TAYLOR DISPERSION — FLOW IN A PIPE AND SPREAD OF A BLOB OF CHEMICAL

CONSIDER THE CONVECTION-DIFFUSION PROBLEM FOR $C = C(X, R, T)$

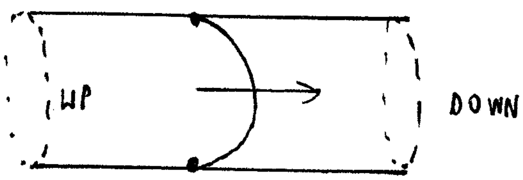
MODELED BY

$$C_T + \frac{\partial}{\partial X}(U C) = D\left[C_{XX} + \frac{1}{R}\partial_R (R C_R)\right], \qquad \text{IN } 0 < R < a$$

$$C_R = 0 \quad \text{ON} \quad R = 0, a, \quad -\infty < X < \infty$$

WE ASSUME $U = U(R) = 2U_0\left(1 - R^2/a^2\right)$ IS PARABOLIC POISEUILLE FLOW.

WE WISH TO DETERMINE WHAT HAPPENS TO AN INITIAL BLOB OF CHEMICAL IN THE STREAM AFTER A LONG TIME AND FAR DOWNSTREAM WHEN THE FLOW IS IN AN INFINITELY LONG CIRCULAR PIPE



X

WE NON-DIMENSIONALIZE: $x = X/L, \quad r = R/a, \quad U = U_0 u, \quad t = \dfrac{T}{(L/U_0)}$

THIS GIVES

$$\frac{U_0}{L} C_t + \frac{U_0}{L}\partial_x (u c) = \frac{D}{L^2} C_{xx} + \frac{D}{a^2}\frac{1}{r}\partial_r (r C_r)$$

WE WRITE AS

$$\left(\frac{U_0 a}{D}\right)\frac{a}{L}\left[C_t + \partial_x (u c)\right] = \frac{a^2}{L^2} C_{xx} + \frac{1}{r}\partial_r (r C_r)$$

WE DEFINE

$$\varepsilon = a/L \ll 1$$

$$P_e = \frac{U_0 a}{D} = O(1) \quad \text{Peclet NUMBER (DIMENSIONLESS)}$$

THIS YIELDS THAT

$$(1)\quad \begin{cases} \varepsilon P_e \left(C_t + \partial_x (U C) \right) = \varepsilon^2 C_{xx} + \dfrac{1}{r} (r C_r)_r & \text{IN } 0 < r < 1, \\ C_r = 0 \quad \text{ON } r = 0, 1 & \end{cases}$$

AND

$$U = 2 \left(1 - r^2 \right).$$

WE NOW LOOK FOR A MULTI-SCALE SOLUTION IN THE FORM

$$(3)\quad C = C_0 (x, t, \tau) + \varepsilon C_1 (x, t, \tau) + \varepsilon^2 C_2 (x, t, \tau)$$

WE WRITE (1) AS AN OPERATOR EQUATION

$$\mathcal{L}_1 C = - \varepsilon^2 C_{xx} + \varepsilon P_e \left(C_t + \partial_x (U C) \right)$$

WITH

$$\mathcal{L}_1 \phi \equiv \frac{1}{r} (r \phi_r)_r .$$

NOW UPON SUBSTITUTING WE OBTAIN THAT WITH $C_t = C_{0_t} + \varepsilon C_{0_\tau} + \dots$

$$(4)\quad \mathcal{L}_1 C_0 = 0 \qquad C_{0r} = 0 \text{ ON } r = 0, 1$$

$$(5)\quad \mathcal{L}_1 C_1 = + P_e \left[C_{0_t} + \partial_x (U C_0) \right] \qquad C_{1r} = 0 \text{ ON } r = 0, 1$$

$$(6)\quad \mathcal{L}_1 C_2 = + P_e \left[C_{1_t} + \partial_x (U C_1) \right] + P_e C_{0_\tau} - C_{0xx} \qquad C_{2r} = 0 \text{ ON } r = 0, 1$$

