SLOW-FAST SYSTEMS

THE PROBLEMS WE CONSIDERED WITH DELAYED BIFURCATIONS HAVE THE FORM

$$x' = F[x, \lambda] \qquad x' = dx/dt.$$
$$\lambda' = \varepsilon$$

LETTING $\tau = \varepsilon t$ WE HAVE

$$\begin{cases} \varepsilon \dot{x} = F[x, \lambda] \\ \dot{\lambda} = 1 \end{cases} \qquad \dot{} = d/d\tau.$$

THIS IS JUST A SPECIAL CASE OF A GENERAL SLOW-FAST SYSTEM WRITTEN AS

$$(*) \quad \begin{cases} \varepsilon \dot{x} = f(x, y) & f \in \mathbb{R}^N \quad x \in \mathbb{R}^N \\ \dot{y} = g(x, y) & g \in \mathbb{R}^M \quad y \in \mathbb{R}^M \end{cases}$$

THE SLOW MANIFOLD(S) FOR (1) IS THE SOLUTION(S) TO $f(x, y) = 0$, WHICH WE WRITE AS $f[x^*(y), y] = 0$.

WE THEN LINEARIZE $\varepsilon \dot{x} = f(x,y)$ ABOUT $x^*(y)$ TO OBTAIN THE JACOBIAN

$$A^*(y) = \frac{\partial}{\partial x} f(x,y)\Big|_{x = x^*(y)} \equiv f_x[x^*(y), y].$$

THE SLOW MANIFOLD IS SAID TO BE <u>HYPERBOLIC</u> IF ALL OF THE EIGENVALUES OF A^* HAVE NON-ZERO REAL PARTS.

THE SLOW MANIFOLD IS ASYMPTOTICALLY STABLE IF ALL THE EIGENVALUES OF A^* HAVE NEGATIVE REAL PART.

THE SLOW MANIFOLD IS UNSTABLE IF $\exists$ AN EIGENVALUE OF A^* WITH POSITIVE REAL PART.

IF THE SLOW MANIFOLD IS ASYMPTOTICALLY STABLE, THEN THE DYNAMICS ON THE SLOW MANIFOLD IS, TO LEADING ORDER,

$$\dot{y} = g[x^*(y), y]$$

<u>EXAMPLE 1</u> CONSIDER $x'' + \gamma x' + V'(x) = 0$ WHERE $V(x)$ IS SOME SCALAR POTENTIAL AND $\gamma >> 1$.

WE WRITE $\frac{d}{dt}[x' + \gamma x] = -V'(x)$.

LET y BE INTRODUCED AS $\gamma y = x' + \gamma x$.

THEN
$$x' = \gamma(y - x)$$
$$y' = -\frac{1}{\gamma} V'(x).$$

NOW LET $\tau = 1/\gamma \, t$ SO $d/dt = d/d\tau \, \frac{1}{\gamma}$.

THEN
$$\frac{1}{\gamma^2}\dot{x} = y - x$$
$$\dot{y} = -V'(x)$$

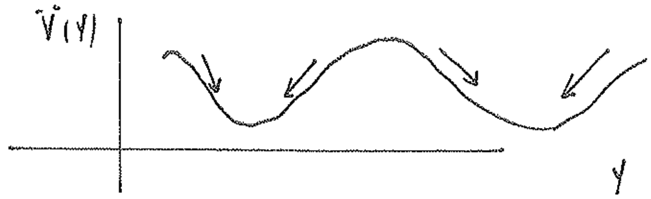
FOR $\gamma >> 1$, WE DEFINE $\varepsilon = 1/\gamma^2$ SO THAT WE GET SLOW-FAST SYSTEM

$$\begin{cases} \varepsilon \dot{x} = y - x = f(x,y) \\ \dot{y} = -V'(x) = g(x,y) \end{cases}$$

THE SLOW MANIFOLD IS $x^*(y) = y$. IT IS ASYMPTOTICALLY STABLE $\forall y$ SINCE $\partial_x[f(x,y)]\big|_{x = x^*(y)} = -1 < 0$. THE DYNAMICS ON THE SLOW MANIFOLD IS $\dot{y} = -V'(y)$ TO LEADING ORDER.

NOW $\frac{d}{d\tau}\tilde{V}(y) = \tilde{V}'(y)\frac{dy}{d\tau} = -\left[\tilde{V}'(y)\right]^2$

