NONLINEAR OSCILLATORS

1. $x'' + x + \varepsilon h(x) = 0 \qquad 0 < \varepsilon \ll 1, \quad h(0) = 0$ CONSERVATIVE SYSTEM

$x(0) = A, \quad x'(0) = 0$

(i) FIND AN APPROXIMATION VALID FOR $t = O(1/\varepsilon)$ AND CALCULATE AN APPROXIMATION TO THE PERIOD $T(\varepsilon) = 2\pi(1 + a_1\varepsilon + a_2\varepsilon^2 + \ldots)$

(here let $\tau = (1 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \ldots)t$)

2. $x'' + \varepsilon h(x, x') + x = 0 \qquad 0 < \varepsilon \ll 1 \qquad h(0,0) = 0$

(i) FIND special initial conditions $x(0) = A(\varepsilon), \; x'(0) = 0$ SO THAT WE CAN FIND A PERIODIC SOLUTION WHICH IS ISOLATED $\longrightarrow$ LIMIT CYCLE. (LET $A = a_0 + \varepsilon a_1 + \ldots \qquad \tau = (1 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \ldots)t$

(ii) FOR ARBITRARY $x(0), x'(0)$ CALCULATE AN APPROXIMATION TO THE SOLUTION VALID FOR $t = O(1/\varepsilon)$.

(HERE THERE ARE TWO-TIME SCALES $\tau = \varepsilon t, t$.)

3. NOW CONSIDER PROBLEMS WITH FORCING

$$x'' + \varepsilon h(x, x') + x = \varepsilon \cos(\omega t)$$

(i) DANGEROUS when $\omega = 1 + O(\varepsilon)$ AS we are near resonance.

(ii) CAN GET BISTABILITY AND HYSTERESIS

(iii) SUBHARMONIC RESONANCE

(iv) CHAOS IS POSSIBLE $\longrightarrow$ MELNIKOV'S METHOD

4. PARAMETRIC RESONANCE IS FOR

$$x'' + (1 + \varepsilon f(t))x = 0 \qquad f(t + 2\pi) = f(t).$$

EXAMPLE OF (1)

WE CONSIDER
$$x'' + x + \varepsilon x^3 = 0$$
$$x(0) = A, \quad x'(0) = 0$$

THE PHASE PLANE INDICATES PERIODIC SOLUTIONS $\forall A$.

WE WANT AN APPROXIMATION VALID FOR $t = O(1/\varepsilon)$. TRY A NAIVE SERIES
$$x = x_0 + \varepsilon x_1 + \dots$$

THIS GIVES:
$$x_0'' + x_0 = 0, \qquad x_0(0) = A, \quad x_0'(0) = 0$$
$$x_1'' + x_1 = -x_0^3, \qquad x_1(0) = 0, \quad x_1'(0) = 0$$

WE GET $x_0 = A\cos t$ AND USING $\cos^3 t = \frac{3}{4}\cos t + \frac{1}{4}\cos 3t$ WE OBTAIN
$$x_1'' + x_1 = -A^3\cos^3 t = \underbrace{\frac{-3A^3}{4}\cos t}_{\text{RESONANT TERM}} - \frac{A^3}{4}\cos 3t$$

WE SOLVE TO GET
$$x_1 = -\frac{3}{4}t\sin t + \frac{1}{32}\cos 3t.$$

HENCE
$$x = A\cos t + \varepsilon\Big(\underbrace{-\frac{3}{4}t\sin t}_{\text{secular term}} + \frac{1}{32}\cos 3t\Big) + \dots$$

THIS IS NOT VALID WHEN $t = O(1/\varepsilon)$, AND YIELDS A NON-PERIODIC APPROXIMATION.

TO GET AN APPROXIMATION VALID FOR $t = O(1/\varepsilon)$ USE POINCARÉ-LINDSTEDT METHOD. WE LET
$$x = x_0(\tau) + \varepsilon x_1(\tau) + \dots \quad \text{WITH} \quad \tau = \omega t, \quad \omega = 1 + \varepsilon\omega_1 + \dots \quad \varepsilon \to 0.$$

NOW $dx/dt = dx/d\tau \; d\tau/dt = \omega \, dx/d\tau$, SO THAT
$$\frac{d^2x}{d\tau^2} + \frac{1}{\omega^2}(x + \varepsilon x^3) = 0 \qquad \omega^2 = 1 + 2\varepsilon\omega_1 + \dots$$
$$\omega^{-2} = 1 - 2\varepsilon\omega_1 + \dots$$

THEREFORE, $$\frac{d^2X}{d\tau^2} + (1-2\varepsilon\omega_1+\dots)(X+\varepsilon X^3)=0$$

WE EXPAND $X = X_0(\tau)+\varepsilon X_1(\tau)+\dots$ TO GET

$$\ddot{X}_0 + X_0 = 0, \quad X_0(0)=A, \quad X_0'(0)=0$$

$$\ddot{X}_1 + X_1 = 2\omega_1 X_0 - X_0^3 \qquad X_1(0)=0, \quad X_1'(0)=0$$

WE OBTAIN THAT $X_0 = A\cos\tau$ AND

$$\ddot{X}_1 + X_1 = 2\omega_1 A\cos\tau - A^3\cos^3\tau = \left(2\omega_1 A - \frac{3}{4}A^3\right)\cos\tau - \frac{A^3}{4}\cos(3\tau)$$

<u>NOTE</u>:

(i) TO GET INITIAL CONDITIONS WE NEED THAT $\frac{dX}{dt} = \frac{dX}{d\tau}\omega = 0$ AT $t=0$.

SO $$(\dot{X}_0+\varepsilon\dot{X}_1)(1+\varepsilon\omega_1) = 0 \rightarrow \dot{X}_0 + \varepsilon(\dot{X}_1+\omega_1\dot{X}_0)=0$$

THIS YIELDS $\dot{X}_0(0)=0, \quad \dot{X}_1(0) = -\omega_1\dot{X}_0(0) = 0.$

(ii) TO ELIMINATE THE SECULAR TERM THAT CAUSES RESONANCE WE TAKE $\omega_1 = \frac{3}{8}A^2$. THIS YIELDS THAT X_1 IS BOUNDED AS $t\to\infty$.

THEN WE CALCULATE $$\tau = \left(1+\frac{3\varepsilon A^2}{8}+\dots\right)t$$

AND $$X = A\cos\left[\left(1+3\varepsilon A^2/8+\dots\right)t\right] + \varepsilon X_1$$

THE PERIOD IS SIMPLY $$T = \frac{2\pi}{(1+3\varepsilon A^2/8+\dots)} \sim 2\pi\left(1-3\varepsilon A^2/8+\dots\right) \quad \varepsilon\to 0.$$

NOTICE THAT THE PERIOD T DEPENDS ON THE AMPLITUDE A.

(iii) THIS FORMULA CAN BE VERIFIED ANALYTICALLY.

$$X'' + X + \varepsilon X^3 = 0$$

$$X(0)=A, \quad X'(0)=0$$

NOW MULTIPLY BY x' AND INTEGRATE

$$\frac{x'^2}{2} + V(x) = E \qquad V(x) = \frac{x^2}{2} + \frac{\varepsilon x^4}{4}, \quad E = V(A).$$

NOW $x(T/2) = -A$ AND ON $0 \le t \le T/2$ WE HAVE $x' < 0$. HENCE,

$$-\sqrt{2}\,dt = \frac{dx}{[V(A) - V(x)]^{1/2}}$$

INTEGRATING FROM $-A$ TO A WE GET

$$-\sqrt{2}\int_{T/2}^{0} dt = \int_{-A}^{A} [V(A) - V(x)]^{-1/2}\,dx = \frac{T}{\sqrt{2}}.$$

THUS,

$$T = 2\sqrt{2}\int_0^A [V(A) - V(x)]^{-1/2}\,dx = 2\sqrt{2}\int_0^A \left[\frac{A^2}{2} + \frac{\varepsilon A^4}{4} - \frac{x^2}{2} - \frac{\varepsilon x^4}{4}\right]^{-1/2} dx$$

