HEAT EQUATION
SIMILARITY SOLUTIONS

$$u_t = u_{xx}$$

let $x = L\bar{x}$, $t = \tau\bar{t}$ AND $\bar{u}(\bar{x},\bar{t}) = u[L\bar{x}, \tau\bar{t}]$

$$\frac{1}{\tau}\bar{u}_{\bar{t}} = \frac{1}{L^2}\bar{u}_{\bar{x}\bar{x}}$$

EQUATION REMAINS THE SAME IF $\tau = L^2$. OR $L = \sqrt{\tau}$.

WE HAVE $\dfrac{\bar{x}}{\sqrt{\bar{t}}} = \dfrac{x}{\sqrt{t}}$ independent of scaling symmetry L.

SO THAT $x/\sqrt{t}$ IS INDEPENDENT OR REMAINS INVARIANT FOR THE TRANSFORMATION.

LOOK FOR $u(x,t) = t^p g(x/\sqrt{t})$ $\quad \eta = x/\sqrt{t}$

NOW

$$u_t = p t^{p-1} g + t^p\left[\frac{d}{dt}(x t^{-1/2})\, g'\right]$$

$$u_t = p t^{p-1} g + t^p\left(-\frac{1}{2}\, x t^{-3/2}\right) g' = p t^{p-1} g + \left(-\frac{1}{2} t^{p-1}\right)\eta\, g'$$

$$u_{xx} = t^{p-1} g''$$

SUBSTITUTE TO CONCLUDE THAT

$$p g - \frac{1}{2}\eta g' = g'' \quad \text{OR} \quad g'' + \frac{1}{2}\eta g' = p g.$$

CASE I CONSIDER $u_t = u_{xx}$, $-\infty < x < \infty$

$u(x,0) = \delta(x)$.

WE MUST HAVE $\displaystyle\int_{-\infty}^{\infty} u(x,t)\,dx = \int_{-\infty}^{\infty} u(x,0)\,dx = 1.$

NOW $\int_{-\infty}^{\infty} t^p g(x/\sqrt{t})\, dx = 1 \quad \forall\ t.$

let $\eta = x/\sqrt{t}, \quad dx = d\eta\sqrt{t}$ so

$$t^{p+1/2}\int_{-\infty}^{\infty} g(\eta)\, d\eta = 1. \longrightarrow \quad p = -1/2.$$

THUS, WITH $p = -1/2$,

$$(1)\ \begin{cases} g'' + \frac{1}{2}[\eta g' + g] = 0. \\ g \to 0 \text{ AS } \eta \to \pm\infty. \\ \int_{-\infty}^{\infty} g(\eta)\, d\eta = 1 \end{cases}$$

WE SOLVE TO GET $\quad g'' + \frac{1}{2}(\eta g)' = 0.$

$$g' + \frac{1}{2}\eta g = 0.$$

$$g = A e^{-\eta^2/4}.$$

NOW $\int_{-\infty}^{\infty} A e^{-\eta^2/4}\, d\eta = 1 \qquad \frac{1}{\sqrt{2\pi}\,\sigma}\int_{-\infty}^{\infty} e^{-x^2/2\sigma^2} dx = 1$

let $2\sigma^2 = 4 \qquad \sigma = \sqrt{2}$

SO $\int_{-\infty}^{\infty} A e^{-\eta^2/4}\, d\eta = A\sqrt{4\pi} = 1 \qquad A = 1/\sqrt{4\pi}.$

SO $g(\eta) = \frac{1}{\sqrt{4\pi}} e^{-\eta^2/4}.$

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t}$$

CASE II

$$u_t = u_{xx}, \quad 0 < x < \infty$$

$$u(x,0) = 0, \quad u(0,t) = 1$$

LET $p = 0$, $\quad u = g(\lambda)$ SO THAT

$$g'' + \frac{1}{2}\lambda g' = 0. \qquad \lambda = x/\sqrt{t}$$

$$g' = A e^{-\lambda^2/4}$$

SO $$g(\lambda) = A\int_0^\lambda e^{-s^2/4}\,ds + B. \qquad \cancel{\mathrm{ERF}(z)} =$$