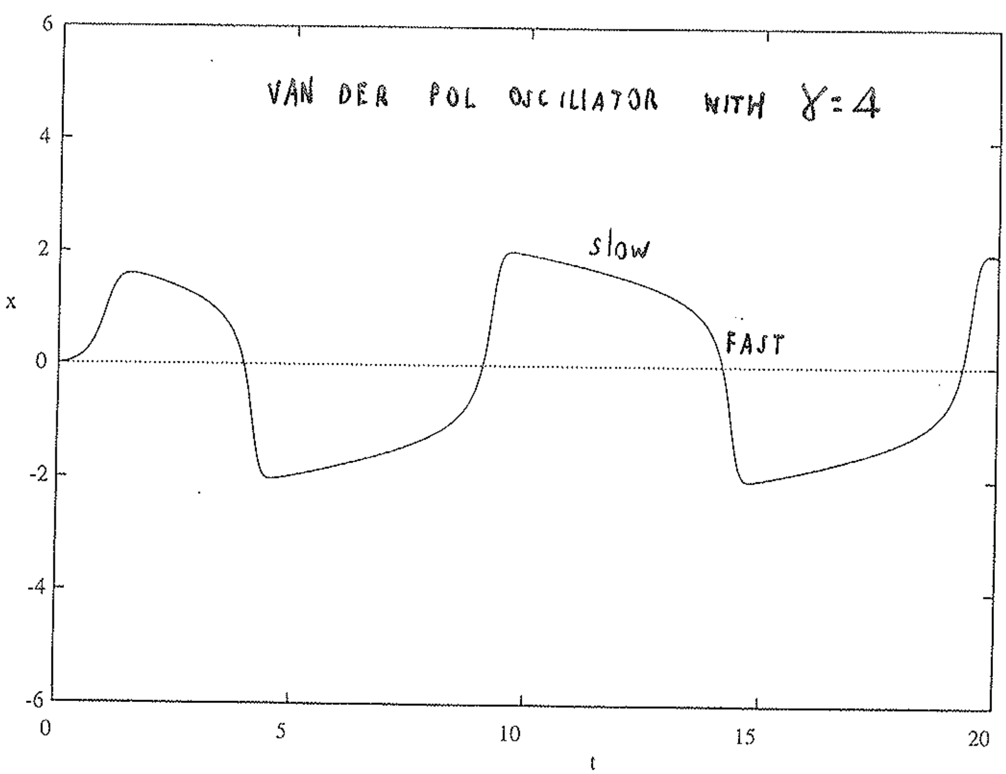
THUS ON SLOW MANIFOLD $\frac{d}{d\tau}\tilde{V}[y(\tau)] < 0$ WITH EQUILIBRIA AT POINTS WHERE $\tilde{V}'(y_e) = 0$. IF $\tilde{V}''(y_e) > 0 \to$ STABLE EQUILIBRIA.



EXAMPLE 2 WE NOW GIVE AN EXAMPLE WHERE TRANSITIONS BETWEEN SLOW MANIFOLDS IS POSSIBLE. THE EXAMPLE IS FOR "RELAXATION OSCILLATIONS" WHICH ARE PERIODIC SOLUTIONS WITH TWO TIME-SCALES. A SLOW BUILD-UP PHASE AND A FAST TRANSIENT PHASE.

WE CONSIDER $x'' + \gamma h(x) x' + x = 0$ WITH $\gamma >> 1$.

THE CLASSIC VAN-DER-POL OSCILLATOR IS $h(x) = x^2 - 1$



CONVERSION TO SLOW-FAST SYSTEM. LET

$$H(x) = \int_0^x h(s)\,ds.$$

THEN $$\frac{d}{dt}\left[x' + \gamma H(x) \right] = -x.$$

WE DEFINE y BY $x' + \gamma H(x) = \gamma y$

SO THAT

$$y' = -\frac{1}{\gamma} x$$

$$x' = \gamma \left[y - H(x) \right]$$

NOW LET $\tau = {}^1/_\gamma\, t$ SO THAT WITH $\varepsilon = {}^1/_{\gamma^2}$

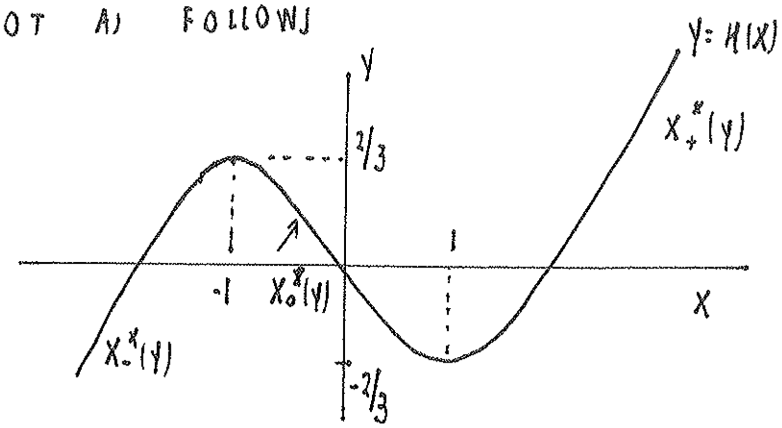
$$(1) \quad \left\{ \begin{aligned} \varepsilon \dot{x} &= y - H(x) = f(x,y) \\ \dot{y} &= -x = g(x,y) \end{aligned} \right.$$

NOW CONSIDER THE VAN-DER-POL OSCILLATOR WITH

$$H(x) = \int_0^x (s^2-1)\,ds = {}^{x^3}/_3 - x.$$

THE SLOW MANIFOLD(S) ARE THE SOLUTIONS $x^*(y)$ TO $y = H(x)$.

WE PLOT AS FOLLOWS



$H'(x) = 0$ AT $x = \pm 1$

$H(1) = -{}^2/_3, \quad H(-1) = {}^2/_3$

WE OBSERVE THAT THERE ARE 3 SLOW MANIFOLDS $x_+^*(y)$, $x_-^*(y)$ AND $x_0^*(y)$.

WE OBSERVE THAT

$$\partial_x f(x,y) = -H'(x) = -[x^2-1].$$

THUS IF $-1 < x < 1$, i.e. ON MANIFOLD $X_0^*(y)$, WE HAVE $\partial_x f[X_0^*(y),y] > 0$, AND HENCE THIS MANIFOLD IS UNSTABLE. THE OTHER MANIFOLDS $X_+^*(y)$, $X_-^*(y)$ ARE STABLE SINCE $|X_\pm^*| > 1 \rightarrow \partial_x[f[X_\pm^*(y),y]] < 0$.