THEREFORE,

$$T = 2\sqrt{2}\int_0^A \sqrt{2}\left[(A^2 - x^2) + \frac{\varepsilon}{2}(A^2 - x^2)(A^2 + x^2)\right]^{-1/2} dx.$$

THIS GIVES,

$$T = 2\sqrt{2}\int_0^A \sqrt{2}\,[A^2 - x^2]^{-1/2}\left[1 + \frac{\varepsilon}{2}(A^2 + x^2)\right]^{-1/2} dx.$$

NOW LET $x = A\sin\varphi$, $dx = A\cos\varphi\,d\varphi$. THEN,

$$T = 4\int_0^{\pi/2} \frac{1}{A\cos\varphi} A\cos\varphi\left[1 + \frac{\varepsilon A^2}{2}(1 + \sin^2\varphi)\right]^{-1/2} d\varphi$$

FOR $\varepsilon \ll 1$

$$T \sim 4\int_0^{\pi/2}\left[1 - \frac{\varepsilon A^2}{4}(1 + \sin^2\varphi)\right] d\varphi$$

$$T \sim 4\left(\frac{\pi}{2} - \frac{\varepsilon A^2}{4}\left(\frac{\pi}{2} + \frac{\pi}{4}\right)\right) = 2\pi\left(1 - 3\varepsilon A^2/8 + \cdots\right)$$

which agrees WITH OUR result.

<u>GENERAL CASE</u> $\qquad x'' + x + \varepsilon h(x) = 0 \qquad x(0) = A,\ x'(0) = 0.$

let $\omega = 1 + \varepsilon\omega_1 + \cdots \qquad \tau = (1 + \varepsilon\omega_1 + \cdots)t \qquad x = x(\tau) \longrightarrow \ddot{x} + (1 - 2\varepsilon\omega_1)(x + \varepsilon h(x)) = 0$

EXPANDING $x = x_0 + \varepsilon x_1 + \cdots \longrightarrow \ddot{x}_0 + x_0 = 0 \qquad x_0(0) = A,\ x_0'(0) = 0 \to x_0 = A\cos\tau$

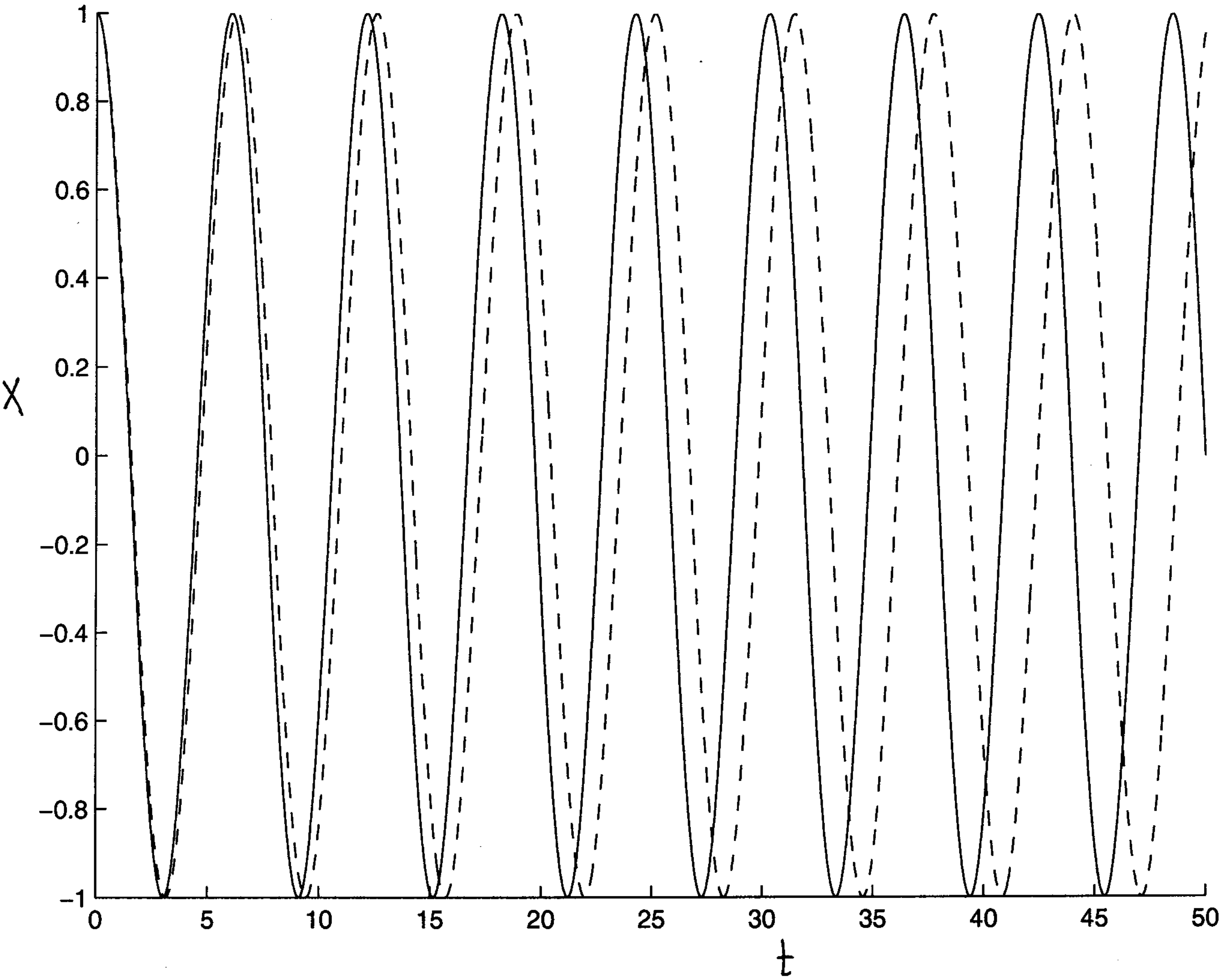
THEN $\ddot{x}_1 + x_1 = 2\omega_1 x_0 - h(x_0)$ NO RESONANCE IF $\int_0^{2\pi}[2\omega_1\cos\tau - h(A\cos\tau)]\cos\tau\,d\tau =$

THIS GIVES $\omega_1 = \omega_1(A)$.

$$x'' + x + \varepsilon x^3 = 0$$

$$x(0) = 1, \quad x'(0) = 0$$

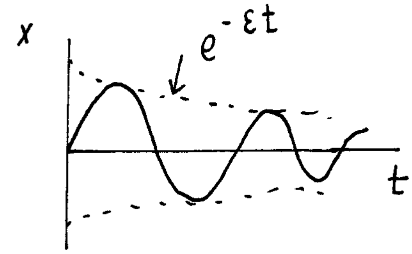
$$T = 2\pi\left(1 - 3\varepsilon/8 + \cdots\right)$$



SOLID CURVE $\varepsilon = 0.1$

DOTTED CURVE $\varepsilon = 0$.

EXAMPLE OF (2)



IF WE CONSIDER
$$x'' + 2\varepsilon x' + x = 0$$
$$x(0) = 0, \quad x'(0) = 1$$

THE EXACT SOLUTION IS $x = (1-\varepsilon^2)^{-1/2} e^{-\varepsilon t} \sin\left[(1-\varepsilon^2)^{1/2} t\right]$

WHICH SUGGESTS TWO TIME-SCALES $\tau = \varepsilon t$ (slow), $T = (1 + a_1\varepsilon^2 + \dots)t$ (FAST).