$$g(0) = 1 \;\rightarrow\; B = 1$$

$$g(\infty) = 0 \qquad A\int_0^\infty e^{-s^2/4}\,ds = -1$$

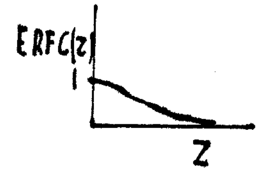
$$y = s/2$$

SO $$g(\lambda) = 1 - \frac{\int_0^\lambda e^{-s^2/4}\,ds}{\int_0^\infty e^{-s^2/4}\,ds} = 1 - \frac{2\int_0^{\lambda/2} e^{-y^2}\,dy}{2\int_0^\infty e^{-y^2}\,dy}$$

NOW $$\mathrm{ERF}(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-y^2}\,dy \qquad \int_0^\infty e^{-y^2}\,dy = \frac{\sqrt{\pi}}{2}.$$

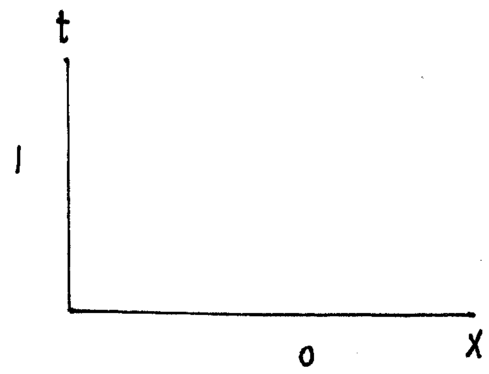
SO $$g(\lambda) = 1 - \frac{2}{\sqrt{\pi}}\int_0^{\lambda/2} e^{-y^2}\,dy = 1 - \mathrm{ERF}(\lambda/2).$$

BUT $$\mathrm{ERFC}(z) = 1 - \mathrm{ERF}(z) = \frac{2}{\sqrt{\pi}}\int_z^\infty e^{-y^2}\,dy$$



SO $$g(\lambda) = \mathrm{ERFC}(\lambda/2)$$

$$\longrightarrow \quad u(x,t) = \mathrm{ERFC}\left(x/2\sqrt{t}\right)$$



U CONSTANT ON LINES $x/\sqrt{t}$ = CONSTANT.

$u \to 1$ AS $t \to \infty$ FOR x FIXED

$u \to 0$ AS $x \to \infty$ FOR t FIXED.

CASE III

$u_t = u_{xx}, \quad 0 < x < \infty$

$u(x,0) = 0, \; x > 0, \quad u \to 0$ AS $x \to \infty$, t FIXED

$u_x(0,t) = -Q.$

LET $u = t^p g(\eta)$. $\quad u_x(0,t) = t^{p-1/2} g'(0) = Q.$

let $p = 1/2$.

$$\begin{cases} g'' + \frac{1}{2}\eta g' = \frac{1}{2} g. \\ g \to 0 \text{ AS } \eta \to \infty \\ g'(0) = -Q \end{cases}$$

$g = \eta$ is a sol'n.

$g = \eta h. \quad g' = h + \eta h', \quad g'' = \eta h'' + 2h'$

$\eta h'' + 2h' + \frac{\eta}{2}[\eta h' + h] = \frac{1}{2}\eta h$

$\to h'' + \left[\frac{2}{\eta} + \frac{\eta}{2}\right] h' = 0$

INT. FACTOR IS $e^{\int 2/\eta + \eta/2} = \eta^2 e^{\eta^2/4}$

$(\eta^2 e^{\eta^2/4} h')' = 0$

$h = \int^{\eta} \frac{1}{s^2} e^{-s^2/4} \, ds$

CAN SOLVE TO OBTAIN

$$g(\eta) = -Q\left[\eta \operatorname{ERFC}\left(\frac{\eta}{2}\right) - \frac{2}{\sqrt{\pi}} \exp\left(-\eta^2/4\right)\right]$$

