND SO ON OTHER BRANCH SLIGHTLY BELOW THE FOLD WE HAVE $\sigma > 0 \rightarrow$ UNSTABLE.

MEMS IN A DISK — PLOTS OF $\bar{u}_s(r)$

$S \equiv -u(0)$





$$\Delta \Phi - \lambda f'(\bar{u}) \Phi = \sigma \Phi$$

RECALL THAT $\sigma \int_\Omega \bar{u}_s \Phi \, d\underline{x} = \lambda_s \int_\Omega \Phi f(\bar{u}) \, d\underline{x}$

SO IN POLAR COORDINATES IF Ω IS UNIT DISK WE HAVE

$\sigma \int_0^1 \bar{u}_s \Phi \, r \, dr = \lambda_s \int_0^1 \Phi f(\bar{u}) \, r \, dr$. WHEN $\lambda_s = 0$, $\Phi = \bar{u}_s$. AND $\Phi_1 = \bar{u}_s$

- WE CONCLUDE ON lower BRANCH where $\lambda_s > 0$ THAT SINCE $\bar{u}_s < 0 \longrightarrow \sigma_1 < 0$.
- AT FIRST FOLD POINT, THE PRINCIPAL EIGENVALUE CROSSES THROUGH ZERO
- AT SECOND FOLD POINT THE SECOND EIGENVALUE CROSSES THROUGH ZERO.

THUS IS BECAUSE $\bar{u}_s$ HAS ONE-ZERO CROSSING $\longrightarrow \Phi_2 = \bar{u}_s$

NOW WE DEVISE A NUMERICAL APPROACH TO CALCULATE $\bar{U}_S(r)$

WHEN $\bar{U}(0) = -S$.

WE RECALL $\bar{U}(0) = -1 + \dfrac{1}{w(\gamma)}$ SO $1 - S = \dfrac{1}{w(\gamma)}$ OR $w(\gamma) = (1-S)^{-1}$.

NOW $\bar{U}(r) = -1 + w(\gamma r)(1-S)$ AND $\lambda = \gamma^2 (1-S)^3$.

WE NOW CALCULATE $w'(\gamma)\gamma_S = (1-S)^{-2}$ SO $\gamma_S = \dfrac{(1-S)^{-2}}{w'(\gamma)} = \dfrac{[w(\gamma)]^2}{w'(\gamma)}$.

WE OBTAIN THAT $\bar{U}_S = w'(\gamma r) r \gamma_S (1-S) - w(\gamma r)$

SUBSTITUTING WE OBTAIN

$$\bar{U}_S(r) = \frac{w(\gamma)}{w'(\gamma)} r w'(\gamma r) - w(\gamma r), \quad \text{ON } 0 < r < 1.$$

$$\lambda_S = -3(1-S)^2 \gamma^2 + 2\gamma\gamma_S (1-S)^3 = \gamma(1-S)^2 \left[2\gamma_S (1-S) - 3\gamma \right].$$

$$\text{SO } \lambda_S = \gamma (1-S)^2 \left[2(1-S) \frac{(w(\gamma))^2}{w'(\gamma)} - 3\gamma \right] \qquad (2)$$

WE WANT TO SHOW THAT $\bar{U}_S(r) < 0$ ON $0 < r < 1$. NOW WE ELIMINATE $(1-S)$ TO OBTAIN:

$$(*) \begin{cases} \bar{U}_S(r) = \dfrac{w(\gamma)}{w'(\gamma)} r w'(\gamma r) - w(\gamma r) \\[2ex] \lambda_S = \dfrac{\gamma}{[w(\gamma)]^2} \left[\dfrac{2\, w(\gamma)}{(w'(\gamma))} - 3\gamma \right] \end{cases}$$

$$\lambda = \frac{\gamma^2}{[w(\gamma)]^3}$$

$$S = 1 - 1/w(\gamma)$$

WE WANT TO PLOT λ_S AND $\bar{U}_S(r)$ ON $[0,1]$.