WE NOW CONSIDER A PROBLEM WHICH CANNOT BE SOLVED ANALYTICALLY: WE LOOK AT

$$x'' + \varepsilon(x^2 - 1)x' + x = 0 \qquad \text{VAN-DER-POL OSCILLATOR} \qquad \varepsilon \ll 1$$

WE LOOK FOR AN APPROXIMATION VALID FOR $t = O(1/\varepsilon)$. WE INTRODUCE NEW VARIABLES

$$T = t \quad \text{AND} \quad \tau = \varepsilon t \qquad x = x(\tau, T)$$

$$\frac{dx}{dt} = \frac{dx}{dT} + \varepsilon\frac{dx}{d\tau} \qquad \frac{d^2x}{dt^2} = X_{TT} + 2\varepsilon X_{\tau T} + \varepsilon^2 X_{\tau\tau}.$$

SUBSTITUTING WE OBTAIN,

$$X_{TT} + 2\varepsilon X_{\tau T} + \varepsilon^2 X_{\tau\tau} + \varepsilon\left((X^2-1)(X_T + \varepsilon X_\tau)\right) + X = 0$$

IF WE EXPAND $X = X_0 + \varepsilon X_1 + \dots$ AND COLLECT POWERS OF ε WE GET

$$X_{0TT} + X_0 = 0$$

$$X_{1TT} + X_1 = -2X_{0T\tau} - (X_0^2 - 1)X_{0T}$$

THE SOLUTION IS $X_0 = R(\tau)\cos(T + \phi(\tau))$. WE THEN CALCULATE

$$X_{0\tau} = R'\cos(T+\phi) - R\phi'\sin(T+\phi) \qquad X_{0T} = -R\sin(T+\phi)$$

$$X_{0T\tau} = -R'\sin(T+\phi) - R\phi'\cos(T+\phi)$$

THUS
$$X_{1TT} + X_1 = 2R'\sin(T+\phi) + 2R\phi'\cos(T+\phi)$$
$$+ R\sin(T+\phi)\left(R^2\cos^2(T+\phi) - 1\right)$$

THIS BECOMES

$$X_{1TT} + X_1 = 2R' \sin(T+\phi) + 2R\phi' \cos(T+\phi) + R^3 \sin(T+\phi)\cos^2(T+\phi) - R\sin(T+\phi).$$

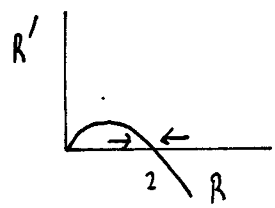
WE THEN USE IDENTITY $\cos^2 X \sin X = \frac{1}{4}\sin X + \frac{1}{4}\sin(3X)$

TO OBTAIN

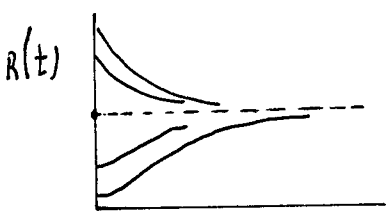
$$X_{1TT} + X_1 = \left(2R' - R + R^3/4\right)\sin(T+\phi) + 2R\phi'\cos(T+\phi) + \frac{R^3}{4}\sin(3(T+\phi)).$$

NOW FOR X_1 TO BE BOUNDED FOR ALL T WE REQUIRE THAT THERE BE NO RESONANT TERMS, i.e.

$$\begin{cases} 2R' = R - R^3/4 \\ 2R\phi' = 0 \end{cases}$$



THIS IMPLIES THAT $\phi = \phi_0$ CONSTANT, AND IN ADDITION

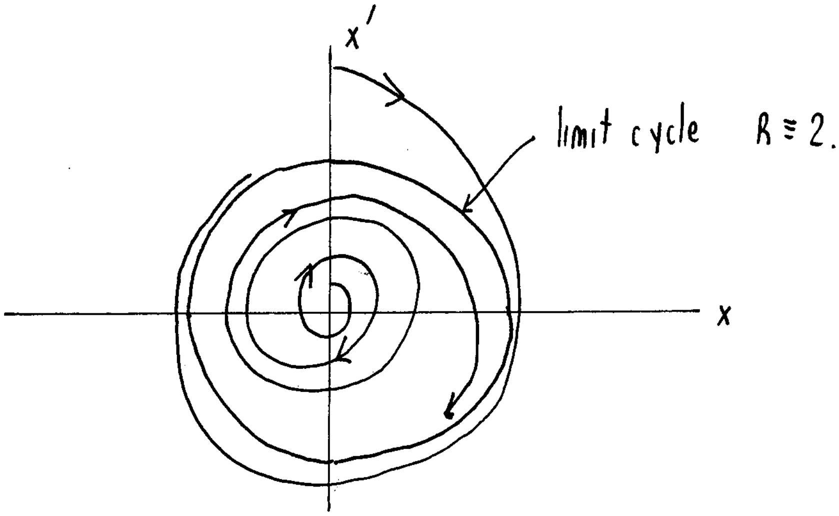


τ

THEREFORE $X \sim R(\tau)\cos(T + \phi_0)$. $X' \sim -R\sin(T+\phi_0$

where $R' = \frac{1}{2}\left(R - R^3/4\right)$ $R \to 2$ AS $\tau \to +\infty$

THERE IS AN ISOLATED PERIODIC SOLUTION WHEN $R = 2$ (THIS IS A LIMIT CYCLE. NOTICE THAT IF $X(0) = 1$ AND $X'(0) = 0$, WE CAN TAKE $R(0)\cos\phi_0 = 1$, $-R\sin\phi_0 = 0 \rightarrow \phi_0 = 0$ AND $R_0 = R(0) = 1$ AS THE INITIAL CONDITION.



<u>GENERAL CASE</u> $x'' + x + \varepsilon h(x, x') = 0.$

WE LET $\tau = \varepsilon t$, $T = t$ AND $x = x(\tau, T)$. THIS YIELDS

$$x_{TT} + 2\varepsilon x_{\tau T} + \varepsilon^2 x_{\tau\tau} + x + \varepsilon h(x, x_T + \varepsilon x_\tau)) = 0$$

WE THEN EXPAND $x = x_0(\tau, T) + \varepsilon x_1(\tau, T) + \cdots$. WE SUBSTITUTE AND COLLECT TERMS TO OBTAIN

$$x_{0TT} + x_0 = 0$$

$$x_{1TT} + x_1 = -2x_{0T\tau} - h(x_0, x_{0T})$$

THE SOLUTION IS THEN $x_0 = R(\tau)\cos(T + \phi(\tau))$. BY CALCULATING $x_{0T\tau}$ WE OBTAIN THAT

$$x_{1TT} + x_1 = 2R'\sin(T+\phi) + 2R\phi'\cos(T+\phi) - h(R\cos(T+\phi), -R\sin(T+\phi)$$

WE LABEL THE RHS AS $B(\varphi)$ WHERE

$$B(\varphi) \equiv 2R'\sin(\varphi) + 2R\phi'\cos(\varphi) - h(R\cos\varphi, -R\sin\varphi)$$

WITH $\varphi = T + \phi$. NOW IF WE EXPAND $B(\varphi)$ IN A FOURIER-SERIES WE HAVE

$$B(\varphi) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos n\varphi + b_n\sin n\varphi)$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} B(\varphi)\cos(n\varphi)\,d\varphi$$

$$b_n = \frac{1}{\pi}\int_0^{\pi} B(\varphi)\sin(n\varphi)\,d\varphi$$

THUS THERE WILL BE NO RESONANCE IF $a_1 = b_1 = 0$ (i.e. NO $\sin\varphi$, $\cos\varphi$ terms).

THIS YIELDS THAT

$$a_1 = 0 \longrightarrow \int_0^{2\pi} 2R\phi' \cos^2\varphi \, d\varphi + \int_0^{2\pi} 2R' \cos\varphi \sin\varphi \, d\varphi - \int_0^{2\pi} h(R\cos\varphi, -R\sin\varphi)\cos\varphi \, d\varphi = 0$$

$$b_1 = 0 \longrightarrow \int_0^{2\pi} 2R\phi' \cos\varphi \sin\varphi + \int_0^{2\pi} 2R' \sin^2\varphi \, d\varphi - \int_0^{2\pi} h(R\cos\varphi, -R\sin\varphi)\sin\varphi \, d\varphi = 0$$

NOW USING THAT $\int_0^{2\pi} \cos^2\varphi \, d\varphi = \int_0^{2\pi} \sin^2\varphi \, d\varphi = \pi$, $\int_0^{2\pi} \cos\varphi \sin\varphi \, d\varphi = 0$,

WE GET

$$2R\phi' = \frac{1}{\pi}\int_0^{2\pi} h(R\cos\varphi, -R\sin\varphi)\cos\varphi \, d\varphi$$

$$2R' = \frac{1}{\pi}\int_0^{2\pi} h(R\cos\varphi, -R\sin\varphi)\sin\varphi \, d\varphi$$

THESE ARE TWO ORDINARY DIFFERENTIAL EQUATIONS FOR R, ϕ.

