Chapter 3

Basic Floquet Theory

3.1 General Results

If we have a problem of the form

$$\mathbf{x}' = \mathbf{A}(t)\mathbf{x} \tag{3.1}$$

where $\mathbf{A}(t)$ is periodic with period T, then $\mathbf{x}$ need not be periodic, however it must be of the form

$$e^{\mu t}\mathbf{p}(t) \tag{3.2}$$

where $\mathbf{p}(t)$ has period T. Additionally, it has n such μ_j and together they satisfy

$$e^{\mu_1 T}e^{\mu_2 T}\cdots e^{\mu_n T} = \exp\left(\int_0^T \operatorname{tr}\left(\mathbf{A}(s)\right)\, ds\right). \tag{3.3}$$

The following theorems prove those results. We follow Ward [28].

Definition (Fundamental Matrix). Let $\mathbf{x}^1(t),\ldots,\mathbf{x}^n(t)$ be n solutions of $\mathbf{x}' = \mathbf{A}(t)\mathbf{x}$. Let

$$\mathbf{X}(t) = \left[\begin{bmatrix}\mathbf{x}^1\end{bmatrix}\cdots\begin{bmatrix}\mathbf{x}^n\end{bmatrix}\right] \tag{3.4}$$

so that $\mathbf{X}(t)$ is an $n\times n$ matrix solution of $\mathbf{X}' = \mathbf{A}\mathbf{X}$.

If $\mathbf{x}^1(t),\ldots,\mathbf{x}^n(t)$ are linearly independent, then $\mathbf{X}(t)$ is non-singular and is called a *fundamental matrix*. If $\mathbf{X}(t_0) = \mathbf{I}$, then $\mathbf{X}(t)$ is the *principal fundamental matrix*.

Lemma 3.1. *If $\mathbf{X}(t)$ is a fundamental matrix then so is $\mathbf{Y}(t) = \mathbf{X}(t)\mathbf{B}$ for any non-singular constant matrix $\mathbf{B}$.*

Proof. Since $\mathbf{X}(t)$ and $\mathbf{B}$ are non-singular then the inverse of $\mathbf{Y}(t)$ is $\mathbf{B}^{-1}\mathbf{X}^{-1}(t)$ and so $\mathbf{Y}(t)$ is non-singular. Also,

$$\mathbf{Y}' = \mathbf{X}'\mathbf{B} = \mathbf{A}\mathbf{X}\mathbf{B} = \mathbf{A}\mathbf{Y} \tag{3.5}$$

so that $\mathbf{Y}'(t) = \mathbf{A}\mathbf{Y}(t)$. □

Lemma 3.2. *Let the* Wronskian $W(t)$ *of* $\mathbf{X}(t)$ *be the determinant of* $\mathbf{X}(t)$. *Then*

$$W(t) = W(t_0)\exp\left(\int_{t_0}^{t} \operatorname{tr}(\mathbf{A}(s))\, ds\right). \tag{3.6}$$

Proof. Let t_0 be some time. Expanding in a Taylor series,

$$\mathbf{X}(t) = \mathbf{X}(t_0) + (t - t_0)\,\mathbf{X}'(t_0) + O\left((t-t_0)^2\right) \tag{3.7}$$

$$= \mathbf{X}(t_0) + (t - t_0)\,\mathbf{A}(t_0)\mathbf{X}(t_0) + O\left((t-t_0)^2\right) \tag{3.8}$$

$$= [\mathbf{I} + (t - t_0)\,\mathbf{A}(t_0)]\,\mathbf{X}(t_0) + O\left((t-t_0)^2\right) \tag{3.9}$$

so that

$$\det(\mathbf{X}(t)) = \det[\mathbf{I} + (t - t_0)\,\mathbf{A}(t_0)]\det(\mathbf{X}(t_0)) \tag{3.10}$$

$$W(t) = \det[\mathbf{I} + (t - t_0)\,\mathbf{A}(t_0)]\,W(t_0). \tag{3.11}$$

Now since

$$\det(\mathbf{I} + \epsilon\mathbf{C}) = 1 + \epsilon\operatorname{tr}(\mathbf{C}) + O\left(\epsilon^2\right), \tag{3.12}$$

we have that

$$W(t) = W(t_0)\left(1 + (t - t_0)\operatorname{tr}(\mathbf{A}(t_0))\right). \tag{3.13}$$

Now by expanding $W(t)$ in a Taylor series, we obtain that

$$W(t) = W(t_0) + (t - t_0)\,W'(t_0) + O\left((t-t_0)^2\right) \tag{3.14}$$

so that

$$W'(t_0) = W(t_0)\operatorname{tr}(\mathbf{A}(t_0)). \tag{3.15}$$

Since we have not made any assumptions about t_0, we can the write

$$W'(t) = W(t)\operatorname{tr}(\mathbf{A}(t)). \tag{3.16}$$

We know that the solution to this equation is

$$W(t) = W(t_0)\exp\left(\int_{t_0}^{t} \operatorname{tr}(\mathbf{A}(s))\, ds\right) \tag{3.17}$$

□

Theorem 3.3. *Let* $\mathbf{A}(t)$ *be a* T*-periodic matrix. If* $\mathbf{X}(t)$ *is a fundamental matrix then so is* $\mathbf{X}(t+T)$ *and there exists a non-singular constant matrix* $\mathbf{B}$ *such that*

i. $\mathbf{X}(t+T) = \mathbf{X}(t)\mathbf{B}$ *for all* t

ii. $\det(\mathbf{B}) = \exp\left(\int_0^T \operatorname{tr}(\mathbf{A}(s))\, ds\right)$

Proof. Begin by showing that $\mathbf{X}(t+T)$ is also a fundamental matrix. Let $\mathbf{Y}(t) = \mathbf{X}(t+T)$. Then

$$\mathbf{Y}'(t) = \mathbf{X}'(t+T) = \mathbf{A}(t+T)\mathbf{X}(t+T) = \mathbf{A}(t)\mathbf{X}(t+T) = \mathbf{A}(t)\mathbf{Y}(t) \quad (3.18)$$

and so $\mathbf{X}(t+T)$ is a fundamental matrix.

i. Let $\mathbf{B}(t) = \mathbf{X}^{-1}(t)\mathbf{Y}(t)$. Then

$$\mathbf{Y}(t) = \mathbf{X}(t)\mathbf{X}^{-1}(t)\mathbf{Y}(t) \quad (3.19)$$

$$= \mathbf{X}(t)\mathbf{B}(t) \quad (3.20)$$

Let $\mathbf{B}_0 = \mathbf{B}(t_0)$. We know by lemma 3.1 that $\mathbf{Y}_0(t) = \mathbf{X}(t)\mathbf{B}_0$ is a fundamental matrix, where, by definition, $\mathbf{Y_0}(t_0) = \mathbf{Y}(t_0)$. Since these are both solutions to $\mathbf{X}' = \mathbf{AX}$, by the uniqueness of the solution, we must then have $\mathbf{Y}_0(t) = \mathbf{Y}(t)$ for all time. As a result, $\mathbf{B}_0 = \mathbf{B}(t)$ and so $\mathbf{B}$ is time-independent.

ii. From lemma 3.2, we have that

$$W(t) = W(t_0)\exp\left(\int_{t_0}^{t} \operatorname{tr}(\mathbf{A}(s))\, ds\right) \quad (3.21)$$

$$W(t+T) = W(t_0)\exp\left(\int_{t_0}^{t} \operatorname{tr}(\mathbf{A}(s))\, ds + \int_{t}^{t+T} \operatorname{tr}(\mathbf{A}(s))\, ds\right) \quad (3.22)$$

$$W(t+T) = W(t)\exp\left(\int_{t}^{t+T} \operatorname{tr}(\mathbf{A}(s))\, ds\right) \quad (3.23)$$

$$W(t+T) = W(t)\exp\left(\int_{0}^{T} \operatorname{tr}(\mathbf{A}(s))\, ds\right). \quad (3.24)$$

We also know that

$$\mathbf{X}(t+T) = \mathbf{X}(t)\mathbf{B} \quad (3.25)$$

$$\det(\mathbf{X}(t+T)) = \det(\mathbf{X}(t))\det(\mathbf{B}) \quad (3.26)$$

$$W(t+T) = W(t)\det(\mathbf{B}) \quad (3.27)$$

and so

$$\det(\mathbf{B}) = \exp\left(\int_{0}^{T} \operatorname{tr}(\mathbf{A}(s))\, ds\right) \quad (3.28)$$

□

Remark. Since $\mathbf{B}$ is time-independent, it can be computed by setting $t = 0$, so that $\mathbf{B} = \mathbf{X}^{-1}(0)\mathbf{X}(T)$. If we took the initial conditions $\mathbf{X}(0) = \mathbf{I}$, then $\mathbf{B} = \mathbf{X}(T)$.