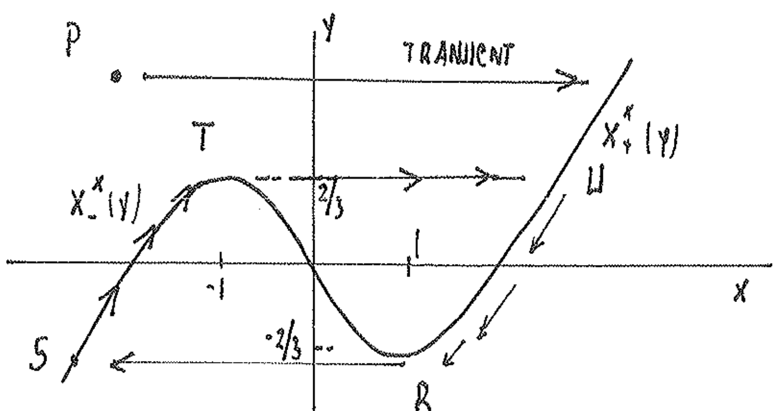
NOW ON THE STABLE SLOW MANIFOLDS, THE SLOW DYNAMICS IS

$$\dot{y} = -X_+^*(y) < 0 \quad \text{SINCE} \quad X_+^*(y) > 0$$

AND

$$\dot{y} = -X_-^*(y) > 0 \quad \text{SINCE} \quad X_-^*(y) < 0.$$

THIS LEADS TO THE FOLLOWING PICTURE FOR A PERIODIC SOLUTION



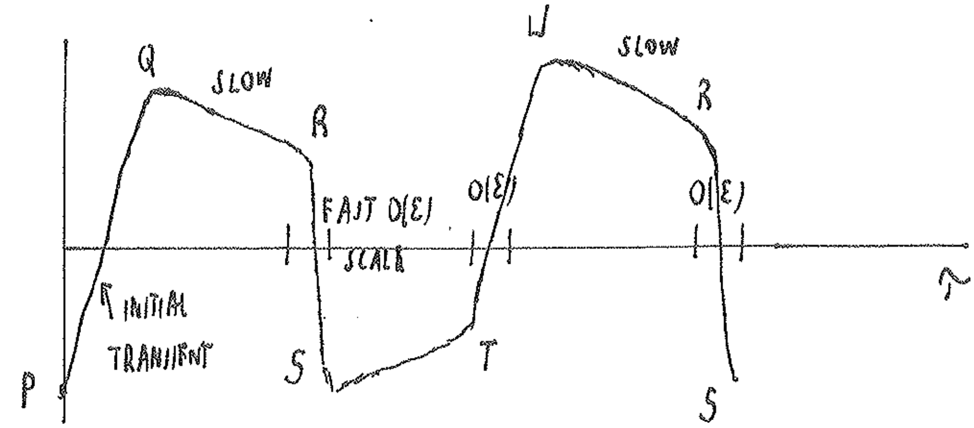
- IF INITIAL CONDITION IS AT POINT P, THEN QUICKLY IN TIME WE REACH POINT Q.
- NOTICE IF $y > H(x)$ THEN $\varepsilon\dot{x} > 0$ SO $\dot{x} = O(1/\varepsilon) > 0$ AND WE MAKE A RAPID EXCURSION FROM T TO U.
- IF $y < H(x)$ THEN $\varepsilon\dot{x} < 0$ SO $\dot{x} = O(1/\varepsilon) < 0$ AND WE MAKE A RAPID EXCURSION FROM R TO S.

NOTICE WHEN $y = 2/3$ THAT $x^3/3 - x = 2/3$ HAS SOLUTION $x = 2$ AS WELL

THUS U HAS COORDINATES $(2, 2/3)$

S HAS COORDINATES $(-2, -2/3)$

THIS GIVES THE SCHEMATIC



NOW CONSIDER FAST TRANSITION FROM R TO S. THEN, LET $\sigma = \tau/\varepsilon$

WE OBTAIN

$$dx/d\sigma = y - H(x)$$

$$dy/d\sigma = -\varepsilon x.$$

WITH $y \sim 2/3$ THEN $x \longrightarrow 2$ AS $\sigma \uparrow$.

NOW TO ESTIMATE THE PERIOD T WE TAKE THE PATH RU AND GET

$$y = H(x)$$

$$\dot{y} = -x$$

BUT $dy/d\tau = H'(x)\, dx/d\tau = (x^2-1)\,\dot{x}$

SO $-x = (x^2-1)\,\dot{x}$

SO $d\tau = -\dfrac{(x^2-1)}{x}\,dx$ $\tau = 0$ WHEN $x = 2$ POINT U

$\tau = T/2$ WHEN $x = 1$ POINT R

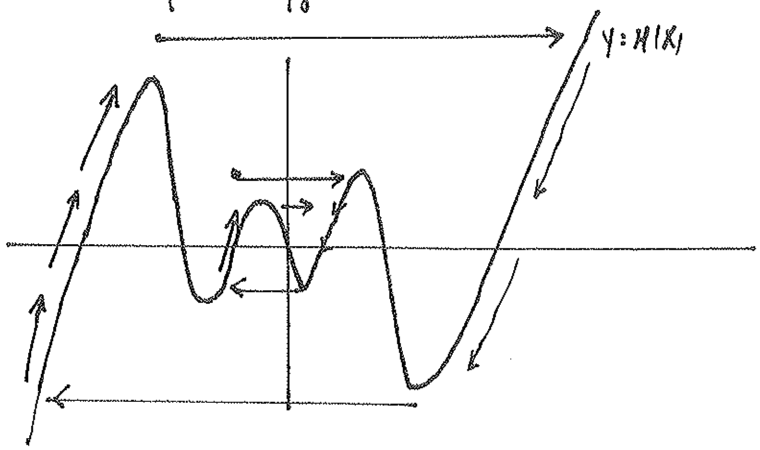
THUS $\displaystyle\int_0^{T/2} d\tau = -\int_2^1 \frac{(1-x^2)}{x}\,dx = \int_1^2 \frac{(x^2-1)}{x}\,dx = \left(x^2/2 - \ln x \right)\Big|_1^2$