FOR EXAMPLE: TAKE THE VAN-DER-POL OSCILLATOR WHERE $h(x, x') = (x^2 - 1)x'$. THEN, $h(R\cos\varphi, -R\sin\varphi) = -(R^2\cos^2\varphi - 1)R\sin\varphi$

SO THAT $$2R\phi' = -\frac{1}{\pi}\int_0^{2\pi} (R^2\cos^2\varphi - 1)R\sin\varphi\cos\varphi \, d\varphi = 0. \longrightarrow \phi' = 0.$$

ALSO, $$2R' = -\frac{1}{\pi}\int_0^{2\pi} (R^2\cos^2\varphi - 1)R\sin^2\varphi \, d\varphi = \frac{R}{\pi}\int_0^{2\pi}\sin^2\varphi \, d\varphi - \frac{R^3}{\pi}\int_0^{2\pi}\cos^2\varphi\sin^2\varphi \, d\varphi$$

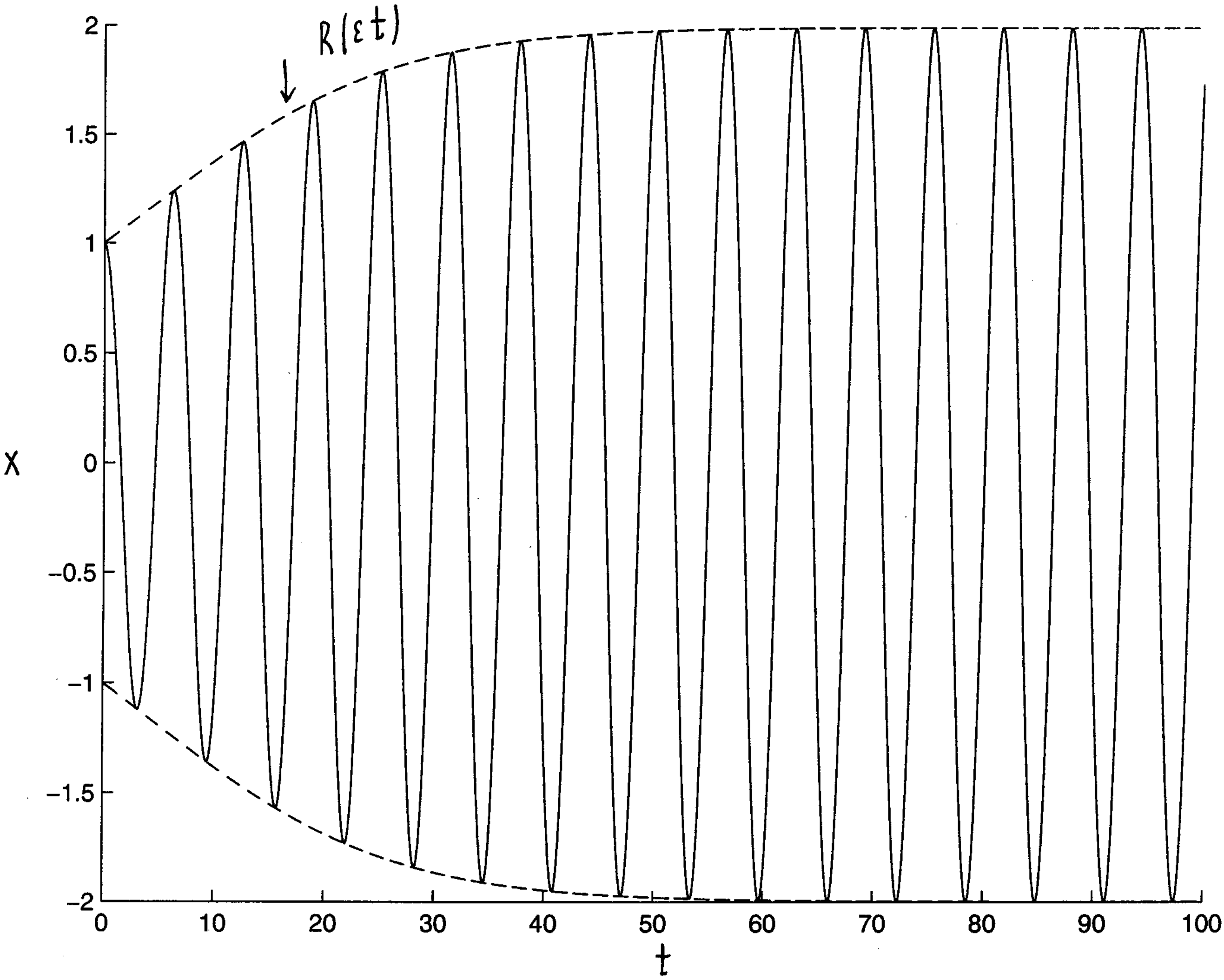
NOW WE CALCULATE: $$R' = \frac{R}{2\pi}\int_0^{2\pi}\sin^2\varphi \, d\varphi - \frac{R^3}{2\pi}\int_0^{2\pi}\cos^2\varphi \, d\varphi + \frac{R^3}{2\pi}\int_0^{2\pi}\cos^4\varphi \, d\varphi.$$

THUS $$R' = \frac{R}{2} - \frac{R^3}{2} + \frac{R^3}{2\pi}\left(\frac{3\pi}{4}\right) \longrightarrow R' = \frac{R}{2} - R^3/8$$ (same as before)

$$x'' + \varepsilon (x^2 - 1) x' + x = 0$$

$$x(0) = 1 \qquad x'(0) = 0$$

$$\varepsilon = .1$$



SOLID CURVE IS FULL NUMERICAL RESULT

DOTTED CURVE IS THE AMPLITUDE $R(\varepsilon t)$,

WHERE $dR/dt = \frac{\varepsilon}{2}(R - R^3/4)$ WITH $R(0) = 1$

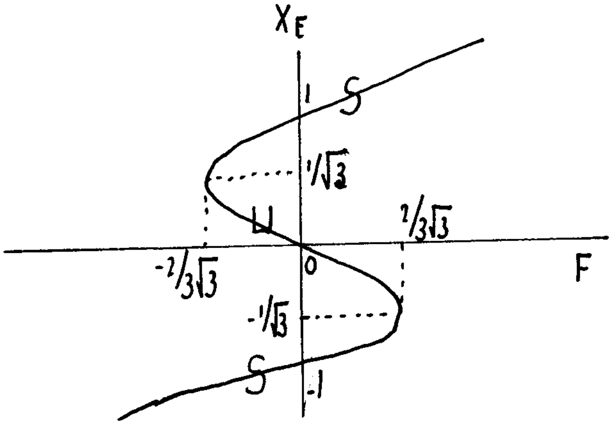
$$x \sim R(\varepsilon t) \cos(t + \phi)$$

HYSTERESIS (NON-REVERSIBLE)

CONSIDER THE FIRST ORDER ODE

$$\frac{dx}{dt} = x - x^3 + F \qquad F \text{ IS A CONSTANT TO BEGIN WITH.}$$

THE PLOT OF THE EQUILIBRIA ARE $F = -X_e + X_e^3$ WHICH CAN BE SHOWN IN THE DIAGRAM BELOW.



SETTING $dF/dX_e = 0 \rightarrow 3X_e^2 = 1 \rightarrow X_E = \pm \frac{1}{\sqrt{3}}$

THIS YIELDS $F = -X_e(1 - X_e^2)$

$$F_{\pm} = \pm \frac{1}{\sqrt{3}} \left(2/3\right) = \pm \frac{2}{3\sqrt{3}}.$$

IT IS EASY TO SEE THAT THE STABILITY OF THE BRANCHES IS AS SHOWN.