Definition (Characteristic Multipliers and Exponents)**.** The eigenvalues $\rho_1, \ldots, \rho_n$ of $\mathbf{B}$ are called the *characteristic multipliers* for $\mathbf{X}'(t) = \mathbf{A}(t)\mathbf{X}(t)$. The *characteristic exponents* or *Floquet exponents* are $\mu_1, \ldots, \mu_n$ satisfying

$$\rho_1 = e^{\mu_1 T}, \qquad \rho_2 = e^{\mu_2 T}, \qquad \ldots \qquad \rho_n = e^{\mu_n T}. \quad (3.29)$$

Note that μ_j for $j \in \mathbb{N}$ may be complex.

Properties.

i. The characteristic multipliers (eigenvalues) $\rho_1, \ldots, \rho_n$ of $\mathbf{B} = \mathbf{X}(T)$ with $\mathbf{X}(0) = \mathbf{I}$ satisfy

$$\det(\mathbf{B}) = \rho_1 \rho_2 \cdots \rho_n = \exp\left(\int_0^T \operatorname{tr}(\mathbf{A}(s))\, ds\right). \tag{3.30}$$

This follows from theorem 3.3ii.

ii. Since the trace is the sum of the eigenvalues, we also have

$$\operatorname{tr}(\mathbf{B}) = \rho_1 + \rho_2 + \cdots + \rho_n. \tag{3.31}$$

iii. The characteristic exponents are not unique since if $\rho_j = e^{\mu_j T}$, then $\rho_j = e^{(\mu_j + 2\pi i/T)T}$.

iv. The characteristic multipliers ρ_j are an intrinsic property of the equation $\mathbf{X}'(t) = \mathbf{A}\mathbf{X}$ and do not depend on the choice of the fundamental matrix.

Proof. Suppose $\hat{\mathbf{X}}(t)$ is another fundamental matrix. Then

$$\hat{\mathbf{X}}(t+T) = \hat{\mathbf{X}}(t)\hat{\mathbf{B}}. \tag{3.32}$$

We have showed in the proof of theorem 3.3 that since $\mathbf{X}(t)$ and $\hat{\mathbf{X}}(t)$ are fundamental matrices then there is a constant non-singular matrix $\mathbf{C}$ such that

$$\hat{\mathbf{X}}(t) = \mathbf{X}(t)\mathbf{C} \tag{3.33}$$

so that

$$\hat{\mathbf{X}}(t+T) = \mathbf{X}(t+T)\mathbf{C} \tag{3.34}$$

$$\left(\hat{\mathbf{X}}(t)\hat{\mathbf{B}}\right) = (\mathbf{X}(t)\mathbf{B})\,\mathbf{C} \tag{3.35}$$

$$\mathbf{X}(t)\mathbf{C}\hat{\mathbf{B}} = \mathbf{X}(t)\mathbf{B}\mathbf{C} \tag{3.36}$$

$$\mathbf{C}\hat{\mathbf{B}} = \mathbf{B}\mathbf{C} \tag{3.37}$$

$$\mathbf{C}\hat{\mathbf{B}}\mathbf{C}^{-1} = \mathbf{B} \tag{3.38}$$

so the eigenvalues of $\mathbf{B}$ and $\hat{\mathbf{B}}$ are the same. □

Theorem 3.4. *Let ρ be a characteristic multiplier and let μ be the corresponding characteristic exponent so that $\rho = e^{\mu T}$. Then there exists a solution $\mathbf{x}(t)$ of $\mathbf{x}' = \mathbf{A}(t)\mathbf{x}$ such that*

i. $\mathbf{x}(t+T) = \rho\mathbf{x}(t)$

ii. There exists a periodic solution $\mathbf{p}(t)$ with period T such that $\mathbf{x}(t) = e^{\mu t}\mathbf{p}(t)$.

Proof.

i. Let $\mathbf{b}$ be an eigenvector of $\mathbf{B}$ corresponding to eigenvalue ρ. Let $\mathbf{x}(t) = \mathbf{X}(t)\mathbf{b}$. Then $\mathbf{x}' = \mathbf{A}\mathbf{x}$ and

$$\begin{aligned}
\mathbf{x}(t+T) &= \mathbf{X}(t+T)\mathbf{b} && (3.39)\\
&= \mathbf{X}(t)\mathbf{B}\mathbf{b} && (3.40)\\
&= \rho\mathbf{X}(t)\mathbf{b} && (3.41)\\
&= \rho\mathbf{x}(t) && (3.42)
\end{aligned}$$

so that $\mathbf{x}(t+T) = \rho\mathbf{x}(t)$.

ii. Let $\mathbf{p}(t) = \mathbf{x}(t)e^{-\mu t}$. We now need to show that $\mathbf{p}(t)$ is T-periodic.

$$\begin{aligned}
\mathbf{p}(t+T) &= \mathbf{x}(t+T)e^{-\mu(t+T)} && (3.43)\\
&= \rho\mathbf{x}(t)e^{-\mu(t+T)} && (3.44)\\
&= \frac{\rho}{e^{\mu T}}\mathbf{x}(t)e^{-\mu t} && (3.45)\\
&= \mathbf{x}(t)e^{-\mu t} && (3.46)\\
&= \mathbf{p}(t) && (3.47)
\end{aligned}$$

As a result, we have a solution of the form $\mathbf{x}(t) = e^{\mu t}\mathbf{p}(t)$ where $\mathbf{p}(t)$ is periodic with period T. □

Remarks.

i. If μ is replaced by $\mu + 2\pi i/T$, then we get

$$\mathbf{x}(t) = e^{\mu t}\mathbf{p}(t)e^{2\pi i t/T} \tag{3.48}$$

where $\mathbf{p}(t)e^{2\pi i t/T}$ is still periodic with period T. As a result, the fact that μ is not unique does not alter our results.

ii. We have that

$$\begin{aligned}
\mathbf{x}_j(t+T) &= \rho_j\mathbf{x}_j(t) && (3.49)\\
\mathbf{x}_j(t+NT) &= \rho_j^N\mathbf{x}_j(t). && (3.50)
\end{aligned}$$

Each characteristic multipliers falls into one of the following categories:

(a) If $|\rho| < 1$, then $\mathrm{Re}(\mu) < 0$ and so $\mathbf{x}(t) \xrightarrow{t\to\infty} 0$.

(b) If $|\rho| = 1$, then $\mathrm{Re}(\mu) = 0$ and so we have a pseudo-periodic solution. If $\rho = \pm 1$, then the solution is periodic with period T.

(c) If $|\rho| > 1$, then $\mathrm{Re}(\mu) > 0$ and so $\mathbf{x}(t) \rightsquigarrow \infty$ as $t \to \infty$.