WE OBTAIN THAT

$$C_0 = C_0 (x, t, \tau)$$

AND SINCE $\mathcal{L}_1$ IS SELF-ADJOINT WITH $\mathcal{L}_1 [1] = 0$, I.E. 1 IS IN NULL-SPACE OF $\mathcal{L}_1$ WE OBTAIN THAT (5) IS SOLVABLE IFF

$$\left(P_e (C_{0_t} + \partial_x (U C_0)), 1 \right) = 0 \qquad \text{WHERE} \quad (U, V) = \int_0^1 U V r \, dr .$$

THUS YIELDS THAT

$$C_{0_t} \int_0^1 r\,dr + \partial_x \left[\left(\int_0^1 u\, r\, dr \right) C_0 \right] = 0.$$

WE LABEL $\langle u \rangle \equiv 2 \int_0^1 u\, r\, dr = 2 \int_0^1 2(1-r^2) r\, dr = 4\left[\frac{1}{2} - \frac{1}{4} \right] = 1.$

THUS ON THE t TIME-SCALE AND FAR DOWNSTREAM WE HAVE

$$(7) \qquad C_{0_t} + \langle u \rangle C_{0x} = 0 \qquad \text{WITH} \quad \langle u \rangle = 1.$$

THUS $C_0 = \mathcal{F}(x-t, \tau)$ WHICH IS DRIFT DOWNSTREAM WITH THE AVERAGE, i.e. A TRAVELLING WAVE MOVING TO THE RIGHT.

NEXT WE WANT TO MEASURE THE DEVIATION FROM THIS PURE DRIFT BEHAVIOR. WE SUBTRACT (5) AND (7) TO OBTAIN UPON SUBSTITUTING $C_{0_t} = -\langle u \rangle C_{0x}$, THAT

$$\mathcal{L}_1 C_1 = P_e \left[u - \langle u \rangle \right] C_{0x} \qquad (\text{NOTICE } \partial_x (u C_0) = u\, C_{0x})$$

WE DEFINE $\tilde{u} = u - \langle u \rangle$ TO BE THE VELOCITY NON-UNIFORMITY AND SO WE WRITE

$$(8) \quad \left\{ \begin{array}{l} \mathcal{L}_1 C_1 = P_e\, \tilde{u}(r)\, C_{0x} \\ C_{1r} = 0 \quad \text{ON} \quad r = 0, 1 \end{array} \right.$$

WHERE $\tilde{u} = u(r) - 1 = 2(1-r^2) - 1 = 1 - 2r^2$, SATISFIES $\langle \tilde{u} \rangle = 0$.

NOW TO SOLVE (8) WE WRITE

$$(9) \qquad C_1 = P_e\, C_{0x}\, V_1(r)$$

WHERE V_1 SOLVES THE "CELL PROBLEM", WHICH IS EASILY SOLVED:

$$(10) \quad \begin{cases} \mathcal{L}_1 V_1 = \hat{u}(r) = 1 - 2r^2 \\ V_{1r} = 0 \quad \text{ON} \quad r = 0, 1 \end{cases} \qquad \text{SO} \quad V_1 = \frac{r^2}{4} - \frac{r^4}{8} + \alpha \qquad (\alpha \text{ ANY CONSTANT})$$

NOW WITH V_1 KNOWN UP TO A CONSTANT, I.E. UP TO A MULTIPLE OF THE ELEMENT IN $\mathcal{N}(\mathcal{L}_1)$ WE PROCEED TO THE EQUATION FOR C_2.

WE HAVE

$$(11) \quad \mathcal{L}_1 C_2 = P_e \left[C_{1t} + u\, C_{1x} \right] + P_e C_{0\tau} - C_{0xx},$$

WITH

$$C_1 = P_e\, C_{0x}\, V_1 \qquad \text{AND} \qquad C_{0t} + \langle u \rangle C_{0x} = 0.$$