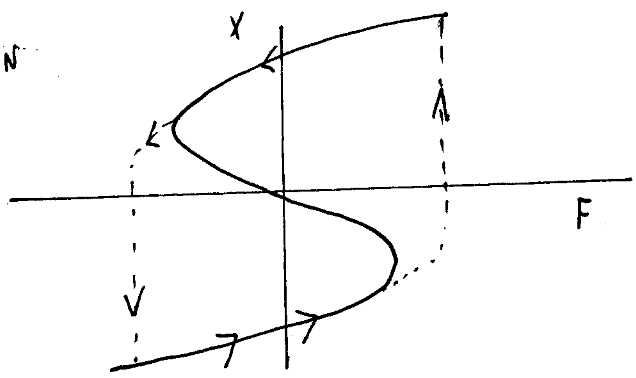
NOW SUPPOSE THAT F DEPENDS SLOWLY ON TIME AS

$$F = \frac{(F_{MIN} + F_{MAX})}{2} + \frac{(F_{MAX} - F_{MIN})}{2} \sin\left(\varepsilon t - \pi/2\right)$$

WHEN $t = 0, 2\pi/\varepsilon, 4\pi/\varepsilon, \ldots$ WE HAVE $F = F_{MIN}$

$t = \pi/\varepsilon, 3\pi/\varepsilon, 5\pi/\varepsilon, \ldots$ WE HAVE $F = F_{MAX}$

IF WE CHOOSE $F_{MAX} > 2/3\sqrt{3}$ AND $F_{MIN} < -2/3\sqrt{3}$ WE GET A HYSTERESIS LOOP AS SHOWN

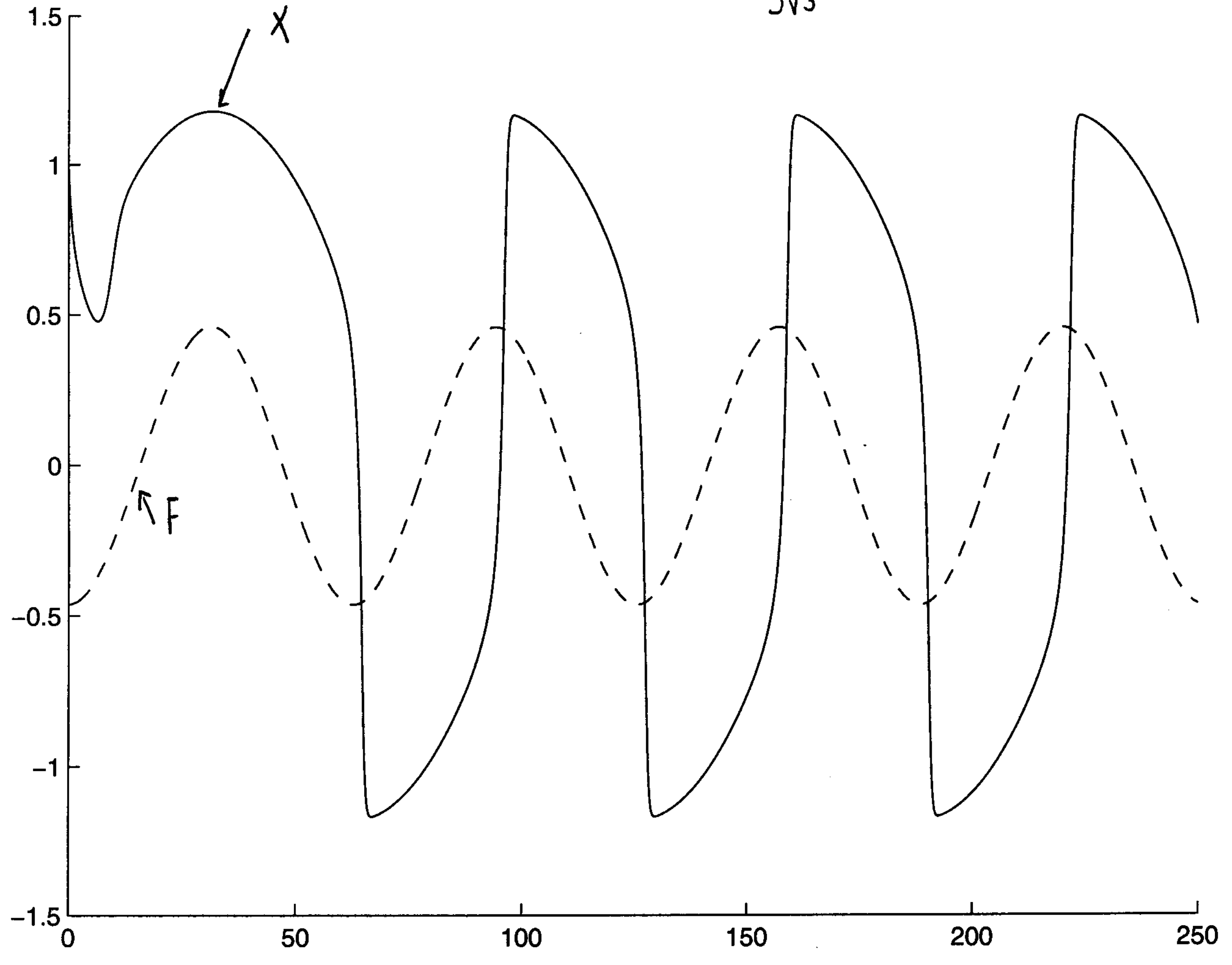


HYSTERESIS LOOP FOR $X' = X - X^3 + F$

$X(0) = 1$ $\qquad \varepsilon = 0.1$

$$F = \left(\frac{F_{MIN} + F_{MAX}}{2}\right) + \left(\frac{F_{MAX} - F_{MIN}}{2}\right) \sin\left(\varepsilon t - \pi/2\right)$$

$$F_{MAX} = \frac{2}{3\sqrt{3}}(1.2) = -F_{MIN}.$$



EXAMPLE OF (3) WE CONSIDER A PROBLEM NEAR RESONANCE THAT IS BEING FORCED. ASSUME THAT THE NONLINEARITY AND DAMPING ARE SMALL.

WE CONSIDER

$$X'' + \varepsilon\lambda X' + X + \varepsilon K X^3 = \varepsilon \cos(\Omega t) \qquad \Omega = 1 + \varepsilon\omega.$$

$$X(0) = X'(0) = 0.$$

REMARK (i) IF $K = 0$, $\lambda = 0$ WE CALCULATE

$$X = \frac{\varepsilon}{1 - \Omega^2}\left[\cos(\Omega t) - \cos t\right]$$

THUS IF $\Omega = 1 + \varepsilon\omega$, THEN $X = -\dfrac{1}{\omega(2 + \varepsilon\omega)}\left[\cos((1+\varepsilon\omega)t) - \cos t\right]$

THUS $O(\varepsilon)$ FORCING $\longrightarrow$ $O(1)$ RESPONSE SINCE WE ARE NEAR RESONANCE.

NOW WE LET $\tau = \varepsilon t$ AND WE LET $X = F(T, \tau)$.
$T = t$

THIS YIELDS

$$F_{TT} + 2\varepsilon F_{T\tau} + \varepsilon^2 F_{\tau\tau} + \varepsilon\lambda(F_T + \varepsilon F_\tau) + \varepsilon K F^3 = \varepsilon \cos(\Omega t)$$

NOW WE DO NOT EXPECT $F = O(\varepsilon)$ WHEN $\Omega = 1 + O(\varepsilon)$. HENCE, WE EXPAND $F = F_0 + \varepsilon F_1 + \ldots$ SUBSTITUTING AND COLLECTING POWERS OF ε WE OBTAIN

$$F_{0TT} + F_0 = 0$$

$$F_{1TT} + F_1 = -2F_{0T\tau} - \lambda F_{0T} - K F_0^3 + \cos(T + \omega\tau)$$

WE OBTAIN THAT $F_0 = A\cos(T + \phi)$ WITH $A = A(\tau)$, $\phi = \phi(\tau)$.