The entire solution is stable if all the characteristic multipliers satisfy $|\rho_j| \leq 1$.

iii. As for the general solution, suppose that $\mathbf{b}_1, \ldots, \mathbf{b}_n$ are n linearly independent eigenvectors of $\mathbf{B}$ corresponding to distinct eigenvalues $\rho_1, \ldots, \rho_n$. Then there are n linearly independent solutions to $\mathbf{x}' = \mathbf{A}\mathbf{x}$, which by the above theorem are given by

$$\mathbf{x}_j(t) = e^{\mu_j t}\mathbf{p}_j(t) \tag{3.51}$$

where $\mathbf{p}_j(t)$ is T-periodic. As a result, we can define

$$\mathbf{X}_0(t) = \left[\begin{bmatrix}\\ \mathbf{x}_1 \\ \\ \end{bmatrix} \cdots \begin{bmatrix}\\ \mathbf{x}_n \\ \\ \end{bmatrix}\right], \quad \mathbf{P}_0(t) = \left[\begin{bmatrix}\\ \mathbf{p}_1 \\ \\ \end{bmatrix} \cdots \begin{bmatrix}\\ \mathbf{p}_n \\ \\ \end{bmatrix}\right], \tag{3.52}$$

$$\mathbf{D}_0(t) = \begin{bmatrix} \mu_1 & & 0 \\ & \ddots & \\ 0 & & \mu_n \end{bmatrix}, \quad \mathbf{Y}_0(t) = \begin{bmatrix} e^{\mu_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\mu_n t} \end{bmatrix}, \tag{3.53}$$

such that

$$\mathbf{X}_0 = \mathbf{P}_0\mathbf{Y}_0, \qquad \mathbf{Y}_0' = \mathbf{D}_0\mathbf{Y}_0. \tag{3.54}$$

iv. Now consider what happens if $\rho < 0$. Suppose $\rho < 0$ real, so that we can write

$$\rho = e^{(\nu + i\pi/T)T} \tag{3.55}$$

where

$$\rho = -e^{\nu T}. \tag{3.56}$$

Then we obtain

$$\mathbf{x}(t) = e^{\mu t}\mathbf{p}(t) \tag{3.57}$$

$$= e^{\nu t}e^{i\pi t/T}\mathbf{p}(t) \tag{3.58}$$

$$= e^{\nu t}\mathbf{q}(t), \tag{3.59}$$

where $\mathbf{q}(t)$ has period T since $\mathbf{p}(t)$ has period T. Since we can choose $\mathbf{x}$ to be real, without loss of generality, we can also choose $\mathbf{q}$ to be real. For the general solution, if $\rho_j < 0$, we can replace $\mathbf{p}_j$ with $\mathbf{q}_j$ and μ_j with ν_j so that

$$\mathbf{P}_0 = \left[\begin{bmatrix}\\ \mathbf{p}_1 \\ \\ \end{bmatrix} \cdots \begin{bmatrix}\\ \mathbf{q}_j \\ \\ \end{bmatrix} \cdots \begin{bmatrix}\\ \mathbf{p}_n \\ \\ \end{bmatrix}\right], \quad \mathbf{Y}_0 = \begin{bmatrix} e^{\mu_1 T} & & & & 0 \\ & \ddots & & & \\ & & e^{\nu_j T} & & \\ & & & \ddots & \\ 0 & & & & e^{\mu_n T} \end{bmatrix} \tag{3.60}$$

and

$$\mathbf{X}_0(t) = \mathbf{P}_0(t)\mathbf{Y}_0(t). \tag{3.61}$$

v. Suppose now that ρ is complex. Then since ρ is an eigenvalue of the real matrix $\mathbf{B}$, $\overline{\rho}$ is as well. The characteristic exponents are μ and $\overline{\mu}$. Let

$$\mu = \nu + i\sigma, \qquad \mathbf{p}(t) = \mathbf{q}(t) + i\mathbf{r}(t) \tag{3.62}$$

where $\mathbf{q}(t)$ and $\mathbf{r}(t)$ must both have period T since $\mathbf{p}(t)$ does. Since $\mathbf{x}(t) = e^{\mu t}\mathbf{p}(t)$ is a solution to $\mathbf{x}' = \mathbf{A}(t)\mathbf{x}$, then by taking the complex conjugate, so is $\overline{\mathbf{x}}(t) = e^{\overline{\mu}t}\overline{\mathbf{p}}(t)$. We can write these as

$$\mathbf{x}(t) = e^{(\nu+i\sigma)t}\left(\mathbf{q}(t) + i\mathbf{r}(t)\right) \tag{3.63}$$
$$= e^{\nu t}\left[\left(\mathbf{q}\cos(\sigma t) - \mathbf{r}\sin(\sigma t)\right) + i\left(\mathbf{r}\cos(\sigma t) + \mathbf{q}\sin(\sigma t)\right)\right] \tag{3.64}$$

and

$$\overline{\mathbf{x}}(t) = e^{(\nu-i\sigma)t}\left(\mathbf{q}(t) - i\mathbf{r}(t)\right) \tag{3.65}$$
$$= e^{\nu t}\left[\left(\mathbf{q}\cos(\sigma t) - \mathbf{r}\sin(\sigma t)\right) - i\left(\mathbf{r}\cos(\sigma t) + \mathbf{q}\sin(\sigma t)\right)\right]. \tag{3.66}$$

We can alternately write the linearly independent real solutions

$$\mathbf{x}_R = \mathrm{Re}\left[e^{\mu t}\mathbf{p}(t)\right] = e^{\nu t}\left[\cos(\sigma t)\,\mathbf{q}(t) - \sin(\sigma t)\,\mathbf{r}(t)\right], \tag{3.67}$$
$$\mathbf{x}_I = \mathrm{Im}\left[e^{\mu t}\mathbf{p}(t)\right] = e^{\nu t}\left[\sin(\sigma t)\,\mathbf{q}(t) + \cos(\sigma t)\,\mathbf{r}(t)\right], \tag{3.68}$$

so that

$$\mathbf{X}_0 = \left[\begin{bmatrix} \\ \mathbf{x}_1 \\ \\ \end{bmatrix} \cdots \begin{bmatrix} \\ \mathbf{x}_R \\ \\ \end{bmatrix} \begin{bmatrix} \\ \mathbf{x}_I \\ \\ \end{bmatrix} \cdots \begin{bmatrix} \\ \mathbf{x}_n \\ \\ \end{bmatrix}\right], \tag{3.69}$$

$$\mathbf{P}_0 = \left[\begin{bmatrix} \\ \mathbf{p}_1 \\ \\ \end{bmatrix} \cdots \begin{bmatrix} \\ \mathbf{q} \\ \\ \end{bmatrix} \begin{bmatrix} \\ \mathbf{r} \\ \\ \end{bmatrix} \cdots \begin{bmatrix} \\ \mathbf{p}_n \\ \\ \end{bmatrix}\right], \tag{3.70}$$

$$\mathbf{Y}_0 = \begin{bmatrix} e^{\mu_1 T} & & & & & 0 \\ & \ddots & & & & \\ & & e^{\nu t}\cos(\sigma t) & e^{\nu t}\sin(\sigma t) & & \\ & & -e^{\nu t}\sin(\sigma t) & e^{\nu t}\cos(\sigma t) & & \\ & & & & \ddots & \\ 0 & & & & & e^{\mu_n T} \end{bmatrix} \tag{3.71}$$

and

$$\mathbf{X}_0(t) = \mathbf{P}_0(t)\mathbf{Y}_0(t). \tag{3.72}$$

3.1.1 Example

For example, consider

$$x_1' = \left(1 + \frac{\cos(t)}{2 + \sin(t)}\right)x_1 \tag{3.73}$$
$$x_2' = x_1 - x_2. \tag{3.74}$$

