NONLINEAR FRICTION AND CONTROL

WE CONSIDER THE TRADITIONAL MASS-ON-A-MOVING BELT MODEL. THE NEGATIVE SLOPE IN THE FRICTION-VELOCITY RELATION LEADS TO SELF-EXCITED OSCILLATIONS.

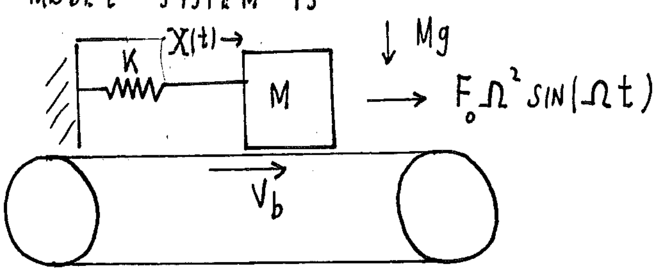
MANY SLIDING MECHANICAL INTERFACES ARE CHARACTERIZED BY SOME FORM OF DRY FRICTION WITH A FORCE-VELOCITY CURVE HAVING A NEGATIVE SLOPE AT LOW VELOCITIES. THEN, FRICTION DECREASES AS THE CONTACTING OBJECT BEGINS TO MOVE, WHEREAS AT HIGHER VELOCITIES THE FRICTION FORCE INCREASES AGAIN. THE NEGATIVE SLOPE CORRESPONDS TO NEGATIVE DAMPING AND MAY CAUSE OSCILLATIONS THAT GROW IN AMPLITUDE. THERE ARE TYPICALLY TWO PHASES OF SUCH OSCILLATIONS:

(i) A STICK PHASE WITH NO SLIPPAGE BETWEEN PARTS AND FRICTION FORCES LIMITED BY STATIC FRICTION.

(ii) A SLIP PHASE WITH A SOMEWHAT LOWER FRICTION.

APPLICATIONS: SQUEAKING DOOR HINGES, SQUEALING BRAKES, CHATTER FROM FRICTION ETC.

THE MODEL SYSTEM IS



REF. J. THOMSEN "USING FAST VIBRATIONS TO QUENCH FRICTION-INDUCED OSCILLATIONS" J. SOUND AND VIBRATION (1999) 228 (5), PP. 1079-1102.

NEWTON'S LAW YIELDS $$MX'' = -KX - CX' - F_f + F_0\Omega^2 \sin(\Omega t)$$

THE FRICTION FORCE F_f has the FORM

$$F_f = +\mu(V_r)Mg$$

where $V_r = dx/dt - V_b$ IS THE Relative Velocity between the mass and the belt. THIS YIELDS AN EQUATION WITH

$$MX'' + CX' + KX + \mu(V_r)Mg = F_0\Omega^2 \sin(\Omega t)$$

NOW WE NON-DIMENSIONALIZE $\Omega = \Omega_0 \omega_0$ $\quad t = \tau/\omega_0$ $\quad X = xL$.

THIS YIELDS,

$$ML\omega_0^2 \ddot{x} + CL\omega_0 \dot{x} + KLx = F_0 \Omega^2 \sin(\Omega\tau/\omega_0) - \mu(V_r) Mg$$

WE DEFINE $\omega_0 = \sqrt{K/M}$ AND $\beta = \frac{C}{2M\omega_0}$. THIS YIELDS

$$\ddot{x} + 2\beta\dot{x} + x = \frac{F_0\Omega_0^2}{ML\omega_0^2}\sin\left(\Omega_0\tau\right) - \mu(V_r)\frac{g}{L\omega_0^2}$$

NOW $v_r \equiv V_r/\omega_0 L$. WE CHOOSE $L = g/\omega_0^2$ AND LET $a = \frac{F_0}{ML\omega_0^2}$

THIS YIELDS THE DIMENSIONLESS SYSTEM

$$\ddot{x} + 2\beta\dot{x} + x + \gamma^2\mu(v_r) = a\sin\left(\Omega_0\tau\right)\Omega_0^2$$

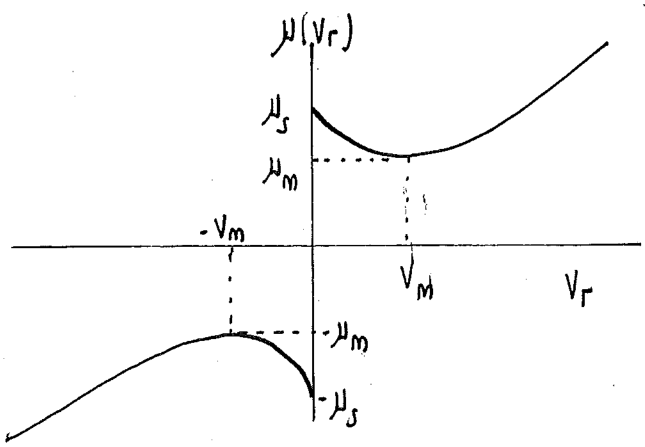
IT IS ASSUMED $\Omega_0 >> 1$ BUT $a\Omega_0 = O(1)$.

here $v_r = \dot{x} - v_b$ IS THE RELATIVE VELOCITY BETWEEN MASS AND BELT.