NOW WE CALCULATE

$$F_{0_T} = -A\sin(T+\phi) \qquad F_{0_{TT}} = -A'\sin(T+\phi) - A\phi'\cos(T+\phi).$$

THIS YIELDS THAT

$$F_{1_{TT}} + F_1 = 2A'\sin(T+\phi) + 2A\phi'\cos(T+\phi) + \lambda A\sin(T+\phi) - \kappa A^3\cos^3(T+\phi) + \cos(T+\omega\tau).$$

NOW WE USE $\cos(T+\omega\tau) = \cos(T+\phi+(\omega\tau-\phi)) = \cos(T+\phi)\cos(\phi-\omega\tau) - \sin(T+\phi)\sin(\omega\tau-\phi).$

$$\cos^3(T+\phi) = \frac{3}{4}\cos(T+\phi) + \frac{1}{4}\cos(3(T+\phi)).$$

THIS YIELDS THAT

$$F_{1_{TT}} + F_1 = \sin(T+\phi)\left(2A' + \lambda A + \sin(\phi-\omega\tau)\right) + \cos(T+\phi)\left(2A\phi' + \cos(\phi-\omega\tau) - \frac{3\kappa A^3}{4}\right) + \frac{1}{4}\cos(3(T+\phi)).$$

THIS YIELDS THAT THERE ARE 2 RESONANANT TERMS. WE OBTAIN THAT F_1 IS BOUNDED AS $T \to \infty$ IFF

$$(*) \quad \begin{cases} 2A' + \lambda A = -\sin(\phi-\omega\tau) \\ 2A\phi' - \dfrac{3\kappa A^3}{4} = -\cos(\phi-\omega\tau) \end{cases}$$

NOW THIS YIELDS TWO ODE FOR $A = A(\tau)$, $\phi = \phi(\tau)$.

THE INITIAL VALUE IS THAT $A(0) = 0$. WE CAN TAKE $\phi(0) = -\pi/2$ SO THAT $A'(0) > 0$.

WE NOW LOOK AT THE SOLUTIONS UNDER (*). THE FINAL SOLUTION WILL HAVE THE FORM

$$X = A(\tau)\cos(T+\phi(\tau))$$

AND SO $A(\tau)$ IS THE "AMPLITUDE" OF THE RESPONSE.

LET'S FIRST FIND ~~A~~ BETTER FORM FOR WRITING (*) WE INTRODUCE

$$\psi = \phi - \omega\tau$$

THIS YIELDS THAT

$$(+)\quad \begin{cases} 2A' + \lambda A = -\sin(\psi) \\ 2A\psi' + 2A\omega - \dfrac{3KA^3}{4} = -\cos(\psi) \end{cases}$$

THIS GIVES TWO AUTONOMOUS EQUATIONS. THE EQUILIBRIA SATISFY

$$\lambda A_e = -\sin(\psi_e)$$

$$2A_e\omega - \frac{3KA_e^3}{4} = -\cos(\psi_e)$$

SQUARING BOTH SIDES AND ADDING WE OBTAIN THAT

$$\lambda^2 A_e^2 + \left(2\omega A_e - 3KA_e^3/4\right)^2 = 1$$

THUS $$A_e^2\left[\lambda^2 + \left(2\omega - \frac{3K}{4}A_e^2\right)^2\right] = 1$$

WE NOW SOLVE FOR ω: $$\left(2\omega - \frac{3K}{4}A_e^2\right)^2 = \frac{1}{A_e^2} - \lambda^2$$

THIS SHOWS THAT $\frac{1}{A_e^2} - \lambda^2 > 0$. THEN WE TAKE THE SQUARE ROOT TO GET

$$\omega = \frac{3K}{8}A_e^2 \pm \frac{1}{2}\left(\frac{1}{A_e^2} - \lambda^2\right)^{1/2} \quad \text{FOR} \quad 1/A_e^2 - \lambda^2 > 0.$$

NOTICE THAT IF $\frac{1}{A_e^2} - \lambda^2 = 0$, I.E. $A_e = 1/\lambda$ THEN $\omega = \frac{3K}{8}\frac{1}{\lambda^2}$. LABEL $A_{e_M} = 1/\lambda$, $\omega_M = 3K/8\lambda^2$.

HENCE FOR ANY $0 < A_e < 1/\lambda$, $\exists$ TWO VALUES OF ω. NOTICE ALSO THAT $A_e \to 0$ AS $|\omega| \to \infty$ WITH $A_e \sim 1/(2|\omega|)$ AS $|\omega| \to \infty$.

NOW WE FIND A POINT WHERE $d\omega/dA = 0$. WE DIFFERENTIATE IMPLICITLY TO OBTAIN

$$2\left(2\omega - \frac{3K}{4}A_e^2\right)\left(2\frac{d\omega}{dA_e} - \frac{3K}{2}A_e\right) = -\frac{2}{A_e^3}.$$

SETTING $d\omega/dA_e = 0$ WE CALCULATE $$\left(2\omega - \frac{3K}{4}A_e^2\right)\frac{3K}{2}A_e^4 = 1.$$

NOW ELIMINATING $\left(2\omega - \frac{3K}{4} A_e^2\right)$ we get

$$\frac{4}{9K^2 A_e^8} = \frac{1}{A_e^2} - \lambda^2 .$$

THIS CAN BE WRITTEN AS $\frac{4}{9K^2 A_e^8} - \frac{1}{A_e^2} + \lambda^2 = 0.$

LET $\varsigma = \frac{1}{\lambda^2 A_e^2}$. SINCE WE HAVE $0 < A_e < 1/\lambda$ WE HAVE $\varsigma > 1$.

WE SUBSTITUTE TO GET $\frac{4\lambda^8}{9K^2}\varsigma^4 - \lambda^2 \varsigma + \lambda^2 = 0.$

DEFINE $b \equiv \frac{4\lambda^6}{9K^2}$. THEN WE LOOK FOR ROOTS OF $g(\varsigma) = 0$

FOR $g(\varsigma) = b\varsigma^4 - \varsigma + 1 = 0$ WITH $\varsigma > 1$.

<u>REMARK</u> $g(1) > 0$ AND $g'(1) = 4b - 1$. SO $g'(1) > 0$ IF $b > \frac{1}{4}$

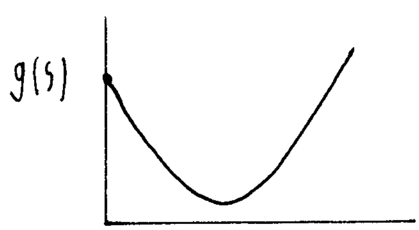
AND $g'(1) < 0$ IF $0 < b < 1/4$.

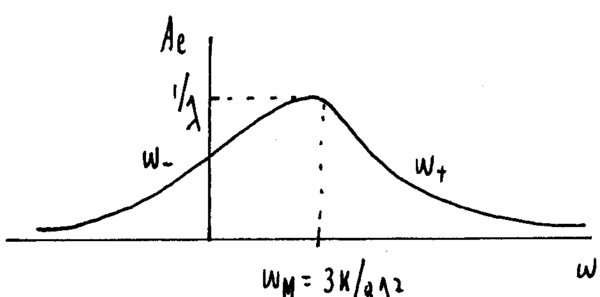
NOW SET $g'(\varsigma) = 0$. WE CALCULATE $4b\varsigma_{MIN}^3 - 1 = 0$ OR $\varsigma_{MIN}^3 = \frac{1}{4b}$.