Here, we know that the solution is in general

$$x_1 = c_1 e^t \left(2 + \sin(t)\right) \tag{3.75}$$

$$x_2 = c_1 e^t \left(2 + \frac{1}{2}\sin(t) - \frac{1}{2}\cos(t)\right) + c_2 e^{-t} \tag{3.76}$$

which we can write as

$$\mathbf{x} = c_1 e^t \begin{bmatrix} 2 + \sin(t) \\ 2 + \frac{1}{2}\sin(t) - \frac{1}{2}\cos(t) \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{3.77}$$

Using all the above definitions, the fundamental matrix is

$$\mathbf{X}(t) = \begin{bmatrix} e^t \left(2 + \sin(t)\right) & 0 \\ e^t \left(2 + \frac{1}{2}\sin(t) - \frac{1}{2}\cos(t)\right) & e^{-t} \end{bmatrix} \tag{3.78}$$

so that

$$\mathbf{B} = \mathbf{X}^{-1}(0)\mathbf{X}(2\pi) \tag{3.79}$$

$$= \begin{bmatrix} 2 & 0 \\ \frac{3}{2} & 1 \end{bmatrix}^{-1} \begin{bmatrix} 2e^{2\pi} & 0 \\ \frac{3}{2}e^{2\pi} & e^{-2\pi} \end{bmatrix} \tag{3.80}$$

$$= \frac{1}{2} \begin{bmatrix} 1 & 0 \\ -\frac{3}{2} & 2 \end{bmatrix} \begin{bmatrix} 2e^{2\pi} & 0 \\ \frac{3}{2}e^{2\pi} & e^{-2\pi} \end{bmatrix} \tag{3.81}$$

$$= \begin{bmatrix} e^{2\pi} & 0 \\ 0 & e^{-2\pi} \end{bmatrix} \tag{3.82}$$

As a result $\rho_1 = e^{2\pi}$, $\rho_2 = e^{-2\pi}$ and so $\mu_1 = 1$ and $\mu_2 = -1$. Theorem 3.4 then tells us that there is a solution of the form

$$\mathbf{x}_1(t) = e^t \mathbf{p}_1(t), \qquad \mathbf{x}_2(t) = e^{-t}\mathbf{p}_2(t) \tag{3.83}$$

where $\mathbf{p}_1(t)$ and $\mathbf{p}_2(t)$ are periodic with period 2π. We know that in fact

$$\mathbf{p}_1(t) = \begin{bmatrix} 2 + \sin(t) \\ 2 + \frac{1}{2}\sin(t) - \frac{1}{2}\cos(t) \end{bmatrix}, \qquad \mathbf{p}_2(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{3.84}$$

3.1.2 Periodic Solution

Consider a problem of the form $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ with $\mathbf{x} \in \mathbb{R}^n$ where there is a periodic solution $\mathbf{x}(t) = \phi(t)$ with period T. Linearise the solution about ϕ by writing $\mathbf{x} = \phi + \mathbf{v}$. We then obtain

$$\mathbf{v}' = \mathbf{A}(t)\mathbf{v} \tag{3.85}$$

where $\mathbf{A}(t)$ is the Jacobian of f (so $A_{ij}(t) = \left.\frac{\partial f_i}{\partial x_j}\right|_{\phi(t)}$). Since $\phi(t)$ has period T, so does $\mathbf{A}(t)$. Now let $\mathbf{X}(t)$ be the principal fundamental matrix of $\mathbf{v}' = \mathbf{A}\mathbf{v}$ (so that $\mathbf{X}(0) = \mathbf{I}$). Then $\mathbf{B} = \mathbf{X}(T)$.

Now by definition,

$$\phi'(t) = \mathbf{f}(\phi(t)) \tag{3.86}$$

so

$$\phi''(t) = \left.\frac{\partial f_i}{\partial x_j}\right|_{\phi(t)} \phi'(t) \tag{3.87}$$

$$\phi''(t) = \mathbf{A}(t)\phi'(t) \tag{3.88}$$

If we let $\mathbf{v} = \phi'$, then

$$\mathbf{v}'(t) = \mathbf{A}(t)\mathbf{v}(t) \tag{3.89}$$

where, since $\phi(t)$ has period T by assumption, $\mathbf{v}(t)$ must also, and so the corresponding characteristic multiplier is 1. As a result, for a nonlinear system with a periodic solution, one characteristic multiplier is always $\rho = 1$.

3.2 General Results for n = 2

3.2.1 Stability of Periodic Solution

Consider a problem of the form $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ with $\mathbf{x} \in \mathbb{R}^2$ where there is a periodic solution $\mathbf{x}(t) = \phi(t)$ with period T. We know from §3.1.2 that we must have $\rho_1 = 1$ and we know from theorem 3.3ii that

$$\rho_1 \rho_2 = \exp\left(\int_0^T \operatorname{tr}(\mathbf{A}(s))\, ds\right) \tag{3.90}$$

$$\rho_2 = \exp\left(\int_0^T \operatorname{tr}(\mathbf{A}(s))\, ds\right). \tag{3.91}$$

From remark (ii) on page 53, we know that for the perturbation to be bounded and hence for the solution to be stable, we must have $\rho_1 \leq 1$ and $\rho_2 \leq 1$ and so, since we know $\rho_1 = 1$ and we wish ρ_1 and ρ_2 to be distinct, we must have

$$0 > \int_0^T \operatorname{tr}(\mathbf{A}(s))\, ds \tag{3.92}$$

$$0 > \int_0^T \operatorname{tr}\left(\left.\frac{\partial f_i}{\partial x_j}\right|_{\phi(s)}\right) ds \tag{3.93}$$

$$0 > \int_0^T \left.\left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}\right)\right|_{\phi(s)} ds \tag{3.94}$$

$$0 > \int_0^T \nabla \cdot \mathbf{f}|_{\mathbf{x}=\phi}\, ds. \tag{3.95}$$

We get instability when

$$0 < \int_0^T \nabla \cdot \mathbf{f}|_{\mathbf{x}=\phi}\, ds. \tag{3.96}$$

3.2.2 Example

Consider

$$x' = x - y - x\left(x^2 + y^2\right) \tag{3.97}$$
$$y' = x + y - y\left(x^2 + y^2\right). \tag{3.98}$$

Let

$$x = r(t)\cos(\theta(t)) \tag{3.99}$$
$$y = r(t)\sin(\theta(t)) \tag{3.100}$$

so that our problem becomes

$$\sin(\theta)\left(r - r\theta'\right) = \cos(\theta)\left(r - r^3 - r'\right) \tag{3.101}$$
$$\cos(\theta)\left(r - r\theta'\right) = -\sin(\theta)\left(r - r^3 - r'\right). \tag{3.102}$$

By squaring and adding these equations, we obtain that

$$\left(r - r\theta'\right)^2 = \left(r - r^3 - r'\right)^2 \tag{3.103}$$

so we can write

$$a = r - r\theta' \tag{3.104}$$
$$sa = r - r^3 - r' \tag{3.105}$$

where $s = \pm 1$. Our equations then become

$$a\sin(\theta) = sa\cos(\theta) \tag{3.106}$$
$$a\cos(\theta) = -sa\sin(\theta) \tag{3.107}$$

which can be rewritten as

$$a\sin(\theta) = sa\cos(\theta) \tag{3.108}$$
$$-s^2 a\sin(\theta) = sa\cos(\theta) \tag{3.109}$$

so that we must have

$$a\sin(\theta) = -a\sin(\theta) \tag{3.110}$$
$$a\sin(\theta) = 0. \tag{3.111}$$

As a result, we have that

$$a\sin(\theta) = sa\cos(\theta) = 0 \tag{3.112}$$

so that we must have $a = 0$. This means that

$$r - r\theta' = r - r^3 - r' = 0. \tag{3.113}$$

We have that

$$r' = r\left(1 - r^3\right) \tag{3.114}$$

and so we have a solution of constant radius when $r = 0$ (the trivial case) and $r = \pm 1$. Without loss of generality, choose $r = 1$. Then since

$$r\theta' = r, \tag{3.115}$$

we have that $\theta' = 1$, so $\theta = t + C$. As a result, our solution has period $T = 2\pi$.