NOW USE IBP:

$$u = e^{-s^2/4} \quad du = -\frac{s}{2} e^{-s^2/4} ds$$

$$\frac{dv}{ds} = \frac{1}{s^2} \qquad v = -\frac{1}{s}$$

$$h = -\frac{1}{\lambda} e^{-\lambda^2/4} - \frac{1}{2}\int^{\lambda} e^{-s^2/4} ds$$

so $g_2 = \lambda h = -e^{-\lambda^2/4} - \frac{\lambda}{2}\int^{\lambda} e^{-s^2/4} ds$ is second solution

NOW $$g = C_1\lambda + C_2\left[-e^{-\lambda^2/4} - \frac{\lambda}{2}\int_0^{\lambda} e^{-s^2/4} ds\right]$$

let $t = s/2$ IN INTEGRAL

$$\int_0^{\lambda} e^{-s^2/4} ds = 2\int_0^{\lambda/2} e^{-t^2} dt = 2\frac{\sqrt{\pi}}{2} \text{ERF}\left(\frac{\lambda}{2}\right)$$

$$g = C_1\lambda + C_2\left[-e^{-\lambda^2/4} - \frac{\lambda\sqrt{\pi}}{2}\text{ERF}(\lambda/2)\right]$$

ERF$(0) = 0$
ERF$(\infty) = 1$.

NOW IMPOSE $g \to 0$ AS $\lambda \to \infty$. $C_1 - C_2\sqrt{\pi}/2 = 0$.

$$g'(0) = -Q \rightarrow C_1 = -Q \text{ so } C_2 = -\frac{2}{\sqrt{\pi}} Q$$

$$g(\lambda) = -Q\left[\lambda - \frac{2}{\sqrt{\pi}} e^{-\lambda^2/4} - \lambda\,\text{ERFC}(\lambda/2)\right]$$

$$= -Q\left[\lambda\underbrace{(1 - \text{ERF}(\lambda/2))}_{\text{ERFC}(\lambda/2)} - \frac{2}{\sqrt{\pi}} e^{-\lambda^2/4}\right]$$

$$g(\lambda) = -Q\left[\lambda\,\text{ERFC}(\lambda/2) - \frac{2}{\sqrt{\pi}} e^{-\lambda^2/4}\right]$$

PUT $\lambda = x/\sqrt{t}$, $u = t^{1/2} g \rightarrow u(x,t) = Q t^{1/2}\left[\frac{2}{\sqrt{\pi}} e^{-x^2/4t} - \frac{x}{\sqrt{t}}\,\text{ERFC}\left(x/2\sqrt{t}\right)\right]$

SO $$u(x,t) = Q t^{1/2} \left[\frac{2}{\sqrt{\pi}} e^{-x^2/4t} - \frac{x}{\sqrt{t}} \, \mathrm{ERFC}\left(\frac{x}{2\sqrt{t}} \right) \right]$$

WITH $$\mathrm{ERFC}(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-t^2} \, dt.$$

IN FACT $$\int_0^\infty u(x,t)\, dx = + Q t.$$

$$\lim_{R\to\infty} \int_0^R u_t \, dx = \int_0^R u_{xx} \, dx = \lim_{R\to\infty} u_x(R,t) - u_x(0,t)$$
$$= Q - \lim_{R\to\infty} u_x(R,t)$$

SO $$\lim_{R\to\infty} \frac{d}{dt}\left(\int_0^R u \, dx \right) = Q - \lim_{R\to\infty} u_x(R,t)$$

NOTICE $$\mathrm{ERFC}(z) \sim e^{-z^2} / \sqrt{\pi}\, z \quad \text{AS} \quad z \to +\infty$$

SO $u_x \to 0$ AS $x \to +\infty$ FOR ANY $t > 0$ FIXED

SO $$\int_0^\infty u \, dx = Q t + C$$

BUT $u(x,0) = 0 \to C = 0$

$$\int_0^\infty u \, dx = Q t.$$

POROUS MEDIUM EQUATION

WE BEGIN WITH

$$u_t = \left(u^m u_x \right)_x , \quad -\infty < x < \infty \quad (m > 0)$$

$$u(x,0) = \delta(x), \quad u \to 0 \text{ AS } |x| \to \infty, \ t \text{ FIXED.}$$