THE FRICTION MODEL $\mu(v_r)$ HAS THE FORM

$$\mu(v_r) = \mu_s \operatorname{sign}(v_r) - \frac{3}{2}(\mu_s - \mu_m)\left(\frac{v_r}{v_m} - \frac{1}{3}\left(\frac{v_r}{v_m}\right)^3\right).$$

IF WE PLOT THIS FUNCTION IT HAS THE FOLLOWING SHAPE



μ_m = MINIMUM OF Kinetic FRICTION

μ_s = COLOUMB FRICTION.

IF $|\mu| \leq \mu_s$ AND $v_r = 0 \rightarrow$

MASS IS AT REST ON MOVING BELT.

(stick-phase)

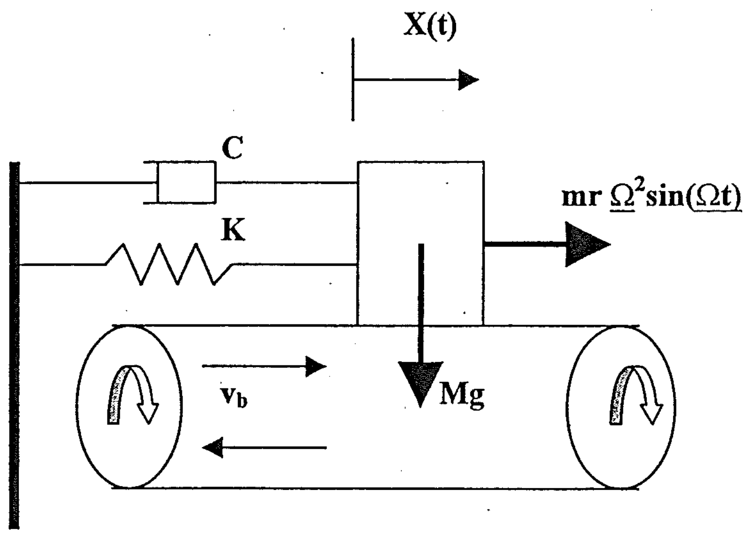
IF IT STARTS MOVING THEN THE FRICTION INITIALLY DECREASES

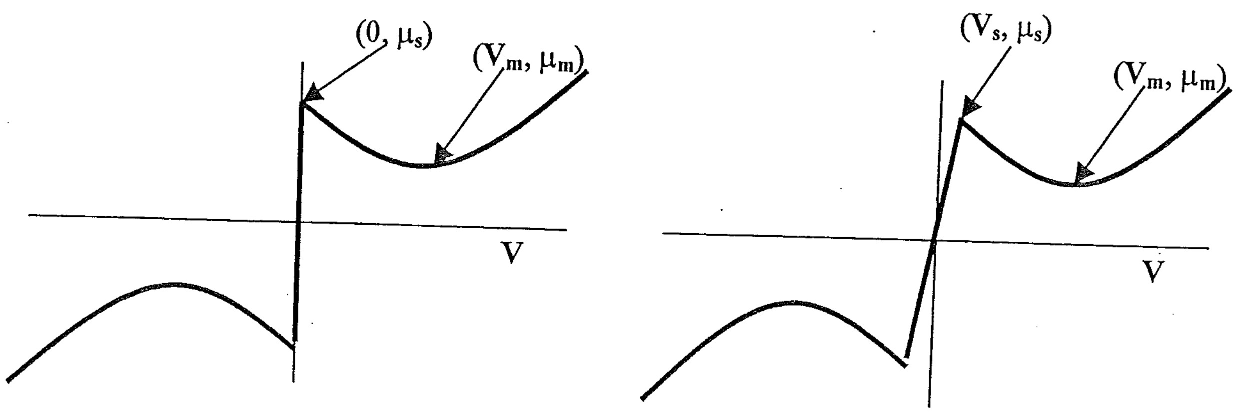
(slip phase).

TO AVOID DISCONTINUITIES WE USE A REGULARIZED FRICTION MODEL OF THE FORM

$$\mu_\delta(v_r) = \begin{cases} \mu_s v_r/\delta & |v_r| < \delta \\ \operatorname{sgn}(v_r)\left[\mu_s - \frac{3}{2}(\mu_s - \mu_m)\left(\frac{|v_r| - \delta}{v_m - \delta} - \frac{1}{3}\left(\frac{|v_r| - \delta}{v_m - \delta}\right)^3\right)\right] & |v_r| \geq \delta. \end{cases}$$

WITH $\delta = 10^{-4}$. THE REGULARIZED AND UNREGULARIZED FRICTION MODELS ARE SHOWN BELOW.