Now

$$\nabla \cdot \mathbf{f}|_{r=1} = \left[\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y}\right]_{r=1} \tag{3.116}$$

$$= \left[\left(1 - 3x^2 - y^2\right) + \left(1 - x^2 - 3y^2\right)\right]_{r=1} \tag{3.117}$$

$$= \left[2 - 4r^2\right]_{r=1} \tag{3.118}$$

$$= -2 \tag{3.119}$$

so that

$$\rho_2 = \exp\left(\int_0^T \operatorname{tr}\left(\mathbf{A}(s)\right)\, ds\right) \tag{3.120}$$

$$= \exp\left(\int_0^{2\pi} -2\, ds\right) \tag{3.121}$$

$$= e^{-4\pi} \tag{3.122}$$

$$< 1. \tag{3.123}$$

As a result, the limit cycle with radius $r = 1$ is stable.

3.2.3 Stability of Second-Order ODE

Consider the second-order ODE

$$x'' + a(t)x = 0 \tag{3.124}$$

where $a(t)$ is periodic with period T. Letting $x_1 = x$ and $x_2 = x_1'$, this can be rewritten as

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a(t) & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{3.125}$$

By choosing the initial condition

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{3.126}$$

we obtain a solution of the form

$$\begin{bmatrix} x_1^{(1)}(t) \\ x_1'^{(1)}(t) \end{bmatrix}. \tag{3.127}$$

Likewise by choosing the initial condition

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{3.128}$$

we obtain a solution of the form

$$\begin{bmatrix} x_1^{(2)}(t) \\ x_1'^{(2)}(t) \end{bmatrix}. \tag{3.129}$$

As a result, we have chosen $\mathbf{X}(0) = \mathbf{I}$ so that

$$\mathbf{B} = \mathbf{X}(T) = \begin{bmatrix} x_1^{(1)}(T) & x_1^{(2)}(T) \\ x_1'^{(1)}(T) & x_1'^{(2)}(T) \end{bmatrix}. \tag{3.130}$$

Now we have from property (i) on page 52 that

$$\rho_1\rho_2 = \exp\left(\int_0^T \operatorname{tr}(\mathbf{A}(s))\, ds\right) \tag{3.131}$$

$$= \exp\left(\int_0^T 0\, ds\right) \tag{3.132}$$

$$= 1 \tag{3.133}$$

and from property (ii) that

$$\rho_1 + \rho_2 = \operatorname{tr}(\mathbf{B}) \tag{3.134}$$

$$= x_1^{(1)}(T) + x_1'^{(2)}(T). \tag{3.135}$$

Let $\phi = \operatorname{tr}(\mathbf{B})/2$ so that

$$\rho_1\rho_2 = 1 \tag{3.136}$$

$$\rho_1 + \rho_2 = 2\phi. \tag{3.137}$$

Solving these, we obtain that

$$\rho = \phi \pm \sqrt{\phi^2 - 1}. \tag{3.138}$$

We can rewrite ρ_i as $\exp(\mu_i T)$, so that

$$\mu_1 + \mu_2 = 0 \tag{3.139}$$

and so

$$e^{\mu_1 T} + e^{\mu_2 T} = 2\phi \tag{3.140}$$

$$e^{\mu_1 T} + e^{-\mu_1 T} = 2\phi \tag{3.141}$$

$$\frac{e^{\mu_1 T} + e^{-\mu_1 T}}{2} = \phi \tag{3.142}$$

$$\cosh(\mu_1 T) = \phi. \tag{3.143}$$

Consider the following cases.

I. Let $-1 < \phi < 1$. We can then define σ by $\phi = \cos(\sigma T)$, where, without loss of generality, $0 < \sigma T < \pi$, so that

$$\rho = \phi \pm \sqrt{\phi^2 - 1} \tag{3.144}$$
$$= \cos(\sigma T) \pm i \sin(\sigma T) \tag{3.145}$$
$$= e^{\pm i\sigma T} \tag{3.146}$$

As in remark (v) on page 55, we can write the general solution as

$$\mathbf{x}(t) = c_1 \operatorname{Re}\left(e^{i\sigma t}\mathbf{p}(t)\right) + c_2 \operatorname{Im}\left(e^{i\sigma t}\mathbf{p}(t)\right) \tag{3.147}$$

and since $|\rho_1| = 1$ and $|\rho_2| = 1$, then from remark (ii) on page 53, the solution is stable and pseudo-periodic.

Now $e^{i\sigma t}$ has period $\hat{T} = \frac{2\pi}{\sigma}$. Now since $\phi \neq 1$ and $\phi \neq -1$, we must have

$$\sigma T \neq m\pi \tag{3.148}$$
$$\frac{2\pi}{\hat{T}} T \neq m\pi \tag{3.149}$$
$$\frac{2T}{m} \neq \hat{T} \tag{3.150}$$

so that $\hat{T} \neq 2T, T, \frac{2}{3}T, \ldots$

Note that for $\hat{T}$ to equal nT, we must have

$$\sigma = \frac{2\pi}{nT} \tag{3.151}$$

for $n \neq 1, 2$ from above.

II. Let $\phi > 1$. Then since $\rho = \phi \pm \sqrt{\phi^2 - 1}$, we must have $\rho_1 > 1$ and since $\rho_1\rho_2 = 1$, we must have $\rho_1 > 1 > \rho_2 > 0$ and $\rho_2 = \frac{1}{\rho_1}$ means $\mu_2 = -\mu_1$. Our solution must therefore be of the form

$$\mathbf{x}(t) = c_1 e^{\mu_1 t}\mathbf{p}_1(t) + c_2 e^{-\mu_1 t}\mathbf{p}_2(t) \tag{3.152}$$

where $\mathbf{p}_1(t)$ and $\mathbf{p}_2(t)$ are both periodic with period T. As a result, the solution is unstable.

III. Let $\phi = 1$. Then $\rho_1 = \rho_2 = 1$. Here, theorem 3.4 only guarantees that we will have *one* solution $\mathbf{x}(t)$ of the form $e^{\mu t}\mathbf{p}(t)$. If $\mathbf{B}$ has two linearly independent eigenvectors, we can find two linearly independent $\mathbf{p}_1(t)$ and $\mathbf{p}_2(t)$ so that the two solutions are both in the standard form. However, if $\mathbf{B}$ only has one eigenvector, we will end up with one solution of the form $\mathbf{p}_1(t)$ (since $\rho = 1$ in this case) and the other of the form $t\mathbf{p}_1(t) + \mathbf{p}_2(t)$. To see this, we replace

$$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \tag{3.153}$$

with the Jordan block

$$\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}. \tag{3.154}$$

As a result, instead of our solution being of the form

$$\mathbf{X}(t) = \mathbf{P}(t)\exp\left(\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} t\right) \tag{3.155}$$

$$= \mathbf{P}(t)\begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} \tag{3.156}$$

$$= \begin{bmatrix} P_1 e^{\lambda_1 t} & P_2 e^{\lambda_2 t} \\ P_3 e^{\lambda_1 t} & P_4 e^{\lambda_2 t} \end{bmatrix}, \tag{3.157}$$

it will be of the form

$$\mathbf{X}(t) = \mathbf{P}(t)\exp\left(\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} t\right) \tag{3.158}$$

$$= \mathbf{P}(t)\begin{bmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{bmatrix} \tag{3.159}$$

$$= \begin{bmatrix} P_1 e^{\lambda t} & P_1 te^{\lambda t} + P_2 e^{\lambda t} \\ P_3 e^{\lambda t} & P_3 te^{\lambda t} + P_4 e^{\lambda t} \end{bmatrix}. \tag{3.160}$$

See the papers by Akhmedov [1] and Wiesel and Pohlen [30].