SCALING $\quad u = \bar{U}\,\bar{u}, \quad x = L\bar{x}, \quad t = \tau \bar{t}$

THEN $\quad \dfrac{\bar{U}}{\tau}\, \bar{u}_{\bar{t}} = \dfrac{\bar{U}^{m+1}}{L^2} \left(\bar{u}^m \bar{u}_{\bar{x}} \right)_{\bar{x}}$

TO BALANCE WE NEED $\quad \dfrac{L^2}{\tau} = \bar{U}^m \rightarrow$ PDE IS INVARIANT.

NOW $\quad u(x,0) = \delta(x) \longrightarrow \bar{U}\, u(L\bar{x},0) = \delta(L\bar{x}) = \dfrac{1}{L}\delta(\bar{x})$

$$\bar{U}\,\bar{u}(\bar{x},0) = \frac{1}{L}\delta(\bar{x}). \rightarrow \bar{U} = 1/L.$$

SO WE HAVE $\quad \dfrac{L^2}{\tau} = \bar{U}^m, \quad \bar{U} = \dfrac{1}{L} \quad \rightarrow \quad L = \tau^{1/(2+m)}$

$$\frac{x}{t^{1/(2+m)}} = \frac{L\bar{x}}{(\tau\bar{t})^{1/(2+m)}} = \left(\frac{L}{\tau^{1/(m+2)}}\right)\frac{\bar{x}}{\bar{t}^{1/2+m}} = \frac{\bar{x}}{\bar{t}^{1/2+m}}$$

AND $\quad U = \dfrac{1}{L} = \dfrac{1}{\tau^{2+m}}.$

THIS SUGGESTS WE LOOK FOR $\quad u(x,t) = t^{-1/2+m} F(\eta) \qquad \eta = \dfrac{x}{t^{1/2+m}}$

NOW LET $p = 1/m+2. \qquad u = t^{-p} F\left(x/t^p\right)$ WITH $\eta = x/t^p. \qquad \eta_x = 1/t^p$

$$u_x = t^{-2p} F'(\eta)$$

$$u_t = -p t^{-p-1} F(\eta) + t^{-p} F'(\eta)\,\eta_t \qquad \eta_t = -\frac{p x}{t^{p+1}} = -\frac{p\,\eta}{t}$$

$$u_t = -p t^{-p-1} \left[F(\eta) + \eta F'(\eta) \right]$$

$$(u^m u_x)_x = t^{-mp-2p} \left(F^m F' \right)' \eta_x = t^{-(m+2)p} \left(F^m F' \right)' t^{-p}$$

$$\text{SO} \quad (u^m u_x)_x = t^{-p-1} \left(F^m F' \right)'$$

NOW $u_t = (u^m u_x)_x$

GIVES $-p\, t^{-p-1}\left(f(\eta) + \eta f'(\eta)\right) = t^{-p-1}(f^m f')'$

WITH $p = \frac{1}{2+m}$ WE GET

$$(f^m f')' = -\frac{1}{2+m}(f + \eta f').$$

WE WANT TO FIND A SOLUTION WHERE $f(\eta) = f(-\eta)$, i.e. $f(\eta)$ IS EVEN.
WE WRITE $f + \eta f' = (\eta f)'$ AND INTEGRATE TO GET

$$f^m f' = -\frac{\eta}{m+2} f + c.$$

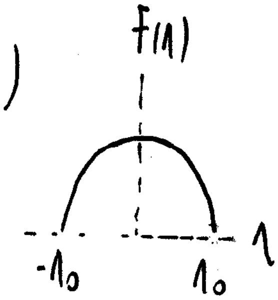
NOW $f'(0) = 0 \rightarrow c = 0$. WE SET $f = 0$ AT $\eta = \eta_0$ AND TAKE $f(\eta) = f(-\eta)$.

WE WILL ASSUME $f^m f' \rightarrow 0$ AS $\eta \rightarrow \eta_0$.