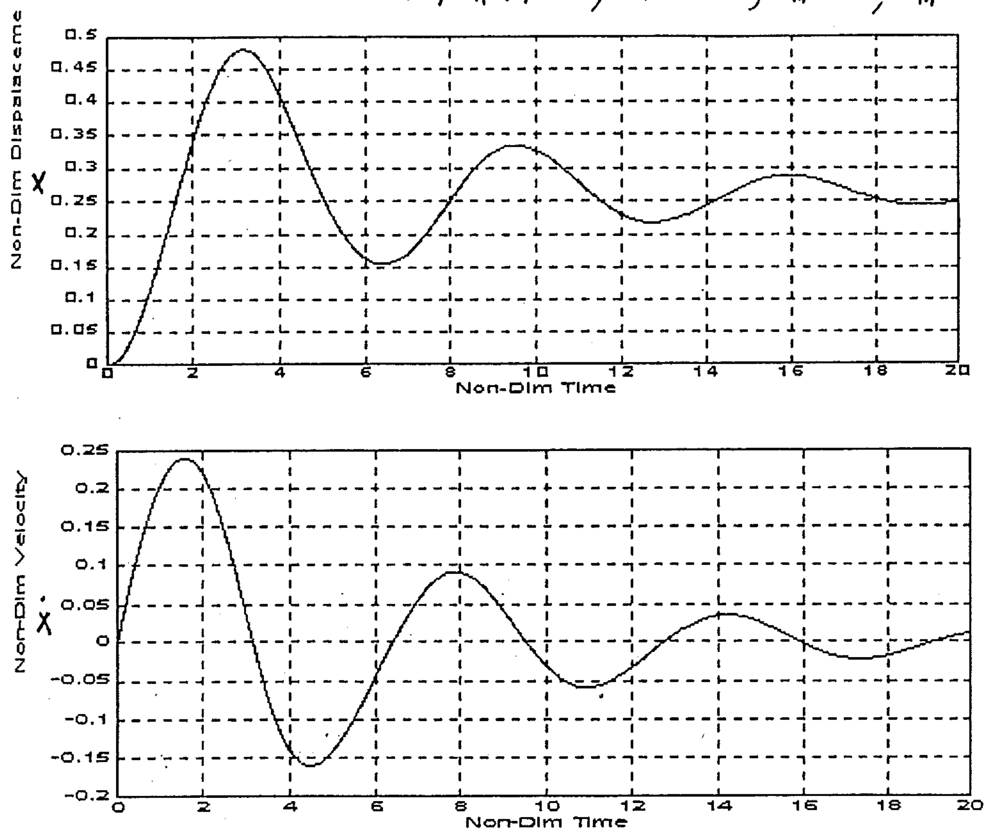




(a) (b)

Figure (5)

$x(0) = x'(0) = 0$, $\beta = 0.05$, $v_m = 0.5$, $\mu_m = 0.25$, $\delta = 10^{-4}$, $\mu_s = 0.4$



$V_b = 0.6$

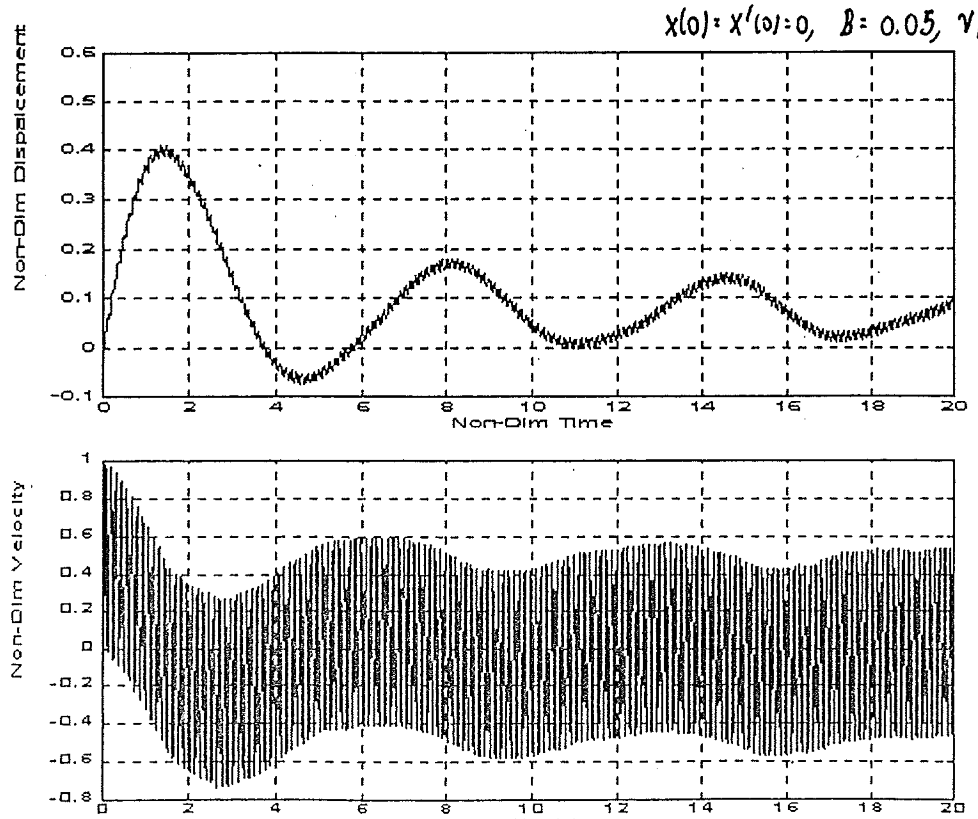
$v_b : 0.6$

Figure (6)

$x(0) = x'(0) = 0$, $\beta = 0.05$, $v_m = 0.5$, $\mu_m = 0.25$, $\delta = 10^{-4}$, $\mu_s = 0.4$