IV. Let $\phi < -1$. Since $\rho = \phi \pm \sqrt{\phi^2 - 1}$, we must have $\rho_1 < -1$ and since $\rho_1\rho_2 = 1$, we must have $\rho_1 < -1 < \rho_2 < 0$ and $\rho_2 = \frac{1}{\rho_1}$ means $\mu_2 = -\mu_1$. Now we can write $\mu_1 = \frac{i\pi}{T} + \gamma$ so that our solution must be of the form

$$\mathbf{x}(t) = c_1 e^{\gamma t} e^{i\pi t/T}\mathbf{p}_1(t) + c_2 e^{-\gamma t} e^{i\pi t/T}\mathbf{p}_2(t) \tag{3.161}$$

where $\mathbf{p}_1(t)$ and $\mathbf{p}_2(t)$ are both periodic with period T and so $e^{i\pi t/T}\mathbf{p}_1(t)$ and $e^{i\pi t/T}\mathbf{p}_2(t)$ are both periodic with period $2T$. As a result, the solution is unstable.

V. Let $\phi = -1$. Then $\rho_1 = \rho_2 = -1$. As in the case when $\phi = 1$, we have one solution which is periodic (this time with period $2T$),

$$\mathbf{x}_1(t) = e^{i\pi t/T}\mathbf{p}_1(t) \tag{3.162}$$

and the other which grows linearly with time,

$$\mathbf{x}_2(t) = te^{i\pi t/T}\mathbf{p}_1(t) + e^{i\pi t/T}\mathbf{p}_2(t). \tag{3.163}$$

We summarise these results in figure 3.1. For $\phi > 1$, we have an unstable solution of the form

$$\mathbf{x}(t) = c_1 e^{\mu_1 t}\mathbf{p}_1(t) + c_2 e^{-\mu_1 t}\mathbf{p}_2(t). \tag{3.164}$$

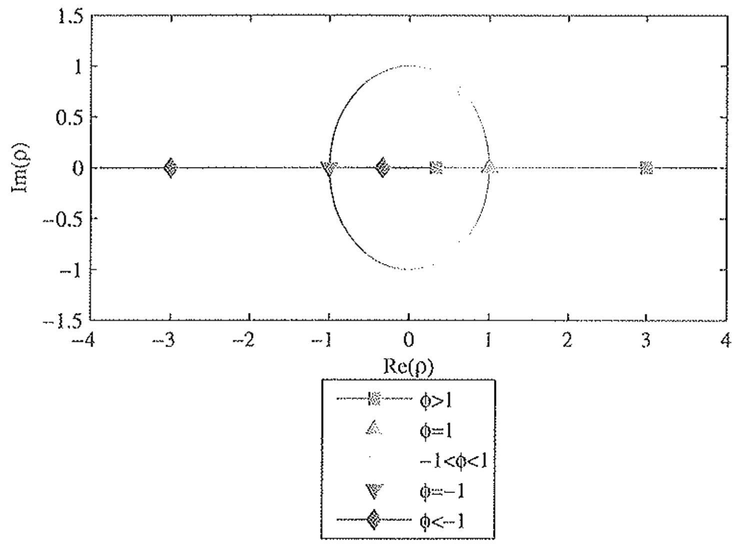

Figure 3.1: The range of ρ_1, ρ_2 for different values of ϕ real. In the region $\phi > 1$, the sample point has $\rho = 1/3, 3$; for $\phi = 1$, we have $\rho = 1$. In $1 < \phi < 1$, the sample point shown is $\rho = 2/3 \pm i\sqrt{5}/3$; for $\phi = -1$, we have $\rho = -1$ and in the region $\phi < -1$, we show $\rho = -1/3, -3$.

For $\phi = 1$, we have an unstable solution of the form

$$\mathbf{x}(t) = (c_1 + tc_2)\,\mathbf{p}_1(t) + c_2\mathbf{p}_2(t). \tag{3.165}$$

For $-1 < \phi < 1$, we have a stable pseudo-periodic solution of the form

$$\mathbf{x}(t) = c_1 \operatorname{Re}\left(e^{i\sigma t}\mathbf{p}(t)\right) + c_2 \operatorname{Im}\left(e^{i\sigma t}\mathbf{p}(t)\right). \tag{3.166}$$

For $\phi = -1$, we have an unstable solution of the form

$$\mathbf{x}(t) = (c_1 + tc_2)\,\mathbf{q}_1(t) + c_2\mathbf{q}_2(t). \tag{3.167}$$

Finally, for $\phi < -1$, we have an unstable solution of the form

$$\mathbf{x}(t) = c_1 e^{\gamma t}\mathbf{q}_1(t) + c_2 e^{-\gamma t}\mathbf{q}_2(t) \tag{3.168}$$

where $\mathbf{p}_i(t)$ represents a function that has period T and $\mathbf{q}_i(t)$ represents a function that has period $2T$.

3.2.4 Application to Hill's Equation

Consider Hill's equation

$$x'' + (\delta + \epsilon b(t)) = 0 \tag{3.169}$$

where $b(t)$ has period T. If $\epsilon = 0$, the solution is stable, however, there are some values of δ for which the solution is only marginally stable, according to the above criteria. As a result, we expect that for ϵ small but nonzero near those values of δ, we will get the beginning of a region of instability. We wish to find those values of δ.

For $\epsilon = 0$, if $\mathbf{X}(0) = \mathbf{I}$, then

$$\mathbf{X}(t) = \begin{bmatrix} \cos\left(\sqrt{\delta}t\right) & \frac{1}{\sqrt{\delta}}\sin\left(\sqrt{\delta}t\right) \\ -\sqrt{\delta}\sin\left(\sqrt{\delta}t\right) & \cos\left(\sqrt{\delta}t\right) \end{bmatrix} \tag{3.170}$$

and so

$$\mathbf{B} = \mathbf{X}(T) = \begin{bmatrix} \cos\left(\sqrt{\delta}T\right) & \frac{1}{\sqrt{\delta}}\sin\left(\sqrt{\delta}T\right) \\ -\sqrt{\delta}\sin\left(\sqrt{\delta}T\right) & \cos\left(\sqrt{\delta}T\right) \end{bmatrix}. \tag{3.171}$$

As a result,

$$\phi = \frac{\operatorname{tr}(\mathbf{B})}{2} = \cos\left(\sqrt{\delta}T\right). \tag{3.172}$$

If $\phi = 1$, then

$$\sqrt{\delta}T = 2m\pi \tag{3.173}$$

$$\delta = \left(2m\frac{\pi}{T}\right)^2 \tag{3.174}$$

where m is a positive integer since $\sqrt{\delta} > 0$. If $\phi = -1$, then

$$\sqrt{\delta}\pi = (2m+1)\,\pi \tag{3.175}$$

$$\delta = \left((2m+1)\,\frac{\pi}{T}\right)^2. \tag{3.176}$$

Now we have from the previous section that $\phi = 1$ corresponds to the existence of a periodic solution of period T and $\phi = -1$ corresponds to the existence of a periodic solution of period $2T$. As a result, we will have the border between stability and instability breaking off from $\epsilon = 0$ at

$$\delta = \left(2m\frac{\pi}{T}\right)^2 \tag{3.177}$$

corresponding to solutions with period T and breaking off from $\epsilon = 0$ at

$$\delta = \left((2m+1)\,\frac{\pi}{T}\right)^2 \tag{3.178}$$

corresponding to solutions with period $2T$.