SO INTEGRATING USING $f(\eta_0) = 0$ GIVES

$$f^{m-1} f' = -\frac{\eta}{m+2} \rightarrow \frac{f^m}{m} = -\frac{1}{2(m+2)}(\eta^2 - \eta_0^2)$$

THIS YIELDS $f^m(\eta) = \begin{cases} \frac{m}{2(m+2)}(\eta_0^2 - \eta^2), & |\eta| \le \eta_0 \\ 0, & |\eta| \ge \eta_0 \end{cases}$



THE FINITE EXTENT OF "DROP" IS DUE TO DEGENERACY AT $f=0$ WHEN $m>0$.
NOTICE THAT $m f^{m-1} f' = O(1)$ AS $\eta \rightarrow \eta_0$. SO $f^m f' = O(f) \rightarrow 0$ AS $\eta \rightarrow \eta_0$.
THIS IS CONSISTENT. FINALLY WE MUST FIX THE AREA OF THE DROP.

$\forall t$: NEED $\int_{-\infty}^{\infty} t^{-\frac{1}{m+2}} f\left(\frac{x}{t^{1/m+2}}\right) dx = 1 \rightarrow \int_{-\eta_0}^{\eta_0} f(\eta)\, d\eta = 1.$

THEREFORE $$\frac{1}{2} = \left(\frac{m}{2(m+2)}\right)^{1/m} \int_0^{\eta_0} (\eta_0^2 - \eta^2)^{1/m}\, d\eta$$

IS THE EXPRESSION FOR η_0 THAT AGREES WITH MASS OF DIRAC.

let $\eta = \eta_0 y$.

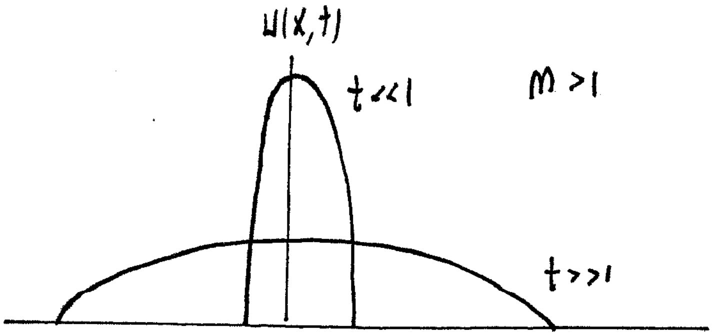
$$1 = 2\left(\frac{m}{2(m+2)}\right)^{1/m}\left[\int_0^1 (1-y^2)^{1/m}\,dy\right]\eta_0^{2/m+1}$$

$$\eta_0 = \left[\frac{1}{2}\left(\frac{2(m+2)}{m}\right)^{1/m}\left(\int_0^1 (1-y^2)^{1/m}\,dy\right)^{-1}\right]^{m/m+2}$$

THUS

$$u(x,t) = t^{-1/(m+2)}\left[\frac{m}{2(2+m)}\left(\eta_0^2 - x^2/t^{2/2+m}\right)_+\right]^{1/m}$$

WITH $(x)_+ = \begin{cases} x, & x > 0 \\ 0, & x \le 0. \end{cases}$



$x = \eta_0 t^{1/m+2}$ IS MOTION OF spreading FRONT.

THIS IS CALLED BARENBLATT'S SOLUTION.

PROBLEM. CONSIDER

$$u_t = (u^m u_x)_x \qquad 0 < x < \infty$$

WITH $\left. u^m u_x \right|_{x=0} = -1, \quad u = 0 \text{ AT } t = 0, \quad u \to 0 \text{ AS } x \to \infty.$

WE LET $u = t^{\beta} f(\eta), \quad \eta = x t^{-\alpha}$. THIS CORRESPONDS TO MASS BEING INJECTED AT $x=0$ FOR ALL t

IF WE SUBSTITUTE INTO THE EQUATION WE OBTAIN

$$t^{\beta-1}\left[\beta f - \alpha \eta f'\right] = t^{\beta(m+1)-2\alpha}\left(f^m f'\right)'.$$