WE THEN CALCULATE $g(\varsigma_{MIN}) = \varsigma(b\varsigma^3 - 1) + 1 = \frac{1}{(4b)^{1/3}}\left(-3/4\right) + 1 = -\frac{3}{4}\varsigma_{MIN} + 1$

(I) HENCE IF $g(\varsigma_{MIN}) > 0$ THEN WE HAVE NO ROOTS TO $d\omega_+/dA_e = 0$

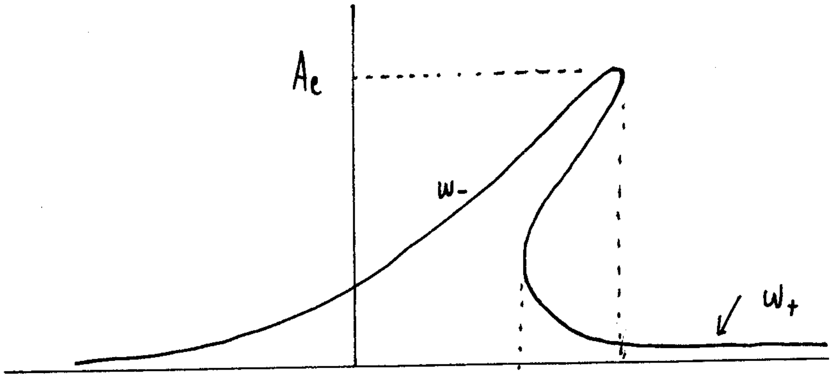
AND WE GET A PLOT AS SHOWN. THIS MEANS $\varsigma_{MIN} < 4/3$





THIS PICTURE HOLDS FOR $\varsigma_{MIN} < 4/3 \rightarrow \frac{1}{4b} < \left(\frac{4}{3}\right)^3 \rightarrow 4b > \left(\frac{3}{4}\right)^3$

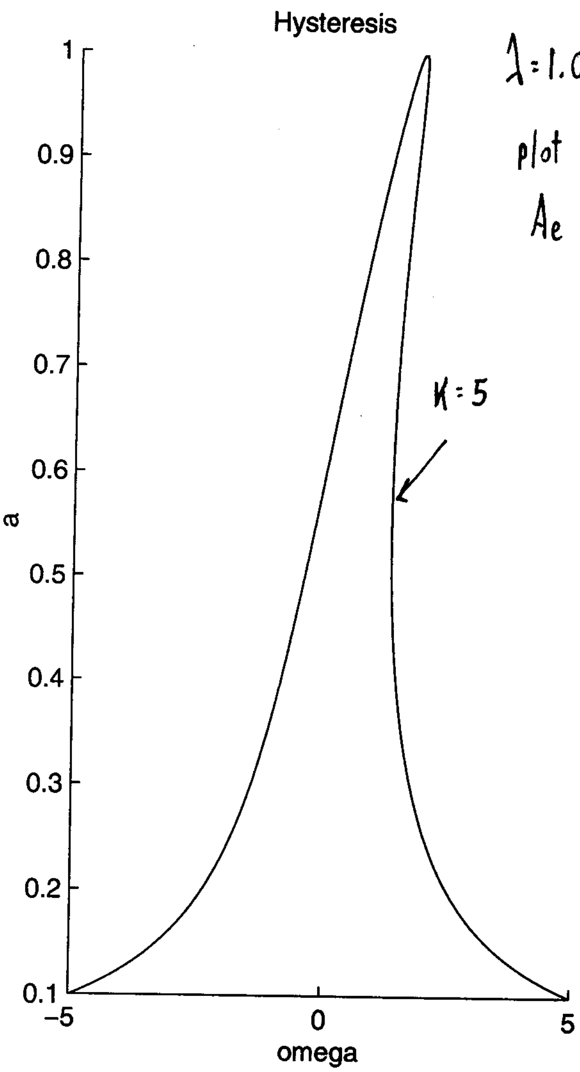
RECALLING $b = \frac{4\lambda^6}{9K^2}$ WE NEED $\boxed{|K| < \left(\frac{4}{3}\right)^{5/2} \lambda^3 .}$



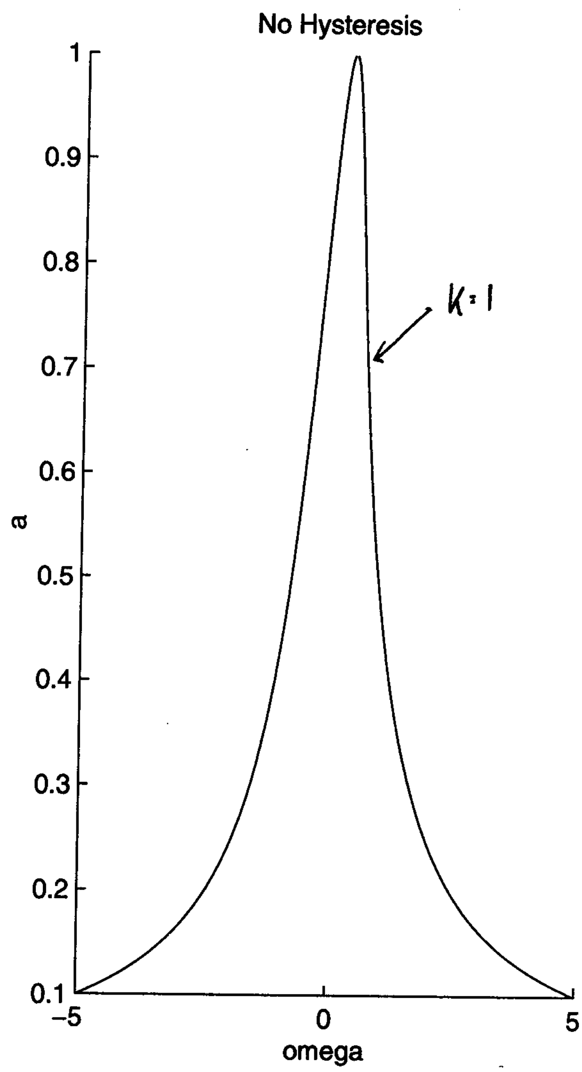
$|K| > \left(\frac{4}{3}\right)^{5/2} \lambda^3$

WITH $K >> 1$

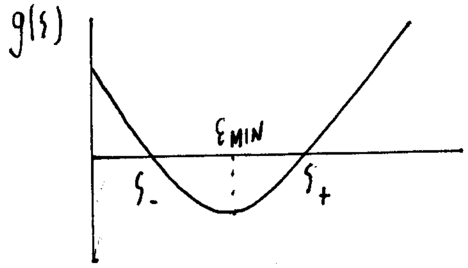
USING A COMPUTER WE CAN OBTAIN THE PICTURES AS SHOWN.



$\lambda = 1.0$, plot of A_e vs ω



(II) NOW SUPPOSE THAT $g(\varsigma_{MIN}) < 0$ SO THAT $\varsigma_{MIN} > 4/3$ AND CONSEQUENTLY THAT $|K| > \left(\frac{4}{3}\right)^{5/2} \lambda^3$. THEN WE GET A PICTURE AS FOLLOWS:



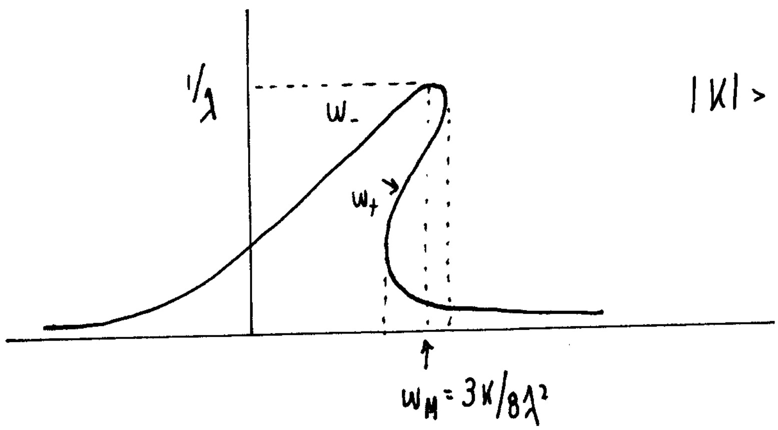
HOWEVER WE NEED $\varsigma > 1$. THEREFORE, WE CONCLUDE

$(I)^0$ $\exists$ TWO ROOTS TO $dw_+/dA_e = 0$ WHEN $\varsigma_- > 1$.

$(II)^0$ $\exists$ ONLY ONE ROOT TO $dw_+/dA_e = 0$ WHEN $\varsigma_- < 1$.

NOTICE THAT WE MUST HAVE $\varsigma_+ > 1$ SINCE $\varsigma_{MIN} > 4/3$.

NOW CASE $(I)^0$ WILL YIELD A PICTURE OF THE FORM



$|K| > \left(\frac{4}{3}\right)^{5/2} \lambda^3$.

NOTICE THAT $(II)^0$ IS NOT POSSIBLE SINCE $g(1) = b > 0$. HENCE WE MUST ALWAYS HAVE CASE $(I)^0$.