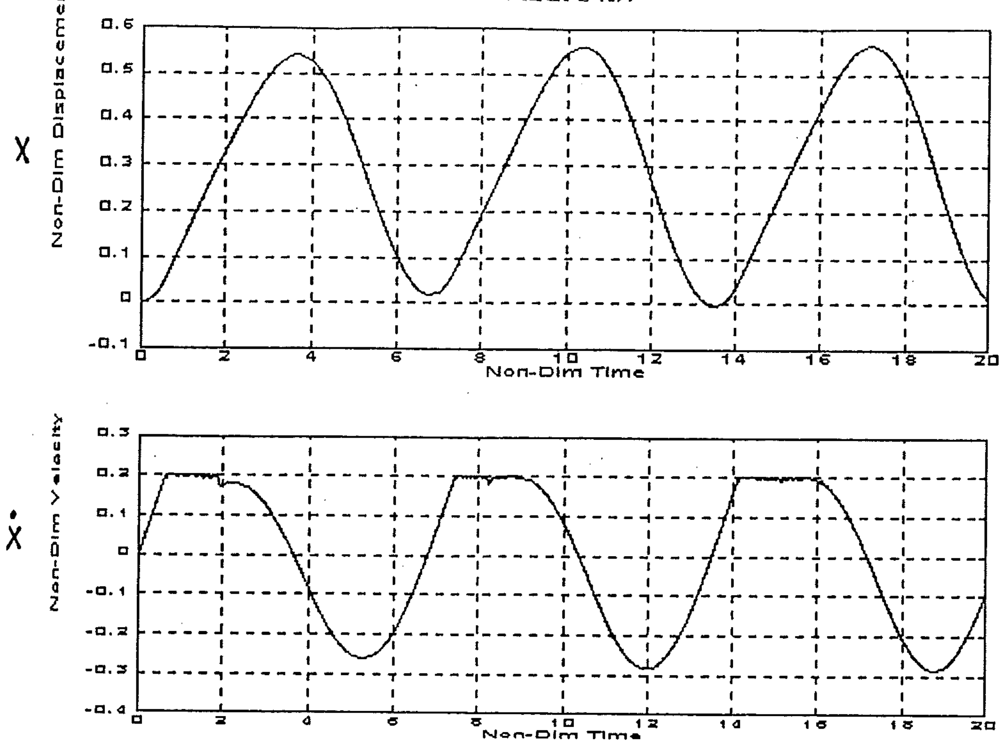
$a = 0.01$, $\Omega_0 = 50$

$v_b = 0.2$



$X(0) = X'(0) = 0,\ \beta = .05,\ V_m = 0.5,\ \mu_m = 0.25,$
$\delta = 10^{-4},\ \mu_s = 0.4$

Figure (3)

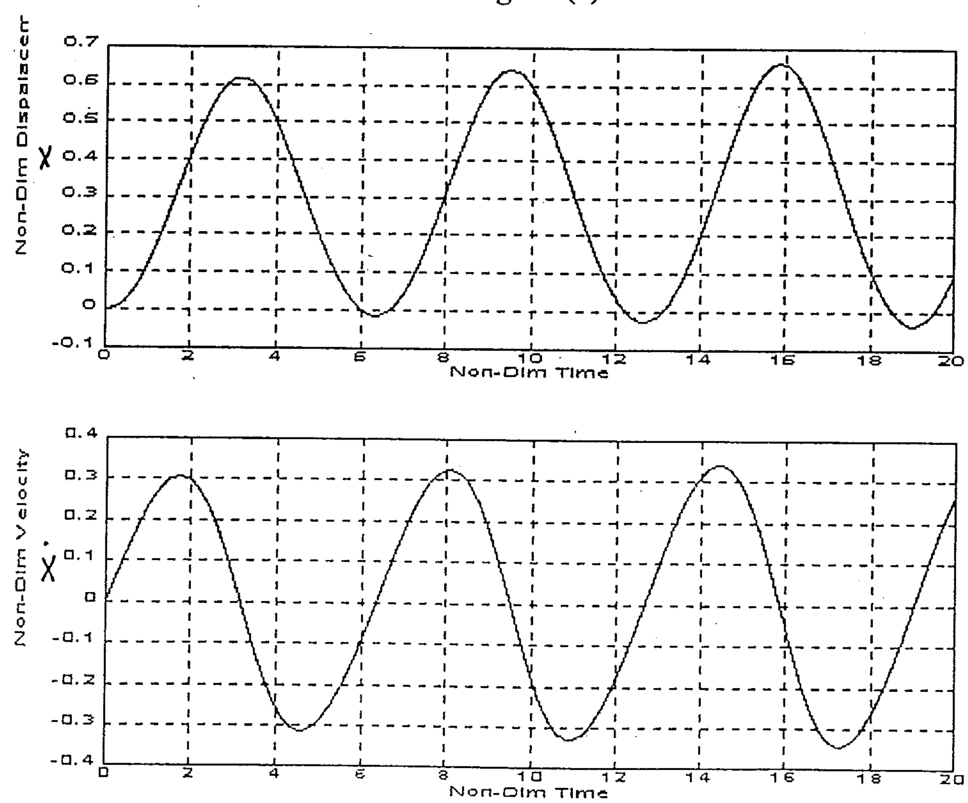


← Vb = 0.2

← Vb = 0.2

stick-slip oscillation

Figure (4)



← Vb = 0.4

sliding oscillation.