NOW WE CHOOSE $\beta - 1 = \beta m + \beta - 2\alpha$ OR $\beta = (2\alpha - 1)/m$,

WHICH YIELDS THAT

$$\left(f^m f'\right)' = \beta f - \alpha \eta f'.$$

NOW IF WE INTEGRATE THE EQUATION WE OBTAIN THAT

$$\frac{d}{dt}\int_0^\infty u \, dx = \int_0^\infty (u^m u_x)_x \, dx = \left. u^m u_x \right|_0^\infty = 1. \implies \frac{d}{dt}\int_0^\infty u \, dx = 1$$

THUS $\int_0^\infty u(x,t)\,dx = t$, WITH $\int_0^\infty u(x,0)\,dx = 0$.

NOW WE SUBSTITUTE $\int_0^\infty t^\beta f\left(x/t^\alpha\right) dx = t \implies \int_0^\infty t^{\beta+\alpha} f(\eta)\,d\eta = t.$

THIS IMPLIES THAT $\beta + \alpha = 1$, AND $\int_0^\infty f(\eta)\,d\eta = 1.$

SOLVING $m\beta = 2\alpha - 1$ AND $\beta + \alpha = 1$, WE OBTAIN THAT

$$\alpha = \frac{m+1}{m+2} = 1 - \frac{1}{m+2}, \qquad \beta = \frac{1}{m+2}.$$

NOW WE WRITE $B = 1 - \alpha$. THIS YIELDS,

$$(f^m f')' = -\alpha(f + \Lambda f') + f$$

NOW $u^m u_x \big|_{x=0} = -1 \longrightarrow t^{Bm + B - \alpha} f^m f' \big|_{\Lambda=0} = -1 \longrightarrow Bm + B - (1-B) = B(m+2) - 1 = 0$

ALSO $u \to 0$ AS $x \to \infty$ YIELDS THAT $f(\infty) = 0$.

THEREFORE, WE HAVE THE SIMILARITY SOLUTION

$$(*) \quad \left\{ \begin{array}{l} (f^m f')' = -\alpha(\Lambda f)' + f \qquad 0 \leq \Lambda \leq \infty \\ f^m f' \big|_0 = -1, \qquad f(\infty) = 0 \\ \int_0^\infty f \, d\Lambda = 1. \end{array} \right. \qquad \alpha = \frac{m+1}{m+2}$$

NOW INTEGRATING ONCE WE OBTAIN THAT

$$f^m f' + \alpha \Lambda f = \int_0^\Lambda f(s)\, ds + C.$$

NOW LET $\Lambda \to 0$ AND ASSUME THAT f BOUNDED AS $\Lambda \to 0$.

THIS YIELDS THAT $\lim_{\Lambda \to 0} f^m f' = C \longrightarrow C = -1$.

THEREFORE WE MUST SOLVE

$$f^m f' + \alpha \Lambda f = \int_0^\Lambda f(s)\, ds - 1 \quad , \quad f(\infty) = 0$$

NOTICE THAT THIS IS CONSISTENT WITH $\int_0^\infty f(s)\, ds = 1$.

NOW IF WE LET $V(\Lambda) = \int_0^\Lambda f(s)\, ds - 1$

WE HAVE $V' = f$, $V'' = f'$ SO THAT

$$\begin{cases} V'^m V'' + \alpha \eta V' = V & \alpha = \dfrac{m+1}{m+2}. \\ V(0) = -1, \quad V(\infty) = 0. \end{cases}$$

NOTICE THAT IF WE LOOK FOR $V(\eta) = -1 + A\eta + B\eta^2 + \ldots$ AS $\eta \to 0$, WE CONCLUDE FROM $V'^m V''|_0 = -1$ THAT $B = -A^{-m}/2$,