NOW WE CALCULATE NOTING THAT V_1 IS INDEPENDENT OF x AND t:

$$\begin{aligned} C_{1t} + u\, C_{1x} &= P_e V_1\, \partial_t C_{0x} + u P_e V_1 C_{0xx} \\ &= P_e V_1\, \partial_x \left[C_{0t} + u\, C_{0x} \right] \\ &= P_e V_1\, \partial_x \left[- C_{0x} \langle u \rangle + u\, C_{0x} \right] \\ &= P_e V_1\, C_{0xx} \left[u - \langle u \rangle \right] \end{aligned}$$

SO

$$C_{1t} + u\, C_{1x} = P_e V_1 C_{0xx}\, \tilde{u} \qquad \text{WITH} \qquad \hat{u} = u - \langle u \rangle.$$

SUBSTITUTING INTO (11) YIELDS THAT

$$(12) \quad \begin{cases} \mathcal{L}_1 C_2 = P_e^2 V_1 \tilde{u}\, C_{0xx} + P_e C_{0\tau} - C_{0xx}. \\ C_{2r} = 0 \quad \text{ON} \quad r = 0, 1 \end{cases}$$

WHERE $V_1 = V_1(r)$, $\hat{u} = \hat{u}(r)$.

FINALLY, THE SOLVABILITY CONDITION FOR (12) IS THAT

$$P_e^2 \, C_{0xx} (V_1 \tilde{U}, 1) + (P_e C_{0\tau}, 1) - (C_{0xx}, 1) = 0$$

WHERE $(U, V) \equiv \int_0^1 r \, U \, V \, r \, dr$. THIS YIELDS,

$$P_e \, C_{0\tau} \, (1,1) = C_{0xx} (1,1) - P_e^2 \, C_{0xx} \left(V_1 \hat{U}, 1 \right)$$

DIVIDING BY $(1,1) = \int_0^1 r \, dr = \frac{1}{2}$, WE GET

$$P_e \, C_{0\tau} = C_{0xx} - P_e^2 \frac{(V_1 \hat{U}, 1)}{(1,1)}$$

WE DEFINE THE SPATIAL AVERAGE $\langle V_1 \hat{U}, 1 \rangle = (V_1 \hat{U}, 1)/(1,1)$, WHICH IS

$$\langle V_1 \tilde{U}, 1 \rangle \equiv \frac{\int_0^1 r V_1 \tilde{U} \, dr}{\int_0^1 r \, dr} = 2 \int_0^1 r \, V_1 \tilde{U} \, dr .$$

THIS GIVES $$P_e \, C_{0\tau} = C_{0xx} \left(1 - P_e^2 \langle \tilde{U} V_1 , 1 \rangle \right) \Big\} \qquad (13)$$

NOW WE CALCULATE $\langle \hat{U} V_1 \rangle$. WE HAVE FROM (10) THAT

$V_1 = \frac{r^2}{4} - \frac{r^4}{8} + \alpha$ AND $\tilde{U} = U - \langle U \rangle = 1 - 2r^2$.

THE CHOICE OF α IS IRRELEVANT SINCE $\langle \hat{U} \rangle = 0$. WE CALCULATE

$$\langle \hat{U} V_1 , 1 \rangle = \langle \tilde{U} \left(\frac{r^2}{4} - \frac{r^4}{8} \right) + \alpha \tilde{U} \rangle = \langle \tilde{U} \left(\frac{r^2}{4} - \frac{r^4}{8} \right) \rangle \quad \text{SINCE } \langle \tilde{U} \rangle = 0.$$

$$= 2 \int_0^1 (1 - 2r^2) \left(\frac{r^2}{4} - \frac{r^4}{8} \right) r \, dr = \frac{1}{4} \int_0^1 (r - 2r^3)(2r^2 - r^4) \, dr$$

$$= \frac{1}{4} \int_0^1 \left[2r^2 - r^5 - 4r^5 + 2r^7 \right] dr$$

$$= \frac{1}{4} \left[\frac{2r^3}{3} \Big|_0^1 - \frac{r^6}{6} \Big|_0^1 - \frac{4r^6}{6} \Big|_0^1 + \frac{2r^8}{8} \Big|_0^1 \right]$$

THIS GIVES

$$\langle \hat{u}, v_1 \rangle = \frac{1}{4}\left(\frac{2}{2} - \frac{1}{6} - \frac{2}{3} + \frac{1}{4}\right) = -\frac{1}{48}$$

IN THIS WAY WE OBTAIN THAT (13) BECOMES

$$P_e\, C_{0\tau} = C_{0xx}\left(1 + \frac{P_e^2}{48}\right) \qquad (14)$$

SO WE HAVE DIFFUSION OVER A LONG TIME-SCALE.