← Vb = 0.4

WHEN THERE IS NO EXTERNAL EXCITATION, THERE ARE ESSENTIALLY THREE TYPES OF BEHAVIORS

(I) AT low belt velocities, there is stick-slip oscillations. (FIG. 3)

(II) AT intermediate belt velocities there is pure-sliding motion (Fig. 4)

(III) AT high belt velocities there is pure sliding motion with essentially NO oscillation. the oscillations are damped.

IN FIG. 6 WE SHOW THAT SMALL AMPLITUDE HIGH FREQUENCY OSCILLATIONS CAN ELIMINATE THE STICK-SLIP VIBRATIONS WHEN $V_b = 0.2$. SELF-EXCITED OSCILLATIONS ON NEGATIVE SLOPE OF FRICTION TERM

ROUGH IDEA OF ANALYSIS FOR $a \ll 1$, $a\Omega_0 = O(1)$, $\Omega_0 \gg 1$.

WE WRITE AS IN (31)

$$\ddot{X} + S(X, \dot{X}) = \Omega\,(a\Omega)\sin(\Omega\tau)$$

WE WRITE $X(\tau) = Z(\tau) + \Omega^{-1}\phi(\tau, \Omega\tau)$ $\quad T = \Omega\tau$ FAST TIME

Z SLOW MOTION, $\Omega\tau$ FAST TIME-SCALE OF EXTERNAL EXCITATIONS.

WE MAKE $\langle \phi(\tau, T)\rangle = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} \phi(\tau, T)\,dT = 0$.

SUBSTITUTING:

$$\dot{X} = \dot{Z} + \Omega^{-1}(\phi_\tau + \Omega\phi_T)$$

$$\ddot{X} = \ddot{Z} + \Omega^{-1}(\phi_{\tau\tau} + 2\Omega\phi_{\tau T} + \Omega^2\phi_{TT})$$

THEN

$$\Omega\phi_{TT} + 2\phi_{\tau T} + \Omega^{-1}\phi_{\tau\tau} + \ddot{Z} + S\left(X, \dot{Z} + \phi_T + \frac{1}{\Omega}\phi_\tau\right)$$
$$= \Omega\,(a\Omega)\sin(\Omega\tau)$$

THEREFORE,

$$\phi_{TT} + \frac{1}{\Omega}\left(2\phi_{\tau T} + \ddot{z} + S\left(z + \Omega^{-1}\phi, \dot{z} + \phi_T + \frac{1}{\Omega}\phi_\tau\right)\right) + \frac{1}{\Omega^2}\phi_{\tau\tau} = (a\Omega)\sin(\Omega\tau).$$

HENCE $\phi_{TT} = a\Omega \sin(T)$

SO $\phi = -a\Omega \sin(T)$ $\langle \phi \rangle = 0.$

SO $\langle f \rangle = \cdot$

THIS GIVES $X(\tau) = z(\tau) - a\sin(T)$

NOW $$\ddot{z} + S\left(z - a\sin(T), \dot{z} - a\Omega\cos T\right) = 0$$

WE THE AVERAGE TO OBTAIN

$$\ddot{z} + \langle S(z - a\sin(\Omega\tau), \dot{z} - a\Omega\cos(\Omega\tau))\rangle = 0$$

RECALL THAT $S(X, \dot{X}) = X + 2B\dot{X} + \gamma^2 \mu(\dot{X} - V_b)$

THIS WILL YIELD THAT

$$\ddot{z} + 2B\dot{z} + z + \gamma^2 \bar{\mu}(\dot{z} - V_b) = 0$$

where $\bar{\mu}$ is an effective friction characteristic defined by

$$\bar{\mu}(V_r) = \langle \mu(V_r - a\Omega\cos(\Omega\tau))\rangle.$$

THIS IS AN AUTONOMOUS EQUATION WHICH IS EASIER TO ANALYSE. NOTICE MIN OCCURS AT V_m SO THAT

$$V_r - a\Omega\cos(\Omega\tau) = V_m$$

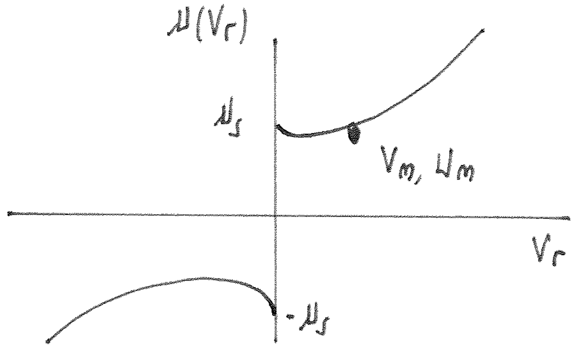
FOR $|V_r| > a\Omega$ EFFECTIVE FRICTION larger than true FRICTION

STICK-SLIP OSCILLATIONS

IN NON-DIMENSIONAL FORM

$$\ddot{X} + 2B\dot{X} + X + \mu(\dot{X} - V_b) = 0 \quad \text{IF} \quad \dot{X} \neq V_b \quad (\text{slip})$$

$$\ddot{X} = 0, \quad X + 2BV_b < \mu_s \quad \text{IF} \quad \dot{X} = V_b \quad (\text{stick})$$



$$\mu(V_r) = \mu_s \, \text{sign}(V_r) - K_1 V_r + K_3 V_r^3$$

$$V_r = \dot{X} - V_b \qquad K_1 = \frac{3}{2}\left(\frac{\mu_s - \mu_m}{V_m}\right) \qquad K_3 = \frac{1}{2}\left(\frac{\mu_s - \mu_m}{V_m^3}\right)$$

- μ_s coefficient dry FRICTION
- V_m velocity corresponding to minimum μ_m of dynamic friction
- $\mu_m < \mu_s$ WITH $K_1 > 0$, $K_3 > 0$.