3.3 Stability Boundary of Mathieu's Equation

3.3.1 Undamped Case

We have from §3.2.3 and §3.2.4 that on the edge of the region of stability, we have either $\phi = 1$ or $\phi = -1$. The former corresponds to the existence of a periodic solution with period T and the latter to a periodic solution with period $2T$. In order to determine the region of stability of the Mathieu equation in the δ-ϵ plane, we then need to determine the conditions on δ and ϵ required in order to have a solution which is periodic with either period π or 2π. We follow McLachlan [17] and Ward [28].

Functions of Period π

We can write a general function of period π as

$$x = \sum_{n=0}^{\infty} a_n \cos(2nt) + \sum_{n=1}^{\infty} b_n \sin(2nt). \tag{3.179}$$

We then obtain

$$0 = x'' + (\delta + \epsilon \cos(2t))\, x \tag{3.180}$$

$$0 = \sum_{n=0}^{\infty} \left(\delta - 4n^2\right) a_n \cos(2nt) + \sum_{n=1}^{\infty} \left(\delta - 4n^2\right) b_n \sin(2nt)$$
$$+ \epsilon \sum_{n=0}^{\infty} a_n \cos(2nt)\cos(2t) + \epsilon \sum_{n=1}^{\infty} b_n \sin(2nt)\cos(2t). \tag{3.181}$$

Using the identities

$$\cos(A)\cos(B) = \frac{1}{2}\left(\cos(A-B) + \cos(A+B)\right) \tag{3.182}$$

$$\sin(A)\cos(B) = \frac{1}{2}\left(\sin(A-B) + \sin(A+B)\right) \tag{3.183}$$

this becomes

$$0 = \sum_{n=0}^{\infty} \left(\delta - 4n^2\right) a_n \cos(2nt) + \sum_{n=1}^{\infty} \left(\delta - 4n^2\right) b_n \sin(2nt)$$
$$+ \frac{\epsilon}{2} \sum_{n=0}^{\infty} a_n \left(\cos(2(n+1)t) + \cos(2(n-1)t)\right)$$
$$+ \frac{\epsilon}{2} \sum_{n=1}^{\infty} b_n \left(\sin(2(n+1)t) + \sin(2(n-1)t)\right) \tag{3.184}$$

and so we must have

$$0 = \sum_{n=0}^{\infty} \left(\delta - 4n^2\right) a_n \cos(2nt) + \frac{\epsilon}{2} \sum_{n=0}^{\infty} a_n \left(\cos(2(n+1)t) + \cos(2(n-1)t)\right) \tag{3.185}$$

$$0 = \left(\delta a_0 + \frac{\epsilon}{2} a_1\right) \cos(0) + \left((\delta - 4) a_1 + \frac{\epsilon}{2}(2a_0 + a_2)\right) \cos(2t) + \sum_{n=2}^{\infty} \left(\left(\delta - 4n^2\right) a_n + \frac{\epsilon}{2}(a_{n-1} + a_{n+1})\right) \cos(2nt) \tag{3.186}$$

and

$$0 = \sum_{n=1}^{\infty} \left(\delta - 4n^2\right) b_n \sin(2nt) + \frac{\epsilon}{2} \sum_{n=1}^{\infty} b_n \left(\sin(2(n+1)t) + \sin(2(n-1)t)\right) \tag{3.187}$$

$$0 = \left((\delta - 4) b_1 + \frac{\epsilon}{2} b_2\right) \sin(2t) + \sum_{n=2}^{\infty} \left(\left(\delta - 4n^2\right) b_n + \frac{\epsilon}{2}(b_{n-1} + b_{n+1})\right) \sin(2nt). \tag{3.188}$$

By orthogonality of the sine and cosine, these can be rewritten as

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix} = \begin{bmatrix} \delta & \frac{\epsilon}{2} & & & & 0 \\ \epsilon & \delta - 4 \cdot 1^2 & \frac{\epsilon}{2} & & & \\ & \frac{\epsilon}{2} & \delta - 4 \cdot 2^2 & \frac{\epsilon}{2} & & \\ & & \frac{\epsilon}{2} & \delta - 4 \cdot 3^2 & \frac{\epsilon}{2} & \\ 0 & & & \ddots & \ddots & \ddots \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \end{bmatrix} \tag{3.189}$$

and

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix} = \begin{bmatrix} \delta - 4 \cdot 1^2 & \frac{\epsilon}{2} & & & 0 \\ \frac{\epsilon}{2} & \delta - 4 \cdot 2^2 & \frac{\epsilon}{2} & & \\ & \frac{\epsilon}{2} & \delta - 4 \cdot 3^2 & \frac{\epsilon}{2} & \\ 0 & & \ddots & \ddots & \ddots \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \end{bmatrix} \tag{3.190}$$

In order to have a non-zero solution, the determinant of at least one of these (infinite) matrices must be zero. This gives us the requirement that ϵ and δ must satisfy in order to be on the borderline between stability and instability. We can approximate the determinants of these matrices by the determinants of the finite $n \times n$ matrices of the same form. The resultant curves in the δ-ϵ plane for different values of n are shown in figure 3.2.

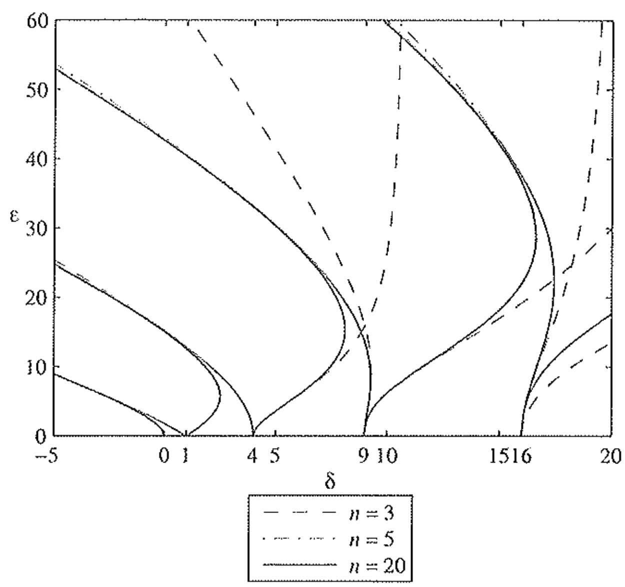

Figure 3.2: The approximation to the border of the region of stability of the Mathieu equation (determined by equations 3.189, 3.190, 3.198, 3.199) where each infinite matrix is approximated by its $n \times n$ counterpart.

Functions of Period 2π

We now perform a similar analysis for functions of period 2π. We can write a general function of period 2π as

$$x = \sum_{n=0}^{\infty} a_n \cos(nt) + \sum_{n=1}^{\infty} b_n \sin(nt). \tag{3.191}$$

We then remove from this all the terms which also have period π since we have already dealt with those. If we included them, we would obtain the lines in the δ-ϵ plane where we obtain solutions that either have period π or have period 2π. As a result, we have

$$x = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} a_n \cos(nt) + \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} b_n \sin(nt). \tag{3.192}$$

so that we obtain

$$0 = x'' + (\delta + \epsilon \cos(2t))\, x \tag{3.193}$$

$$\begin{aligned} 0 = & \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \left(\delta - n^2\right) a_n \cos(nt) + \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \left(\delta - n^2\right) b_n \sin(nt) \\ & + \epsilon \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} a_n \cos(nt)\cos(2t) + \epsilon \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} b_n \sin(nt)\cos(2t) \end{aligned} \tag{3.194}$$

$$\begin{aligned} 0 = & \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \left(\delta - n^2\right) a_n \cos(nt) + \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \left(\delta - n^2\right) b_n \sin(nt) \\ & + \frac{\epsilon}{2} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} a_n \left(\cos((n+2)t) + \cos((n-2)t)\right) \\ & + \frac{\epsilon}{2} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} b_n \left(\sin((n+2)t) + \sin((n-2)t)\right). \end{aligned} \tag{3.195}$$