TO EXPRESS OUR FINAL RESULT WE USE

$$\left.\begin{aligned} &P_e\left(C_{0t} + \langle u \rangle C_{0x}\right) = 0 \\ \text{AND} \quad &\varepsilon P_e\, C_{0\tau} = \varepsilon\, C_{0xx}\left(1 + P_e^2/48\right) \end{aligned}\right\} \quad (*)$$

AND ADD TOGETHER TO OBTAIN

$$P_e\left(C_{0t} + \varepsilon\, C_{0\tau}\right) + P_e \langle u \rangle C_{0x} = \varepsilon\, C_{0xx}\left(1 + P_e^2/48\right)$$

SO WE HAVE OUR APPROXIMATION $C_0'(x,t) = C_0(x, t, \tau)$

SATISFIES

$$P_e\, C'_{0t} + P_e \langle u \rangle C'_{0x} = \varepsilon\, C'_{0xx}\left(1 + P_e^2/48\right)$$

NOW DIVIDE BY PECLET NUMBER $P_e = U_0\, a/D$ TO OBTAIN

$$C'_{0t} + \langle u \rangle C'_{0x} = \frac{\varepsilon}{P_e}\, C'_{0xx}\left(1 + P_e^2/48\right).$$

NOW CONVERT TO ORIGINAL VARIABLES THAT WERE DIMENSIONAL.

$$T = \frac{U_0}{L} t, \qquad x = \frac{X}{L}, \qquad U = U_0\, u, \qquad \varepsilon = a/L$$

THIS GIVES

$$\frac{L}{U_0} C_{0T} + \frac{L}{U_0} \langle U \rangle C_{0X} = \left(\frac{a}{L}\right) \frac{L^2}{\left(U_0 a / D\right)} C_{0XX} \left(1 + \frac{P_e^2}{48}\right)$$

THIS YIELDS THAT OVER LONG-SCALES

$$C_{0T} + \langle U \rangle C_{0X} = D\, C_{0XX} \left(1 + \frac{P_e^2}{48}\right).$$

NOW WE DEFINE THE <u>EFFECTIVE</u> DIFFUSIVITY

$$D_{EFF} = D\left(1 + \frac{P_e^2}{48}\right) = D\left(1 + \frac{U_0^2 a^2}{48 D^2}\right) = D + \underbrace{\frac{U_0^2 a^2}{48 D}}_{\text{"EXTRA"}}$$

NOTICE THAT THE EXTRA TERM ~~SEEM~~ INCREASES AS D DECREASES. AS D DECREASES THE EFFECT OF THE FLOW IS MORE ONE DIMENSIONAL DOWN THE AXIS OF THE PIPE WHICH ENHANCES THE EFFECTIVE DIFFUSIVITY, STRETCHING OUT THE INITIAL BLOB OF DYE. THE EFFECT IS MORE PRONOUNCED AT HIGHER PECLECT NUMBERS

FOR SALT IN WATER $D = 1 \times 10^{-5}\ cm^2/sec$, $U_0 = \frac{1}{2}\ cm/sec$ $a = 0.2\ cm$

SO $\dfrac{U_0^2 a^2}{48 D} = 21\ cm^2/sec$ WHICH IS A LARGE EFFECT.

<u>REMARK</u> THE ONLY UNSATISFYING PART OF THE DERIVATION IS ADDING THE TWO EQUATIONS IN (*). IF ONE INSTEAD REPEATS THIS ANALYSIS BUT IN A MOVING REFERENCE FRAME WITH

$$z = x - \langle u \rangle t$$

AND LOOK FOR A MULTI-SCALE SOLUTION $C = C_0(z, \tau)$ ONE READILY CAN DERIVE (14).