THIS YIELDS $T/2 \approx 3/2 - \ln 2.$ PERIOD OF OSCILLATION IN ORIGINAL VARIABLE IS $\approx (3 - 2\ln 2)\,\gamma.$

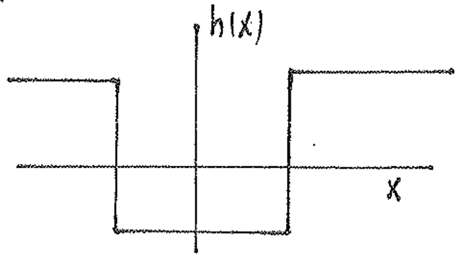
REMARK 1 SUPPOSE WE WANTED A SOLUTION WITH TWO PERIODIC SOLUTIONS. THEN FOR

$$x'' + \gamma h(x) x' + x = 0 \quad \text{WITH} \quad \gamma >> 1$$

WE LET $H(x) = \int_0^x h(s)\,ds$. 2-LIMIT CYCLES WOULD BE POSSIBLE FOR



REMARK 2 SUPPOSE $h(x)$ IS AS SHOWN



DOES $x'' + \gamma h(x) x' + x = 0$ WITH $\gamma >> 1$ HAVE A LIMIT CYCLE EXHIBITING RELAXATION DYNAMICS?

IF SO, CALCULATE THE APPROXIMATE PERIOD OF OSCILLATIONS.

LINEAR SYSTEMS

CONSIDER $\quad \underset{\sim}{u}' = \mathcal{L}^{\varepsilon} \underset{\sim}{u} \qquad \underset{\sim}{u}(0) = \underset{\sim}{v}$

WHERE $\quad \mathcal{L}^{\varepsilon} = \mathcal{L}_0 + \frac{1}{\varepsilon} \mathcal{L}_1 \qquad$ ASSUME $\quad \dim[\mathcal{N}(L_1)] = 1 \qquad$ KERNEL $(L_1) = \text{span}\{\underset{\sim}{\phi}\}$

KERNEL $(L_1^*) = \text{span}\{\underset{\sim}{\psi}\}$

WE EXPAND $\quad \underset{\sim}{u} = \underset{\sim}{u}_0 + \varepsilon \underset{\sim}{u}_1 + \varepsilon^2 \underset{\sim}{u}_2 + \cdots$ TO OBTAIN

$$\frac{1}{\varepsilon} \mathcal{L}_1 \underset{\sim}{u}_0 + [\mathcal{L}_1 \underset{\sim}{u}_1 + \mathcal{L}_0 \underset{\sim}{u}_0 - \underset{\sim}{u}_0'] + \varepsilon[\mathcal{L}_1 \underset{\sim}{u}_2 + \mathcal{L}_0 \underset{\sim}{u}_1 - \underset{\sim}{u}_1'] + O(\varepsilon^2) = 0$$

THIS GIVES THE HIERARCHY:

(1) $\quad \mathcal{L}_1 \underset{\sim}{u}_0 = 0$

(2) $\quad \mathcal{L}_1 \underset{\sim}{u}_1 = \underset{\sim}{u}_0' - \mathcal{L}_0 \underset{\sim}{u}_0$

(3) $\quad \mathcal{L}_1 \underset{\sim}{u}_2 = \underset{\sim}{u}_1' - \mathcal{L}_0 \underset{\sim}{u}_1$

SUPPOSE $\quad \dim \mathcal{N}(L_1) = 1 \quad$ SO $\quad \underset{\sim}{u}_0 = \alpha(t) \phi_0.$

THEN

$$\mathcal{L}_1 \underset{\sim}{u}_1 = \underset{\sim}{u}_0' - \alpha \mathcal{L}_0 \underset{\sim}{\phi}_0.$$

NOW WITH INNER PRODUCT $\langle a, b \rangle$ WE HAVE

$$\langle \underset{\sim}{\psi}, \mathcal{L}_1 \underset{\sim}{u}_1 \rangle = \langle \underset{\sim}{\psi}, \underset{\sim}{u}_0' \rangle - \alpha \langle \underset{\sim}{\psi}, \mathcal{L}_0 \underset{\sim}{\phi}_0 \rangle$$

$$\langle \mathcal{L}_1^* \underset{\sim}{\psi}, \underset{\sim}{u}_1 \rangle = \langle \underset{\sim}{\psi}, \underset{\sim}{u}_0' \rangle - \alpha \langle \psi, \mathcal{L}_0 \underset{\sim}{\phi}_0 \rangle$$

THUS SINCE $\mathcal{L}_1^* \psi = 0$, WE HAVE BY FREDHOLM ALTERNATIVE THAT

$$\alpha' \langle \underset{\sim}{\psi}, \underset{\sim}{\phi} \rangle = \alpha \langle \underset{\sim}{\psi}, \mathcal{L}_0 \underset{\sim}{\phi} \rangle$$

THUS $\quad \alpha' = \bar{\mathcal{L}} \alpha \qquad \bar{\mathcal{L}} = \dfrac{\langle \underset{\sim}{\psi}, \mathcal{L}_0 \underset{\sim}{\phi} \rangle}{\langle \underset{\sim}{\psi}, \underset{\sim}{\phi} \rangle}.$

THIS GIVES THE SLOW DYNAMICS.

DEGENERATE CASE

SUPPOSE THAT $\alpha' = 0$ SO THAT $\langle \underline{\psi}, L_0 \underline{\phi} \rangle = 0$. THUS THE DYNAMICS MUST OCCUR ON A LONGER TIME-SCALE.