TOUCHDOWN BEHAVIOR — DETAILED LOCAL ANALYSIS NEAR SINGULARITY

WHAT IS THE LOCAL BEHAVIOR NEAR THE SINGULARITY FOR

$$u_t = u_{xx} - \frac{\lambda}{(1+u)^2} \qquad u(\pm 1/2, t) = 0, \quad u(x,0) = 0.$$

WE LET $w = 1 + u$ SO THAT

$$w_t = w_{xx} - \frac{\lambda}{w^2} \qquad \begin{array}{l} w = 1 \text{ ON } x = \pm 1/2 \\ w = 1 \text{ AT } t = 0. \end{array}$$

NOW WE SCALE BY $w = L\hat{w}$, $x = X\hat{x}$, $t - T_x = \tau\hat{t}$ where T_x IS THE TOUCHDOWN TIME. THIS YIELDS THAT,

$$\frac{L}{\tau}\hat{w}_{\hat{t}} = \frac{L}{X^2}\hat{w}_{\hat{x}\hat{x}} - \frac{\lambda}{L^2\hat{w}^2}.$$

TO BALANCE ALL OF THE TERMS $\rightarrow$ $X = \tau^{1/2}$, $L = \tau^{1/3}$. THIS SUGGESTS THAT WE SHOULD LOOK FOR A QUASI-SIMILARITY SOLUTION OF THE FORM :

$$w(x,t) = (T_x - t)^{1/3} H\left[\frac{x}{\sqrt{T_x - t}} ; s\right] \qquad \text{where } s = -\log(T_x - t).$$

IN OTHER WORDS WE ASSUME THAT TOUCHDOWN OCCURS NEAR $x = 0$.

WE CALCULATE

$$w_{xx} = \gamma^{-2/3} H_{\Lambda\Lambda} \qquad w_t = -\frac{1}{3}\gamma^{-2/3} H + \gamma^{1/3}\left[\frac{1}{2} x \gamma^{-3/2} H_\Lambda + \frac{H_s}{\gamma}\right] \qquad \gamma = (T_x - t).$$

THEREFORE,

$$-\frac{1}{3}\gamma^{-2/3} H + \gamma^{-2/3}\left[\frac{\Lambda}{2} H_\Lambda + H_s\right] = \gamma^{-2/3} H_{\Lambda\Lambda} - \lambda\gamma^{-2/3} H^{-2}.$$

THIS YIELDS THE FOLLOWING PDE FOR $H(\Lambda; s)$

$$\left\{ \begin{array}{ll} H_s = H_{\Lambda\Lambda} - \dfrac{\Lambda}{2} H_\Lambda + \dfrac{1}{3} H - \dfrac{\lambda}{H^2} & 0 \le \Lambda < \infty, \quad 0 < s < \infty. \\ H_\Lambda = 0 \text{ ON } \Lambda = 0 & (\text{i.e. } H \text{ IS even IN } x). \end{array} \right.$$

FOR A FIXED S, WE LOOK FOR THE ASYMPTOTIC BEHAVIOR AS $\Lambda \to +\infty$. WE SUBSTITUTE

$$H \sim A\Lambda^{P} + \ldots \quad \text{INTO THE EQUATION TO OBTAIN}$$

$$P(P-1)A\Lambda^{P-2} - P\frac{A}{2}\Lambda^{P} + \frac{1}{3}A\Lambda^{P} - \frac{\lambda}{A^{2}}\Lambda^{-2P} = 0.$$

THEREFORE, $P = 2/3$ AND $H \sim A\Lambda^{2/3}$ AS $\Lambda \to +\infty$. HOWEVER, WE NEED THE BEHAVIOR FOR FIXED Λ AS $S \to +\infty$.

NOW TO DETERMINE THE BEHAVIOR NEAR THE "BLOW-UP" TIME WE LET $t \to T_*^-$ SO THAT $S \to +\infty$. FOR A FIXED Λ WE ASK WHAT IS THE LIMITING BEHAVIOR AS $S \to +\infty$.