SO THAT

$$V \sim -1 + A\eta - \frac{A^{-m}\eta^2}{2} \quad \text{FOR } \eta \ll 1.$$

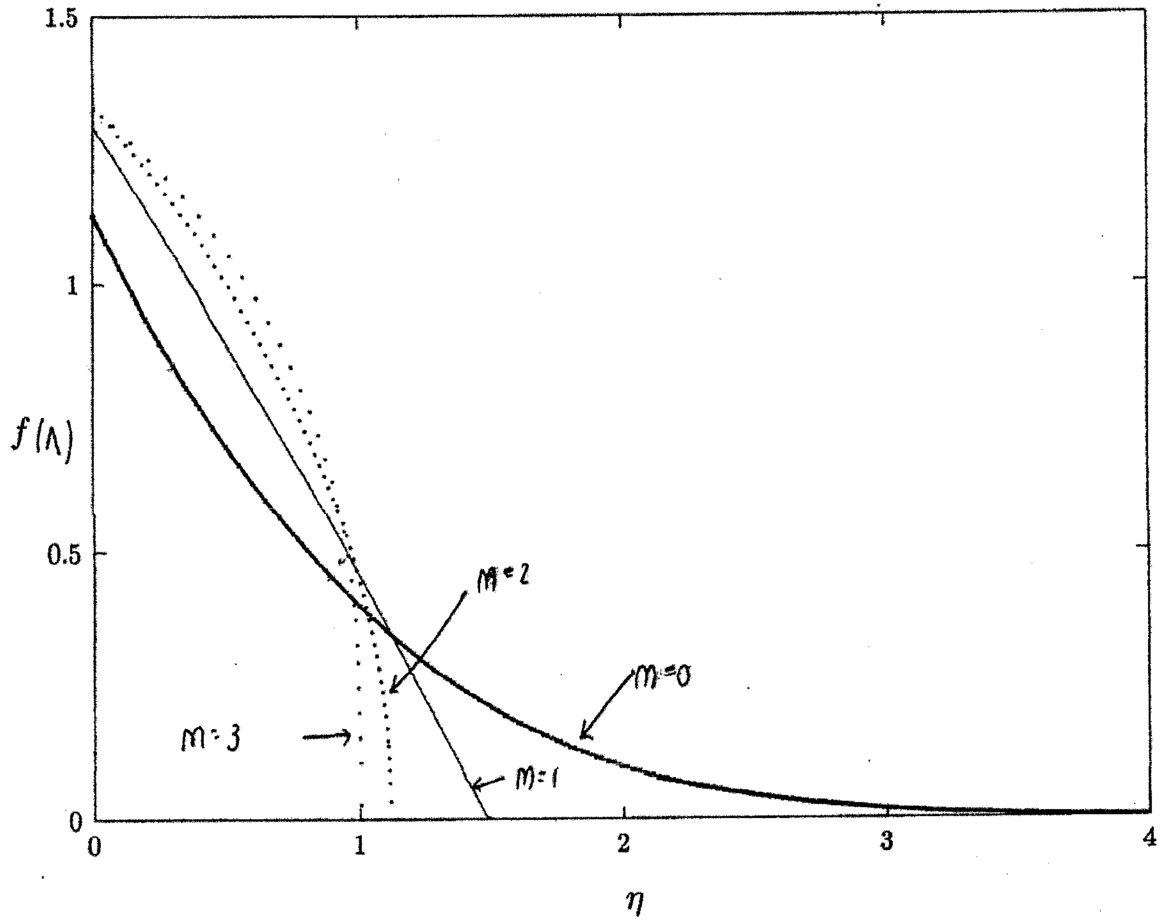
WE THEN LET $V(0) = -1$, $V'(0) = A$ AND TRY TO ADJUST THE CONSTANT A SO THAT $V(\infty) = 0$.

NUMERICALLY, WE FIND THAT

$$u = t^{1/m+2} f\left(x / t^{(m+1)/(m+2)} \right)$$

A	m
1.13	0
1.29	1
1.33	2
1.33	3

FROM next page we see that the solution has compact support when $m = 1, 2, 3$, BUT NOT WITH $m = 0$.



NOTICE COMPACT SUPPORT UNLESS $m=0$.

THE SIMILARITY SOLUTION IS

$$u(x,t) \sim t^{1/2m+2}\, f\left(x \big/ t^{(m+1)/(m+2)} \right).$$