WE LET $\tau = \varepsilon t$ SO THAT $t = O(1/\varepsilon)$ IMPLIES $\tau = O(1)$.

THEN $$\dot{\underline{u}} = \left(\frac{1}{\varepsilon^2} L_1 + \frac{1}{\varepsilon} L_0 \right) \underline{u} . \qquad \dot{} = d/d\tau .$$

WE ASSUME THAT $\langle \underline{\psi}, L_0 \underline{\phi} \rangle = 0$.

NOW WE EXPAND $u = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \dots$

WE OBTAIN

(4) $L_1 u_0 = 0 \qquad u_0 = B(\tau)\phi$

(5) $L_1 u_1 = - L_0 u_0$

(6) $L_1 u_2 = \dot{u}_0 - L_0 u_1$

NOW WE ASSUME $\langle \psi, L_0 \phi_0 \rangle = 0$ SO THAT (5) IS SOLVABLE.

WE THEN OBTAIN THAT $u_1 = B \Lambda_1$ WHERE Λ_1 SOLVES THE "CELL PROBLEM"

(7) $L_1 \Lambda_1 = - L_0 \phi$

WE IMPOSE THE NORMALIZATION CONDITION THAT $\langle \Lambda_1, \phi \rangle = 0$.

THE NORMALIZATION CONDITION DOES NOT INFLUENCE THE DYNAMICS.

AT NEXT ORDER THE SOLVABILITY CONDITION IS

$$\langle \psi, \dot{u}_0 - L_0 u_1 \rangle = \dot{B} \langle \psi, \phi \rangle - \langle \psi, L_0 u_1 \rangle = 0$$

THIS YIELDS $$\dot{B} = B \frac{\langle \psi, L_0 \Lambda_1 \rangle}{\langle \psi, \phi \rangle} \qquad \Lambda_1 = - L_1^{-1} L_0 \phi + \gamma \phi$$

FOR ANY γ

WE CALCULATE

$$\frac{\langle \psi, L_0 \Lambda_1 \rangle}{\langle \psi, \phi \rangle} = \frac{\langle \psi, -L_0 L_1^{-1} L_0 \phi + \gamma L_0 \phi \rangle}{\langle \psi, \phi \rangle} = \frac{\langle \psi, -L_0 L_1^{-1} L_0 \phi \rangle}{\langle \psi, \phi \rangle}$$

WHICH IS INDEPENDENT OF γ.

THUS THE DYNAMICS,

$$\dot{B} = B \frac{\langle \psi, L_0 \Lambda_1 \rangle}{\langle \psi, \phi \rangle}$$

IS <u>INDEPENDENT</u> OF THE NORMALIZATION CONDITION USED TO SOLVE THE CELL PROBLEM (7).

<u>EXAMPLE</u> (CONTINUOUS TIME MARKOV CHAIN).

$$\begin{tikzcd} P_1 & P_2 \arrow[l, "2\varepsilon"'] \arrow[r, shift left, "\frac{1}{2}-\varepsilon"] & P_3 \arrow[l, shift left, "\frac{1}{2}"] \end{tikzcd}$$

$P_i(t)$ = probability of being in state i AT TIME t

$$P_1' = 2\varepsilon P_2$$

$$P_2' = \frac{1}{2} P_3 - \left(\frac{1}{2} + \varepsilon\right) P_2$$

$$P_3' = \left(\frac{1}{2} - \varepsilon\right) P_2 - \frac{1}{2} P_3$$

WE WRITE THIS AS

$$\underline{P}' = \begin{pmatrix} 0 & 2\varepsilon & 0 \\ 0 & -\left(\frac{1}{2}+\varepsilon\right) & 1/2 \\ 0 & \frac{1}{2}-\varepsilon & -\frac{1}{2} \end{pmatrix} \underline{P} \qquad \underline{P} = \begin{pmatrix} P_1 \\ P_2 \\ P_3 \end{pmatrix}$$

DEFINE $\mathcal{L}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1/2 & 1/2 \\ 0 & 1/2 & -1/2 \end{pmatrix}$ $\quad \mathcal{L}_0 = \begin{pmatrix} 0 & 2 & 0 \\ 0 & -1 & 0 \\ 0 & -1 & 0 \end{pmatrix}$

THUS $\underline{p}' = (\mathcal{L}_1 + \varepsilon \mathcal{L}_0)\, \underline{p}$

WE WANT TO KNOW WHAT THE DYNAMICS ARE ON THE LONG TIME SCALE. LET $\tau = \varepsilon t$ SO $t : O(1/\varepsilon) \to \tau : O(1)$.

$$\dot{\underline{p}} = \left(\frac{1}{\varepsilon} \mathcal{L}_1 + \mathcal{L}_0 \right) \underline{p} = 0 \qquad \dot{} = d/d\tau .$$

NOTICE THAT $\mathcal{L}_1$ HAS A ZERO eigenvalue OF multiplicity 2 together with an eigenvalue OF -1 WITH multiplicity one.

KERNEL $\mathcal{L}_1 = \operatorname{span}\{\underline{\phi}_1, \underline{\phi}_2\}$ $\quad \underline{\phi}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \underline{\phi}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

BUT MATRIX $\mathcal{L}_1$ IS SYMMETRIC SO $\quad \underline{\phi}_1^T \underline{\phi}_2 = 0$

KERNEL $\mathcal{L}_1^* =$ KERNEL $\mathcal{L}_1 = \operatorname{span}\{\phi_1, \phi_2\}$.