THE CONJECTURE IS THAT $H(S,\Lambda) \to H_e$ AS $S \to \infty$ ON Λ FIXED, WHERE H_e IS THE EQUILIBRIUM SOLUTION

$$\frac{1}{3}H_e = \frac{\lambda}{H_e^{2}} \quad \text{OR} \quad H_e^{3} = 3\lambda \quad \text{OR} \quad H_e = \sqrt[3]{3\lambda}.$$

THIS LEADS TO THE STATEMENT THAT

$$W(X,t) \longrightarrow (T_* - t)^{1/3}(3\lambda)^{1/3} \quad \text{AS} \quad t \to T^* \quad \text{FOR } |X| \leq C(T_* - t)^{1/2}.$$

THIS TURNS OUT TO BE CORRECT. THE FOLLOWING IS A RIGOROUS RESULT:

<u>PROPOSITION</u> CONSIDER $W_t = W_{XX} - \dfrac{1}{W^{B}}$ $\quad |X| \leq L,\ t \geq 0,\ B \geq 1$

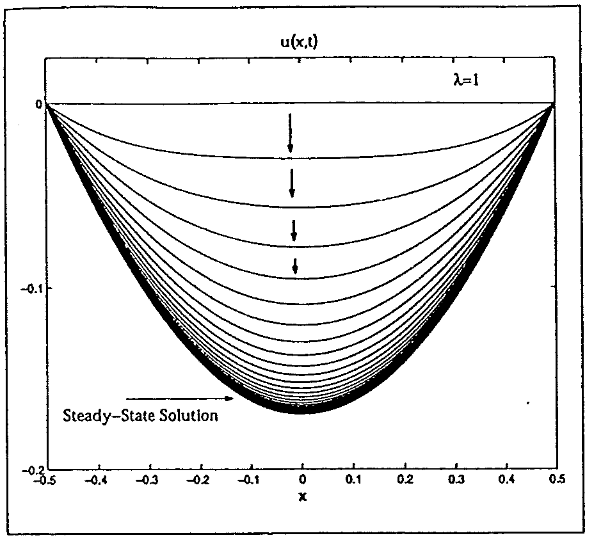
WITH $W(\pm L, t) = 1$, $W(X,0) = W_0(X) > 0$ WITH $W_0(\pm L) = 1$. LET W BE A SOLUTION THAT QUENCHES AT A FINITE TIME T_*, AND LET X_0 BE THE QUENCHING POINT. THEN,

$$\lim_{t \to T^-} W(X,t)\,(T_* - t)^{-1/(B+1)} = (B+1)^{1/B+1} \quad \text{FOR } |X - X_0| \leq C(T_* - t)^{1/2}.$$

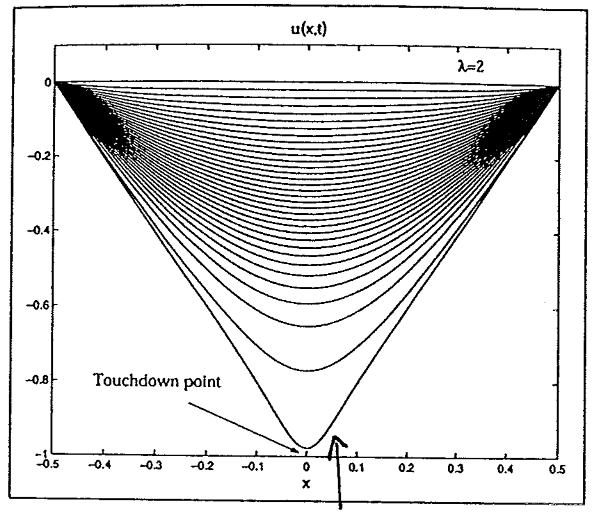
<u>REMARK</u>: FOR OUR PROBLEM $B = 2$ AND $\lambda = 1$ SO $H_{EQ} = 3^{1/3}$ WHICH AGREES WITH THE RESULT ABOVE.

NUMERICAL RESULTS SHOWING BLOW-UP SINGULARITY.

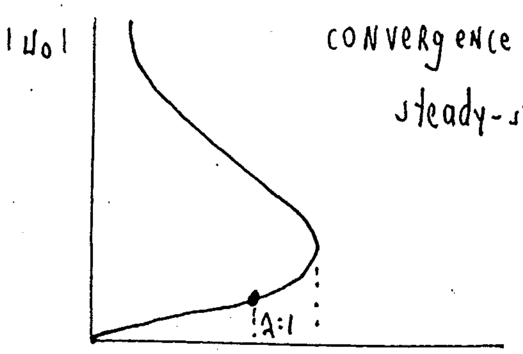
$\lambda = 1$



$\lambda = 2$ (No steady-state)