STICK phase $|\mu| < \mu_s$ WHEN MASS IS ON moving belt at Rest WITH $V_r = 0$.

WHEN MASS STARTS SLIDING THE FRICTION FORCE initially decreases with increasing velocity $V_r \neq 0$ slip phase.

TYPES OF MOTION

FOR $\ddot{X} + 2B\dot{X} + X + \mu(\dot{X} - V_b) = 0$

WE LOOK FOR AN EQUILIBRIUM SOLUTION WITH $\dot{X} = \ddot{X} = 0$ AND $X = X_e$.

WE OBTAIN

$$X_e = -\mu(-V_b) = \mu_s - K_1 V_b + K_3 V_b^3.$$

Belt moves BUT MASS stationary

NOW WE LET $u = X(t) - X_e$.

THEN FOR SLIP: $\ddot{u} + 2B\dot{u} + u + X_e + \mu(\dot{u} - V_b) = 0$

BUT $\ddot{u} + 2B\dot{u} + u + \mu(V_b) + \mu(\dot{u} - V_b) = 0.$

WE WRITE THIS AS $\qquad \mu(V_b) = -\mu(-V_b)$

$$\begin{cases} \ddot{u} + u + \varepsilon h(\dot{u}) = 0 \qquad \text{FOR } \dot{u} \neq V_b \\ \varepsilon h(\dot{u}) = 2B\dot{u} + \mu(\dot{u} - V_b) - \mu(-V_b) \\ \varepsilon h(\dot{u}) = \mu_s(1 + \operatorname{sgn}(\dot{u} - V_b)) + [2B - K_1 + 3K_3 V_b^2]\dot{u} \\ \qquad - 3K_3 V_b \dot{u}^2 + K_3 \dot{u}^3. \end{cases}$$

THE ASSUMPTIONS ARE THAT

$$2B - K_1 = O(\varepsilon), \qquad K_3 = O(\varepsilon), \qquad V_b = O(1).$$

i.e. $\mu_s - \mu_m = O(\varepsilon)$.

NOW FOR THE STICK-PHASE $X = X_e + u = \mu(V_b) + u$.

SO $X + 2BV_b < \mu_s$ BECOMES $u + 2BV_b + (\mu(V_b) - \mu_s) < 0$.

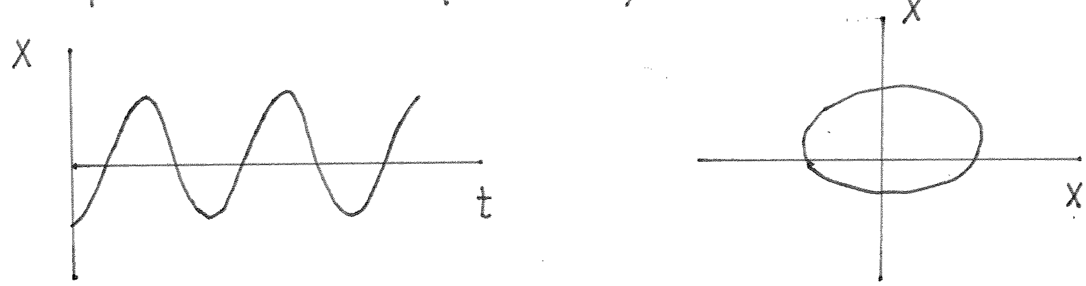
THIS IMPLIES THAT $\Big\{ u + 2BV_b - [K_1 V_b - K_3 V_b^3] < 0$ FOR $\dot{u} = $ \

NOTE: 1) $u = \dot{u} = 0$ steady-state sliding $\longrightarrow$ MASS IS AT REST WITH THE BELT SLIDING AT CONSTANT VELOCITY V_b BELOW IT.

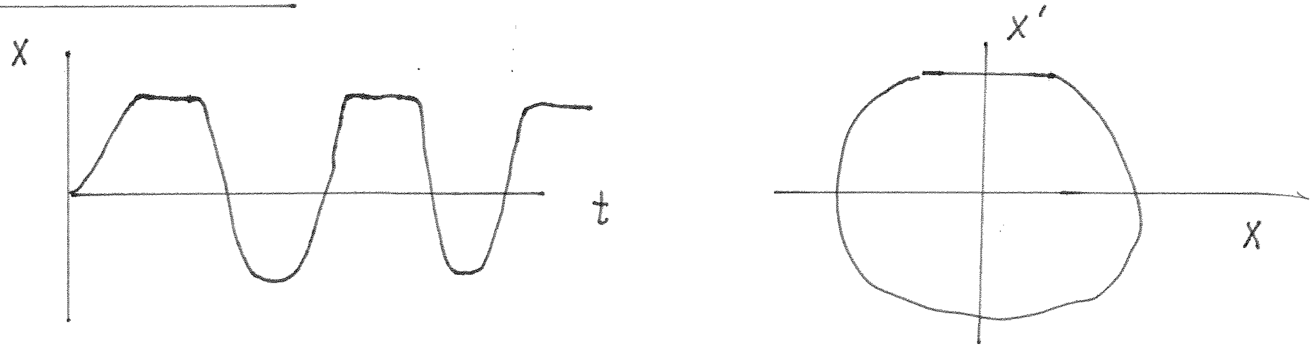
2) negative FRICTION gradient produces negative damping.