NOW WE EXPAND

$$\underline{p} = \underline{p}_0 + \varepsilon \underline{p}_1 + \ldots$$

WE OBTAIN $\quad \underline{p}_0 = \alpha_1 \underline{\phi}_1 + \alpha_2 \underline{\phi}_2$ WITH $\underline{\phi}_1, \underline{\phi}_2$ AS ABOVE

AND $\alpha_1 = \alpha_1(\tau)$, $\alpha_2 = \alpha_2(\tau)$.

AT NEXT ORDER

$$\mathcal{L}_1 \underline{p}_1 = \dot{\underline{p}}_0 - \mathcal{L}_0 \underline{p}_0 . \qquad \langle \underline{u}, \underline{v} \rangle = \underline{u}^T \underline{v}$$

THE SOLVABILITY CONDITION IS THAT $\underline{\phi}_j^T [\dot{\underline{p}}_0 - \mathcal{L}_0 \underline{p}_0] = 0$. $j = 1, 2,$

THUS FOR $j = 1$,

$$\underline{\phi}_1^T [\dot{\alpha}_1 \underline{\phi}_1 + \dot{\alpha}_2 \underline{\phi}_2] = \underline{\phi}_1^T [\alpha_1 \mathcal{L}_0 \underline{\phi}_1 + \alpha_2 \mathcal{L}_0 \underline{\phi}_2]$$

BUT

$$\mathcal{L}_0 \underline{\phi}_1 = \begin{pmatrix} 0 & 2 & 0 \\ 0 & -1 & 0 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \underline{0}$$

$$\mathcal{L}_0 \underline{\phi}_2 = \begin{pmatrix} 0 & 2 & 0 \\ 0 & -1 & 0 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$$

HENCE

$$\dot{\alpha}_1 \underline{\phi}_1^T \underline{\phi}_1 = \alpha_2 \underline{\phi}_1^T \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

$$\underline{\phi}_1^T \underline{\phi}_1 = 1 \qquad \underline{\phi}_1^T \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = 2$$

WE OBTAIN $\dot{\alpha}_1 = 2 \alpha_2$

<u>FOR $j = 2$</u> $\qquad \dot{\alpha}_2 \underline{\phi}_2^T \underline{\phi}_2 = \underline{\phi}_2^T \alpha_2 \mathcal{L}_0 \underline{\phi}_2 = \alpha_2 (- 2) \qquad \underline{\phi}_2^T \underline{\phi}_2 = 2$

THUS $\dot{\alpha}_2 = - \alpha_2 .$

WE CONCLUDE THAT FOR $t = O(1/\varepsilon)$ WE HAVE

$$\underline{P} \sim \alpha_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

WHERE

$$\alpha_1' = 2\varepsilon \alpha_2 \qquad ' = d/dt$$

$$\alpha_2' = -\varepsilon \alpha_2$$

THUS THE system reaches a quasi-steady BALANCE between states P_2 AND P_3 rapidly ON AN $O(1)$ time scale with probability "leaking" FROM states P_2 AND P_3 INTO absorbing state P_1 ON long time-scale.

TAYLOR DISPERSION

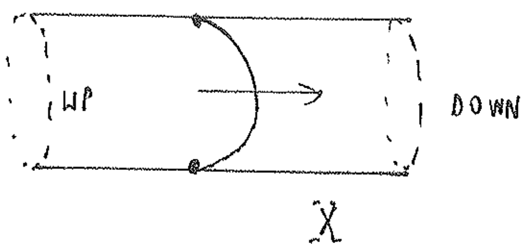
CONSIDER THE CONVECTION-DIFFUSION PROBLEM FOR $C = C(X, R, T)$ MODELED BY

$$C_T + \frac{\partial}{\partial X}(U C) = D\left[C_{XX} + \frac{1}{R}\partial_R(R C_R)\right], \qquad \text{IN } 0 < R < a$$

$$C_R = 0 \quad \text{ON} \quad R = 0, a, \quad -\infty < X < \infty$$

WE ASSUME $U = U(R) = 2U_0\left(1 - R^2/a^2\right)$ IS PARABOLIC POISEUILLE FLOW.

WE WISH TO DETERMINE WHAT HAPPENS TO AN INITIAL BLOB OF CHEMICAL IN THE STREAM AFTER A LONG TIME AND FAR DOWNSTREAM



WE NON-DIMENSIONALIZE: $x = X/L$, $r = R/a$, $U = U_0 u$, $t = \dfrac{T}{(L/U_0)}$

THUS GIVES

$$\frac{U_0}{L} c_t + \frac{U_0}{L}\partial_x(u c) = \frac{D}{L^2} c_{xx} + \frac{D}{a^2}\frac{1}{r}\partial_r(r c_r)$$

WE WRITE AS

$$\left(\frac{U_0 a}{D}\right)\frac{a}{L}\left[c_t + \partial_x(u c)\right] = \frac{a^2}{L^2} c_{xx} + \frac{1}{r}\partial_r(r c_r)$$

WE DEFINE

$$\varepsilon = a/L \ll 1$$

$$P_e = \frac{U_0 a}{D} = O(1) \quad \text{Peclet NUMBER.}$$

THIS YIELDS THAT

$$(1)\quad \begin{cases} \varepsilon P_e \left(C_t + \partial_x (U C) \right) = \varepsilon^2 C_{xx} + \frac{1}{r} (r C_r)_r & \text{IN } 0<r<1, \\ C_r = 0 \quad \text{ON} \quad r = 0, 1 \end{cases}$$

AND $\qquad U = 2 \left(1 - r^2 \right).$

WE NOW LOOK FOR A MULTI-SCALE SOLUTION IN THE FORM

$$(3)\quad C = C_0 (x, t, \tau) + \varepsilon C_1 (x, t, \tau) + \varepsilon^2 C_2 (x, t, \tau)$$