HOWEVER, NOTICE THAT AS $b \to 0$ WE HAVE $\varsigma_- \to 1$ AND SO THE PICTURE WILL HAVE THE FORM AS SHOWN ON THE NEXT PAGE; WHERE ONE ZERO OF $dw_+/dA_e = 0$ OCCURS NEAR $A_e = 1/\lambda$.

REMARKS

(i) CONSIDER THE LINEAR EQUATION WITH $K=0$. WE WANT TO SOLVE

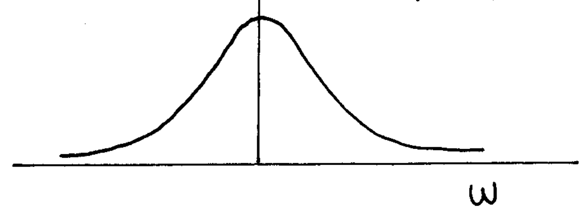
$$x'' + \varepsilon\lambda x' + x = \varepsilon \cos((1+\varepsilon\omega)t)$$

WE REPLACE $\cos$ BY $\mathrm{Re}[e^{i(1+\varepsilon\omega)t}]$ AND LET $\tilde{x} = \tilde{A} e^{i(1+\varepsilon\omega)t}$.

TAKING THE REAL PART WE OBTAIN, LETTING $\varepsilon \ll 1$

$$x_p = R(\omega)\cos((1+\varepsilon\omega)t + \phi)$$

$$R(\omega) = \frac{1}{\sqrt{4\omega^2 + \lambda^2}}$$



NOTICE THAT NONLINEAR TERM CHANGES RESONANT PEAK TO

$\omega_M = \dfrac{3K}{8\lambda^2}$ AND IF $|K| > \left(\dfrac{4}{3}\right)^{5/2}\lambda^3$ IT INTRODUCES

A SOLUTION MULTIPLICITY.

(ii) TO DETERMINE THE STABILITY OF THESE EQUILIBRIA WE MUST CALCULATE THE JACOBIAN FOR

$$A' = -\frac{\lambda}{2}A - \frac{1}{2}\sin(\psi) = f(A,\psi)$$

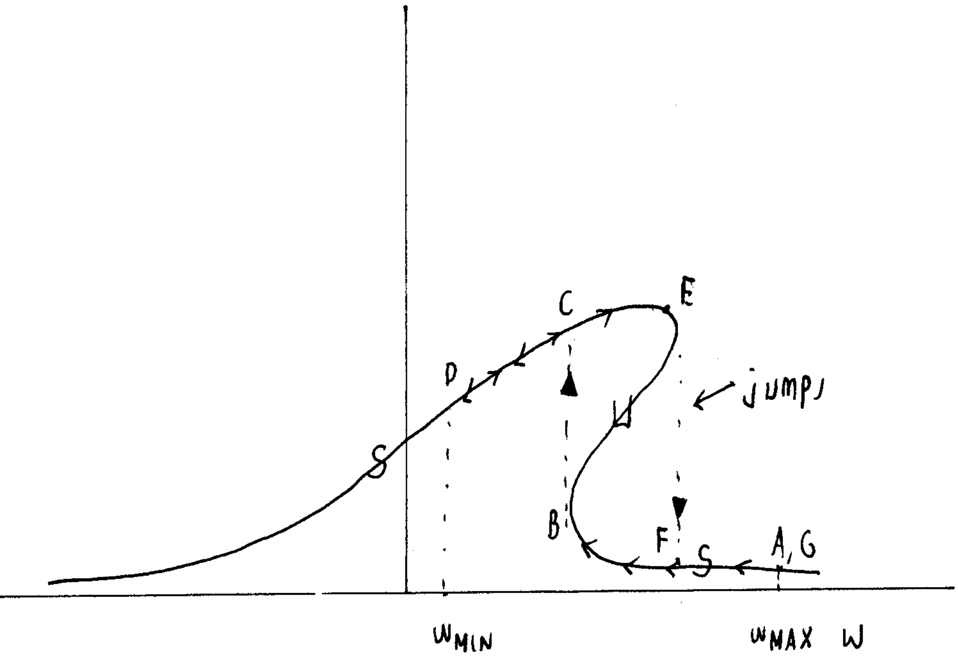
$$A\psi' = -A\omega + \frac{3KA^3}{8} - \frac{1}{2}\cos\psi \equiv A g(A,\psi)$$

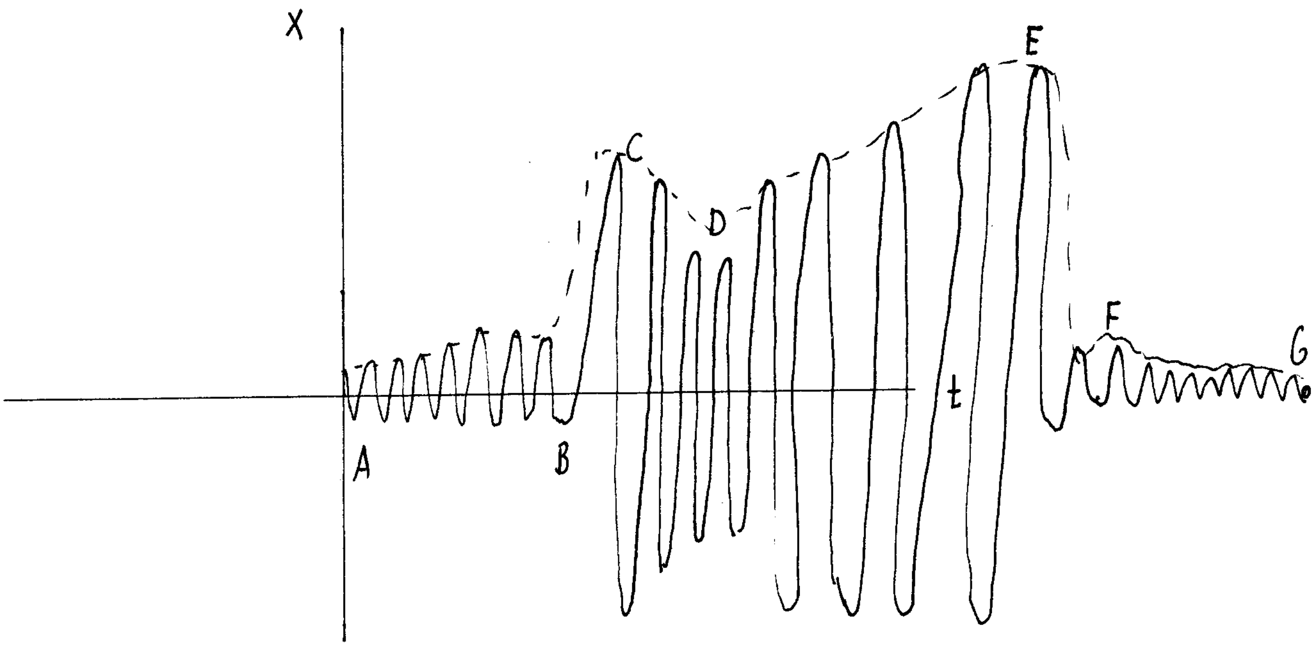
THE JACOBIAN IS $J = \begin{pmatrix} f_A & f_\psi \\ g_A & g_\psi \end{pmatrix} = \begin{pmatrix} -\lambda/2 & -\frac{1}{2}\cos\psi \\ \frac{3KA}{4} + \frac{\cos\psi}{2A^2} & \frac{\sin\psi}{2A} \end{pmatrix}$

BY CALCULATING THE EIGENVALUES THROUGH trace J AND det J

WE GET THE FOLLOWING PICTURE SUGGESTING A HYSTERESIS LOOP

THIS SUGGESTS THE FOLLOWING HYSTERESIS LOOP





$$\omega = \frac{\omega_{MIN} + \omega_{MAX}}{2} + \frac{\omega_{MAX} - \omega_{MIN}}{2} \sin\left(\delta t + \pi/2\right)$$

$$0 < \delta \ll \varepsilon$$