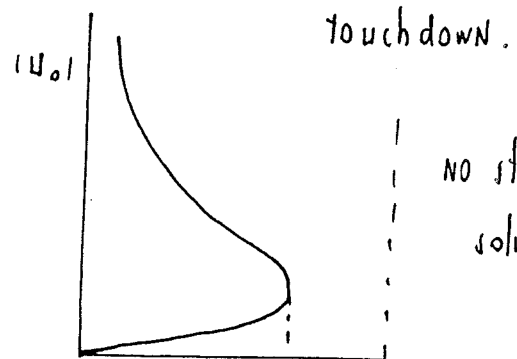


"LOCAL Behavior" to be analyzed touchdown.



convergence to steady-state

λ = 1.4



No steady-state solution

λ 1.4 2

QUESTION: WHAT IS THE LOCAL BEHAVIOR NEAR THE TOUCHDOWN TIME AND THE TOUCHDOWN POINT?

OUCHDOWN IN MEMS

WE CONSIDER THE PDE

$$W_t = W_{xx} - \lambda/_{W^2}\,, \quad \frac{-1}{2} < x < \frac{1}{2}$$

$$W = 1 \text{ ON } x = \pm 1/2, \qquad W = 1 \text{ AT } t = 0$$

WE LOOK FOR A QUASI-SIMILARITY SOLUTION OF THE FORM

$$W(x,t) = (T_* - t)^{1/3} H\left[\frac{x}{\sqrt{T^* - t}}\,;\, s\right] \qquad s = -\log(T_* - t).$$

WE LET $\Lambda = x/\sqrt{T^* - t}$ AND OBTAIN

$$\left\{\begin{array}{l} H_s = H_{\Lambda\Lambda} - \dfrac{\Lambda}{2} H_\Lambda + \dfrac{1}{3} H - \dfrac{\lambda}{H^2}, \qquad 0 \le \Lambda < \infty, \quad 0 < s < \infty \\ H_\Lambda = 0 \quad \text{ON} \quad \Lambda = 0 \end{array}\right.$$

WE WANT TO DETERMINE THE LONG-TIME BEHAVIOR AS $s \to +\infty$.

DEFINE $g(H) = \dfrac{H}{3} - \dfrac{\lambda}{H^2}$ AND NOTICE THAT $g(H_e) = 0$

WHERE $H_e = (3\lambda)^{1/3}$. THEN $V = H - H_e$

$$g'(H_e) = \frac{1}{3} + \frac{2\lambda}{H_e^3} = 1$$

$$g''(H_e) = \frac{-6\lambda}{H_e^4}$$

$$g(H) \sim g'(H_e)V + \frac{g''(H_e)}{2} V^2 + \ldots$$

SO THAT

$$V_s = V_{\Lambda\Lambda} - \frac{\Lambda}{2} V_\Lambda + V - \frac{3\lambda}{H_e^4} V^2 + \ldots$$

WE THEN WRITE THIS PROBLEM IN THE FORM

$$V_s = \mathcal{L} V - w V^2 \qquad w = \frac{3\lambda}{He^4} = \frac{3\lambda}{(3\lambda)^{4/3}} = \frac{1}{(3\lambda)^{1/3}}$$

where $$\mathcal{L} V = \frac{1}{p}\frac{\partial}{\partial \Lambda}(p V_\Lambda) + V \qquad p = e^{-\Lambda^2/4}.$$

THIS IS HERMITE'S OPERATOR!

WE FIRST CONSIDER THE EIGENVALUE PROBLEM

$$(E) \left\{ \begin{array}{l} \mathcal{L}\phi = \lambda \phi, \quad -\infty < \Lambda < \infty \\ \int_{-\infty}^{\infty} p \phi^2 \, d\Lambda < \infty \end{array} \right. \qquad p = e^{-\Lambda^2/4}.$$

THIS PROBLEM IS $\phi_{\Lambda\Lambda} - \frac{\Lambda}{2}\phi_\Lambda + \phi = \lambda\phi$, AND ADMITS THE SPECTRUM

(i) $\lambda_1^+ = 1, \quad \lambda_2^+ = \frac{1}{2}, \quad \phi_1^+ = c_1^+, \quad \phi_2^+ = c_2^+ \Lambda.$