IF $u = \dot{u} = 0$ IS UNSTABLE $\longrightarrow$ periodic solutions are possible

PURE-SLIP OSCILLATIONS velocity OF mass is always smaller than velocity OF belt $(\dot{x} < V_b)$



STICK-SLIP OSCILLATIONS MASS OCCASIONALLY sticks to the belt.



PURE-SLIP OSCILLATIONS WE REQUIRE $\dot{u} < V_b \quad \forall t$

SO THAT DISCONTINUITY IN FRICTION FUNCTION U NEVER MET OR CROSSED. WE WRITE

$$\varepsilon h(\dot{u}) = h_1 \dot{u} + h_2 \dot{u}^2 + h_3 \dot{u}^3$$

$h_1 = 2B - K_1 + 3K_3 V_b^2$

$h_2 = -3K_3 V_b$

$h_3 = K_3$.

THEN $\ddot{u} + u + \varepsilon h(\dot{u}) = 0$.

WE USE THE METHOD OF MULTIPLE-SCALES AS IN THE NOTES.

LET $T = t$, $\hat{T} = \varepsilon t$ AND WRITE

$$u = U_0(T, \hat{T}) + \varepsilon U_1(T, \hat{T})$$

$$U_{0TT} + 2\varepsilon U_{0T\hat{T}} + \varepsilon U_{1TT} + U_0 + \varepsilon U_1 + \varepsilon h(U_{0T}) + \dots = 0.$$

SO $$U_{0TT} + U_0 = 0$$

$$U_{1TT} + U_1 = -2U_{0T\hat{T}} - h(U_{0T})$$

SO $$U_0 = A(\hat{T})\cos(T + \phi(\hat{T})).$$

THEN $$U_{1TT} + U_1 = 2A'\sin(T+\phi) + 2A\phi'\cos(T+\phi) - h(-A\sin(T+\phi))$$

NOW TO SUPPRESS RESONANCE

$$2A'\int_0^{2\pi}\sin^2(T+\phi)\,d(T+\phi) = \int_0^{2\pi} h(-A\sin(T+\phi))\sin(T+\phi)\,d(T+\phi)$$

$$2A\phi'\int_0^{2\pi}\cos^2(T+\phi)\,d(T+\phi) = \int_0^{2\pi} h(-A\sin(T+\phi))\cos(T+\phi)\,d(T+\phi)$$

HENCE $$A' = \frac{1}{2\pi}\int_0^{2\pi} h(-A\sin\psi)\sin\psi\,d\psi$$

$$A\phi' = \frac{1}{2\pi}\int_0^{2\pi} h(-A\sin\psi)\cos\psi\,d\psi$$

$$h(-A\sin\psi) = -Ah_1\sin\psi + A^2h_2\sin^2\psi - A^3h_3\sin^3\psi.$$

PERFORMING THE CALCULATION WE OBTAIN

$$\dot{A} = -\frac{1}{2} A \left(h_1 + \frac{3}{4} h_3 A^2 \right)$$

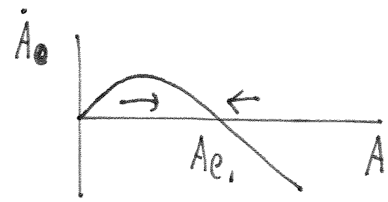
THERE ARE TWO EQUILIBRIUM SOLUTIONS:

$$A_{e_1} = 0 \longrightarrow u = 0 \quad \text{(MASS AT REST)}$$

$$A_{e_2} = \sqrt{\frac{-4 h_1}{3 h_3}} \longrightarrow \text{MASS MAKES PERIODIC OSCILLATIONS.}$$

NOTICE: $h_3 = \frac{1}{2} \left(\frac{\mu_s - \mu_m}{V_m^3} \right) > 0$ SINCE $\mu_s > \mu_m$

HENCE IF $h_1 < 0$ THEN



SETTING $h_1 = 0$ GIVES $2B - K_1 + 3 K_3 V_b^2 = 0$.

WITH $K_1 = \frac{3}{2} \left(\frac{\mu_s - \mu_m}{V_m} \right) \qquad K_3 = \frac{1}{2} \left(\frac{\mu_s - \mu_m}{V_m^3} \right)$

SOLVING FOR V_b WE OBTAIN THAT $h_1 < 0$ WHEN

$$V_b < \sqrt{\frac{K_1 - 2B}{3 K_3}} = \sqrt{\frac{\frac{3}{2}\left(\frac{\Delta\mu}{V_m}\right) - 2B}{\frac{3}{2}\, \Delta\mu / V_m^3}} = V_m \left[1 - \frac{4B}{3} \frac{V_m}{\Delta\mu} \right]^{1/2}$$

HENCE $u = 0$ IS UNSTABLE WHEN

$$V_b < V_m \left[1 - \frac{4B}{3} \frac{V_m}{(\mu_s - \mu_m)} \right]^{1/2} \equiv V_{b_1}$$

NOTICE:

M6

1) periodic solution if $h_1 < 0$ AND $h_3 > 0$ (always satisfied).