WE WRITE (1) AS AN OPERATOR EQUATION

$$\mathcal{L}_1 C = - \varepsilon^2 C_{xx} + \varepsilon P_e \left(C_t + \partial_x (U C) \right)$$

WITH $\qquad \mathcal{L}_1 \equiv \frac{1}{r} (r \, \_)_r .$

NOW UPON SUBSTITUTING WE OBTAIN THAT WITH $C_t = C_{0_t} + \varepsilon C_{0_\tau} + \dots$

$$(4)\quad \mathcal{L}_1 C_0 = 0 \qquad\qquad C_{0r} = 0 \quad \text{ON} \quad r = 0, 1$$

$$(5)\quad \mathcal{L}_1 C_1 = + P_e \left[C_{0_t} + \partial_x (U C_0) \right] \qquad\qquad C_{1r} = 0 \quad \text{ON} \quad r = 0, 1$$

$$(6)\quad \mathcal{L}_1 C_2 = + P_e \left[C_{1_t} + \partial_x (U C_1) \right] + P_e C_{0_\tau} - C_{0xx} \qquad C_{2r} = 0 \quad \text{ON} \quad r = 0, 1$$

WE OBTAIN THAT

$$C_0 = C_0 (x, t, \tau)$$

AND SINCE $\mathcal{L}_1$ IS SELF-ADJOINT WITH $\mathcal{L}_1 [1] = 0$, i.e. 1 IS IN NULL-SPACE OF $\mathcal{L}_1$ WE OBTAIN THAT (5) IS SOLVABLE IFF

$$\left(P_e (C_{0_t} + \partial_x (U C_0)), 1 \right) = 0 \qquad \text{WHERE} \quad (U, V) = \int_0^1 U V r \, dr .$$

THIS YIELDS THAT

$$C_{0_t} \int_0^1 r\,dr + \partial_x \left[\left(\int_0^1 u\, r\, dr \right) C_0 \right] = 0.$$

WE LABEL $\langle u \rangle \equiv 2 \int_0^1 u\, r\, dr = 2 \int_0^1 2(1-r^2) r\, dr = 4\left[\frac{1}{2} - \frac{1}{4}\right] = 1.$

THUS ON THE t TIME-SCALE AND FOR DOWNSTREAM WE HAVE

$$(7) \quad C_{0_t} + \langle u \rangle C_{0x} = 0 \qquad \text{WITH} \quad \langle u \rangle = 1.$$

THUS $C_0 = \mathcal{F}(x - t, \tau)$ WHICH IS DRIFT DOWNSTREAM WITH THE AVERAGE.

NEXT WE WANT TO MEASURE THE DEVIATION FROM THIS PURE DRIFT BEHAVIOR. WE SUBTRACT (5) AND (7) TO OBTAIN

$$\mathcal{L}_1 C_1 = P_e \left[u - \langle u \rangle \right] C_{0x}$$

WE DEFINE $\tilde{u} = u - \langle u \rangle$ TO BE THE VELOCITY NON-UNIFORMITY AND SO WE WRITE

$$(8) \quad \begin{cases} \mathcal{L}_1 C_1 = P_e\, \tilde{u}(r)\, C_{0x} \\ C_{1r} = 0 \quad \text{ON} \quad r = 0, 1 \end{cases}$$

WHERE $\tilde{u} = u(r) - 1 = 2(1-r^2) - 1 = 1 - 2r^2$, SATISFIES $\langle \tilde{u} \rangle = 0$.

NOW TO SOLVE (8) WE WRITE

$$(9) \quad C_1 = P_e\, C_{0x}\, V_1(r)$$

WHERE V_1 SOLVES THE "CELL PROBLEM"

$$(10)\quad \begin{cases} \mathcal{L}_1 V_1 = \hat{u}(r) = 1 - 2r^2 \\ V_{1r} = 0 \quad \text{ON } r = 0, 1 \end{cases} \qquad \text{SO} \quad V_1 = \frac{r^2}{4} - \frac{r^4}{8} + \alpha \qquad (\alpha \text{ ANY CONSTANT})$$

NOW WITH V_1 KNOWN UP TO A CONSTANT, I.E. UP TO A MULTIPLE OF THE ELEMENT IN $\mathcal{N}(\mathcal{L}_1)$ WE PROCEED TO THE EQUATION FOR C_2.

WE HAVE

$$(11)\quad \mathcal{L}_1 C_2 = P_e \left[C_{1t} + u\, C_{1x} \right] + P_e C_{0\tau} - C_{0xx},$$

WITH $\qquad C_1 = P_e C_{0x} V_1 \qquad$ AND $\qquad C_{0t} + \langle u \rangle C_{0x} = 0.$

NOW WE CALCULATE NOTING THAT V_1

$$\begin{aligned} C_{1t} + u\, C_{1x} &= P_e V_1 \partial_t C_{0x} + u P_e V_{1r} C_{0xx} \\ &= P_e V_1 \partial_x \left[C_{0t} + u\, C_{0x} \right] \\ &= P_e V_1 \partial_x \left[- C_{0x} \langle u \rangle + u\, C_{0x} \right] \\ &= P_e V_1 C_{0xx} \left[u - \langle u \rangle \right] \end{aligned}$$

SO

$$C_{1t} + u C_{1x} = P_e V_1 C_{0xx} \tilde{u} \qquad \text{WITH} \quad \hat{u} = u - \langle u \rangle .$$

SUBSTITUTING INTO (11) YIELDS THAT

$$(12)\quad \begin{cases} \mathcal{L}_1 C_2 = P_e^2 V_1 \tilde{u} C_{0xx} + P_e C_{0\tau} - C_{0xx}. \\ C_{2r} = 0 \quad \text{ON } r = 0, 1 \end{cases}$$