(ii) $\lambda_1^0 = 0, \quad \phi_1^0 = c_1^0 \left(\frac{\Lambda^2}{2} - 1\right)$

(iii) $\lambda_j^- = 1 - \frac{(j+2)}{2}, \quad j = 1, 2, \ldots \qquad \phi_j^- = c_j^- H_{j+2}(\Lambda/2) \quad j = 1, 2, \ldots$

where $$H_j(x) = (-1)^j e^{x^2} \frac{d^j}{dx^j}\left(e^{-x^2}\right) \qquad j = 0, 1, \ldots$$

HERMITE POLYNOMIALS

NORMALIZATION OF $\int_{-\infty}^{\infty} p(\phi_1^0)^2 \, dR = \int_0^{\infty} e^{-\Lambda^2/4} (\phi_1^0)^2 \, d\Lambda = 1$ YIELDS

THAT $c_1^0 = \dfrac{1}{2\pi^{1/4}}$.

NOW WE EXPAND

$$V = B_1(s)\,\phi_1^+(\Lambda) + B_2(s)\,\phi_2^+(\Lambda) + \alpha_1(s)\,\phi_1^0(\Lambda) + [\,\gamma_1(s)\,\phi_1^-(\Lambda) + \dots\,]$$

$\longleftarrow$ growing $\longrightarrow$ $\qquad$ $\longleftrightarrow$ NEUTRAL $\qquad$ $\longleftrightarrow$ decaying

WE WANT $V \longrightarrow 0$ AS $s \longrightarrow \infty$. HENCE $B_1 = B_2 = 0$.

WE SUBSTITUTE INTO

$$V_s = \mathcal{L}V - wV^2$$

TO OBTAIN, $$\dot{\alpha}_1\,\phi_1^0 = -w\,\alpha_1^2(\phi_1^0)^2 + \text{decaying modes.}$$

WE THEN USE ORTHOGONALITY OF EIGENFUNCTIONS TO OBTAIN

$$\dot{\alpha}_1 \int_{-\infty}^{\infty} (\phi_1^0)^2\, p\, d\Lambda = -w\,\alpha_1^2 \int_{-\infty}^{\infty} (\phi_1^0)^3\, p\, d\Lambda$$

WE CALCULATE $$\int_{-\infty}^{\infty} p(\phi_1^0)^2\, d\Lambda = 1 \quad \text{NORMALIZATION}$$

$$\int_{-\infty}^{\infty} p\,(\phi_1^0)^3\, d\Lambda = \int_{-\infty}^{\infty} p\,(c_1^0)^3\left(\frac{\Lambda^2}{2}-1\right)^3 d\Lambda = 4c_1^0.$$

HENCE $$\dot{\alpha}_1 = -4w\,c_1^0\,\alpha_1^2$$

THUS $\alpha_1 \sim A/s$ WITH $A = 1/4wc_1^0$ AS $s \rightarrow \infty$.

$$w = \frac{1}{(32)^{1/3}}$$

WE CONCLUDE THAT

$$H \sim H_e + \frac{A}{s}\,c_1^0\left(\frac{\Lambda^2}{2}-1\right) + \dots \quad \text{AS } s \rightarrow +\infty$$

$$c_1^0 A/s = \frac{1}{4ws}.$$

THEREFORE,

$$H \sim H_e + \frac{1}{4ws}\left(\frac{\Lambda^2}{2} - 1\right) \quad \text{AS} \quad s \to \infty, \ \Lambda \ \text{FIXED.}$$

HOWEVER, $H_e = (3\lambda)^{1/3}$, $w^{-1} = (3\lambda)^{1/3}$, $\Lambda = \dfrac{x}{\sqrt{T^* - t}}$, $s = |\log(T^* - t)|$.

THEN $W \sim (T^* - t)^{1/3} H$ yields,

$$W \sim [3\lambda(T^* - t)]^{1/3}\left(1 + \frac{x^2}{8(T^* - t)|\log(T^* - t)|} - \frac{1}{4|\log(T^* - t)|}\right)$$

AS $T^* - t \to 0$ FOR $\dfrac{x^2}{T^* - t}$ FIXED, AND $U = -1 + W$.

THIS IS THE QUENCH PROFILE.