The amplitude is

$$A_e = \sqrt{\frac{-4h_1}{3h_3}} = 2\sqrt{Vb_1^2 - Vb^2}$$

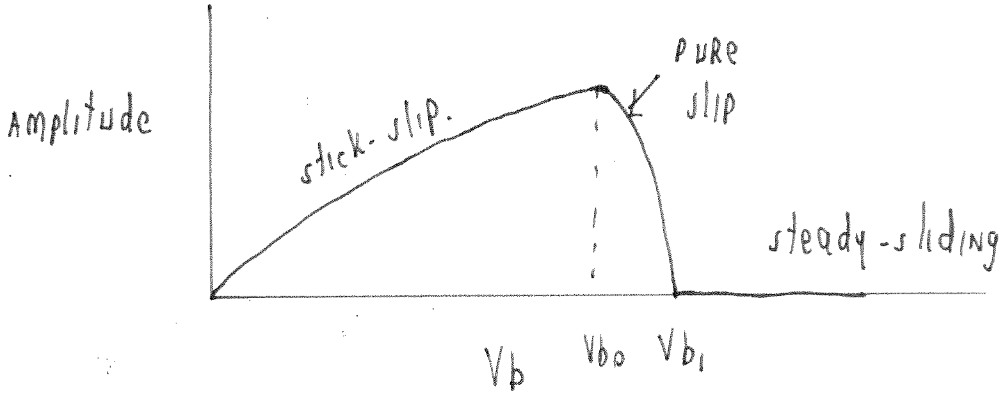
VALID FOR $Vb_0 < Vb < Vb_1$

HERE Vb_0 IS WHERE STICK FIRST OCCURS

SET $\mathrm{MAX}(\dot{u}) = A_e = Vb$. SOLVING FOR Vb_0 WE OBTAIN $2\sqrt{Vb_1^2 - Vb_0^2} = Vb_0 \longrightarrow$

$$Vb_0 = \sqrt{\frac{4}{5}}\, Vb_1$$

- pure-slip oscillation when $Vb_0 < Vb < Vb_1$
- $\dot{x} = \ddot{x} = 0$ is stable when $Vb > Vb_1$
- STICK-SLIP OCCURS WHEN $0 < Vb < Vb_0$.



STICK- SLIP PHASE (STICK $\dot{u} = V_b$)

ASSUME THAT AT $t = 0$ STICK- PHASE ENDS AND SLIP BEGINS.

HENCE,

$$u(0) = -2\beta V_b + K_1 V_b - K_3 V_b^3, \qquad \dot{u}(0) = V_b \quad (\text{SLIP starts})$$

$$\ddot{u} + u + \varepsilon h(\dot{u}) = 0.$$

WE SOLVE ON A FINITE TIME- INTERVAL WITH $\beta, K_1, K_3 = O(\varepsilon)$.

WE WRITE $\quad u = u_0(t) + \varepsilon u_1(t) + \ldots \quad 0 < t < t_{s_1}$

$$\ddot{u}_0 + u_0 = 0, \qquad u_0(0) = 0, \quad \dot{u}_0(0) = V_b$$

$$\ddot{u}_1 + u_1 = -h(\dot{u}_0), \qquad u_1(0) = -2\beta_0 V_b + K_{10} V_b - K_{30} V_b^3,$$

$$\dot{u}_1(0) = 0$$

$$\beta_0 = 2\beta/\varepsilon, \quad K_{10} = K_1/\varepsilon, \quad K_{30} = K_3/\varepsilon$$

HENCE $\quad u_0 = V_b \sin t$

$$u_1 = \frac{3}{2} C_3 - \frac{C_1 t}{2} \sin t + \left(\frac{55}{32} C_3 - C_1\right) \cos t - \frac{1}{2} C_3 \cos 2t + \frac{1}{32} C_3 \cos 3t$$

$$C_3 = K_{30} V_b^3, \qquad C_{10} = 2\beta_0 V_b - K_{10} V_b + \frac{15}{4} C_3.$$

$$u = u_0(t) + \varepsilon u_1(t) \qquad 0 < t < t_{s_1}$$

NOW STICK-STARTS WHEN

$$\dot{u}(t_{s_1}) = V_b \quad \text{FOR} \quad t_{s_1} > 0.$$

THIS GIVES $\quad t_{s_1} = 2\pi + O(\varepsilon^{1/2})$

STICK-PHASE

$$\dot{u} = V_b \quad \text{FOR} \quad t_{s_1} < t < T$$

SO $\quad u(t) = u(t_{s_1}) + V_b(t - t_{s_1}).$

THE END OF STICK IS WHEN $\quad u(T) = u(0)$

SO $$T = t_{s_1} + \frac{[u(0) - u(t_{s_1})]}{V_b}.$$