WHERE $\quad V_1 = V_1(r), \quad \hat{u} = \hat{u}(r).$

FINALLY, THE SOLVABILITY CONDITION FOR (12) IS THAT

$$P_e^2 \, Co_{xx} \, (V_1 \hat{U}, 1) + (P_e \, Co_{\tau}, 1) - (Co_{xx}, 1) = 0$$

WHERE $(U, V) = \int_0^1 r \, U \, V \, dr$. THIS YIELDS

$$(P_e \, Co_{\tau}, 1) = Co_{xx}\left[(1, 1) - P_e^2 (V_1 \hat{U}, 1) \right]$$

OR WRITING MORE FULLY $(1,1) = \int_0^1 r \, dr = \frac{1}{2}$, $(P_e \, Co_{\tau}, 1) = P_e \, Co_{\tau} \frac{1}{2}$.

THUS

$$P_e \, Co_{\tau} = Co_{xx}\left(1 - 2 P_e^2 (V_1 \hat{U}, 1) \right)$$

BUT $(V_1 \hat{U}, 1) = \int_0^1 r \, V_1 \hat{U} \, dr = \frac{1}{2} \langle V_1 \hat{U} \rangle$ WHERE $\langle V_1 \hat{U} \rangle = 2\int_0^1 \hat{U} V_1 r \, dr$

THUS WE HAVE

$$(13) \qquad P_e \, Co_{\tau} = Co_{xx}\left(1 - P_e^2 \langle \hat{U} \, V_1 \rangle \right).$$

WE NOW CALCULATE $\langle \hat{U} V_1 \rangle$. WE HAVE FROM (10) THAT V_1 IS KNOWN AND $\tilde{U} = 1 - 2r^2$. THUS

$$\langle \tilde{U} \, V_1 \rangle = \left\langle \tilde{U}\left(\frac{r^2}{4} - \frac{r^4}{8} \right) + \alpha \tilde{U} \right\rangle = \left\langle \tilde{U}\left(\frac{r^2}{4} - \frac{r^4}{8} \right) \right\rangle \quad \text{SINCE } \langle \tilde{U} \rangle = 0$$

$$= 2\int_0^1 (1 - 2r^2)\left(\frac{r^2}{4} - \frac{r^4}{8} \right) r \, dr$$

$$\langle \hat{U} \, V_1 \rangle = \frac{1}{4}\int_0^1 (r - 2r^3)(2r^2 - r^4) \, dr = \frac{1}{4}\int_0^1 \left[2r^3 - r^5 - 4r^5 + 2r^7 \right] dr.$$

THIS YIELDS

$$\langle \hat{u} V_1 \rangle = \frac{1}{4}\left[\frac{1}{2} - \frac{1}{6} - \frac{2}{3} + \frac{1}{4} \right] = -\frac{1}{48}.$$

THUS, (13) BECOMES

$$P_e C_{0\tau} = C_{0xx}\left[1 + \frac{P_e^2}{48} \right] \qquad (14)$$

WITH $\quad C_1 = P_e V_1(r) C_{0x}$

TO EXPRESS OUR FINAL RESULT, WE TAKE

$$P_e (C_{0_t} + \langle u \rangle C_{0x}) = 0$$

AND ADD TO $\quad \varepsilon P_e C_{0\tau} = \varepsilon C_{0xx}(1 + P_e^2/48).$

THIS GIVES

$$P_e [C_{0_t} + \varepsilon C_{0\tau}] + P_e \langle u \rangle C_{0x} = \varepsilon C_{0xx}(1 + P_e^2/48)$$

NOW THAT WE ARE FINISHED WITH OUR MULTI-SCALE EXPANSION WE WRITE $\quad C_0'(x, t) = C_0(x, t, \tau).$

THUS $\quad P_e C_{0_t}' + P_e \langle u \rangle C_{0x}' = \varepsilon C_{0xx}' (1 + P_e^2/48)$

DIVIDING BY $\quad P_e = U_0 a/D \quad$ WE OBTAIN

$$C_{0_t}' + \langle u \rangle C_{0x}' = \frac{\varepsilon}{P_e} C_{0xx}' \left(1 + \frac{P_e^2}{48} \right)$$

FINALLY, PUT IN TERMS OF DIMENSIONAL VARIABLES.

$$T = U_0/_L\, t, \qquad X = X/_L \quad \text{AND} \quad U = U_0\, u, \qquad \varepsilon = a/_L$$

THUS $$\frac{L}{U_0} C_{0_T} + \frac{L}{U_0} \langle U \rangle C_{0X} = \frac{a}{L(U_0 a/D)} L^2 C_{0XX} \left(1 + Pe^2/48\right)$$

SO $$C_{0_T} + \langle U \rangle C_{0X} = D C_{0XX} \left(1 + Pe^2/48\right)$$

WHICH IS VALID ON LONG TIME-SCALES FOR $P_e = O(1)$

NOTICE THAT THE EFFECTIVE DIFFUSIVITY IS

$$D_{EFF} = D\left(1 + \frac{Pe^2}{49}\right) = D\left(1 + \frac{U_0^2 a^2}{48 D^2}\right) = D + \frac{U_0^2 a^2}{48 D}$$

NOTICE $D_{EFF} \uparrow$ AS $D \downarrow$

FOR SALT IN WATER, $D = 1 \times 10^{-5}\ cm^2/SEC$, $U_0 = \frac{1}{2}\ cm/SEC$, $a = 0.2\ cm$

THEN $$\frac{U_0^2 a^2}{49 D^2} = 22\ cm^2/SEC.$$