We must then have

$$\begin{aligned} 0 = & \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \left((\delta - 1)\, a_1 + \frac{\epsilon}{2} (a_1 + a_3) \right) \cos(t) \\ & + \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \left(\left(\delta - n^2\right) a_n + \frac{\epsilon}{2} (a_{n-2} + a_{n+2}) \right) \cos(nt) \end{aligned} \tag{3.196}$$

and

$$0 = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \left((\delta - 1) b_1 + \frac{\epsilon}{2} (-b_1 + b_3) \right) \sin(t)$$

$$+ \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \left((\delta - n^2) b_n + \frac{\epsilon}{2} (b_{n-2} + b_{n+2}) \right) \sin(nt) \tag{3.197}$$

which we can write as

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix} = \begin{bmatrix} \delta - 1^2 + \frac{\epsilon}{2} & \frac{\epsilon}{2} & & & 0 \\ \frac{\epsilon}{2} & \delta - 3^2 & \frac{\epsilon}{2} & & \\ & \frac{\epsilon}{2} & \delta - 5^2 & \frac{\epsilon}{2} & \\ 0 & & \ddots & \ddots & \ddots \end{bmatrix} \begin{bmatrix} a_1 \\ a_3 \\ a_5 \\ \vdots \end{bmatrix} \tag{3.198}$$

and

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix} = \begin{bmatrix} \delta - 1^2 - \frac{\epsilon}{2} & \frac{\epsilon}{2} & & & 0 \\ \frac{\epsilon}{2} & \delta - 3^2 & \frac{\epsilon}{2} & & \\ & \frac{\epsilon}{2} & \delta - 5^2 & \frac{\epsilon}{2} & \\ 0 & & \ddots & \ddots & \ddots \end{bmatrix} \begin{bmatrix} b_1 \\ b_3 \\ b_5 \\ \vdots \end{bmatrix} \tag{3.199}$$

As before, in order to obtain a nonzero solution, we must have the determinant of at least one of the matrices being zero. This constrains δ and ϵ.

The resultant region of stability is shown in figure 3.3

3.3.2 Undamped Case with ϵ small

Consider now when ϵ is small. We have from §3.2.4 that for ϵ small, we will have the border between stability and instability near

$$\delta = (2m)^2 \tag{3.200}$$

and

$$\delta = (2m+1)^2 . \tag{3.201}$$

As a result, we seek periodic solutions near $\delta = n^2$ to the equation

$$x'' + (\delta + \epsilon \cos(2t))\, x = 0. \tag{3.202}$$

Let

$$x = x_0(t) + \epsilon x_1(t) + \epsilon^2 x_2(t) + \dots , \tag{3.203}$$

$$\delta = n^2 + \epsilon\delta_1 + \epsilon^2 \delta_2 + \dots \tag{3.204}$$

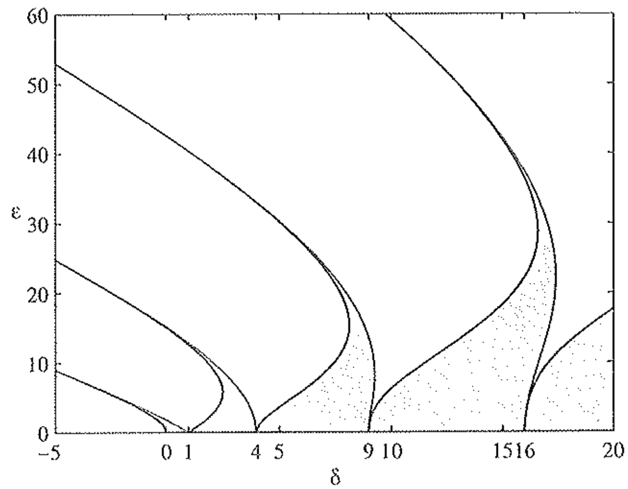

Figure 3.3: The region of stability of the Mathieu equation.

Substituting these into Mathieu's equation, we obtain

$$x_0'' + n^2 x_0 = 0 \tag{3.205}$$

$$x_1'' + n^2 x_1 = -\delta_1 x_0 - x_0 \cos(2t) \tag{3.206}$$

$$x_2'' + n^2 x_2 = -\delta_1 x_1 - \delta_2 x_0 - x_1 \cos(2t) \,. \tag{3.207}$$

For $n \neq 0$, the solution to equation 3.205 is

$$x_0 = a \cos(nt) + b \sin(nt) \,. \tag{3.208}$$

Inserting this into equation 3.206, we obtain

$$x_1'' + n^2 x_1 = -\delta_1 x_0 - x_0 \cos(2t) \tag{3.209}$$

$$= -\delta_1 \left(a \cos(nt) + b \sin(nt)\right) - \left(a \cos(nt) + b \sin(nt)\right) \cos(2t) \tag{3.210}$$

$$= -\delta_1 a \cos(nt) - \delta_1 b \sin(nt) - \frac{a}{2} \cos((n+2)t) - \frac{a}{2} \cos((n-2)t) - \frac{b}{2} \sin((n+2)t) - \frac{b}{2} \sin((n-2)t) \tag{3.211}$$

Under the assumption that $n \neq 1$, in order to eliminate secular terms, we must have

$$-\delta_1 a = 0, \qquad -\delta_1 b = 0. \tag{3.212}$$

As a result, in order to avoid x_0 being the zero solution, we must have $\delta_1 = 0$. We then have

$$x_1'' + n^2 x_1 = -\frac{a}{2}\cos((n+2)t) - \frac{a}{2}\cos((n-2)t) - \frac{b}{2}\sin((n+2)t) - \frac{b}{2}\sin((n-2)t). \tag{3.213}$$

Letting

$$x_1 = \sum_{i=0}^{\infty} c_i \sin(it) + d_i \cos(it), \tag{3.214}$$

this becomes

$$-\sum_{i=1}^{\infty} c_i i^2 \sin(it) + d_i i^2 \cos(it) + \sum_{i=0}^{\infty} c_i n^2 n^2 \sin(it) + d_i n^2 \cos(it) = -\frac{a}{2}\cos((n+2)t) - \frac{a}{2}\cos((n-2)t) - \frac{b}{2}\sin((n+2)t) - \frac{b}{2}\sin((n-2)t). \tag{3.215}$$

Equating coefficients of the sines and cosines, we obtain that

$$c_{n-2} = \frac{b}{8(-n+1)}, \qquad d_{n-2} = \frac{a}{8(-n+1)} \tag{3.216}$$

$$c_{n+2} = \frac{b}{8(n+1)}, \qquad d_{n+2} = \frac{a}{8(n+1)}. \tag{3.217}$$

We can assume that all the $\sin(nt)$ and $\cos(nt)$ component is already in x_0, so we can choose $c_n = 0$, $d_n = 0$. All remaining c_i and d_i are zero. As a result,

$$x_1 = \frac{b}{8(-n+1)}\sin((n-2)t) + \frac{b}{8(n+1)}\sin((n+2)t) + \frac{a}{8(-n+1)}\cos((n-2)t) + \frac{a}{8(n+1)}\cos((n+2)t). \tag{3.218}$$

Finally, inserting this into equation 3.207, we obtain that

$$\begin{aligned} x_2'' + n^2 x_2 = {} & -\delta_2 (a\cos(nt) + b\sin(nt)) \\ & - \frac{b}{16(-n+1)}(\sin(nt) + \sin((n-4)t)) \\ & - \frac{b}{16(n+1)}(\sin((n+4)t) + \sin(nt)) \\ & - \frac{a}{16(-n+1)}(\cos(nt) + \cos((n-4)t)) \\ & - \frac{a}{16(n+1)}(\cos((n+4)t) + \cos(nt)). \end{aligned} \tag{3.219}$$

Under the assumption that $n \neq 2$, in order to eliminate the secular terms, we must have

$$0 = -\delta_2 a - \frac{a}{16(-n+1)} - \frac{a}{16(n+1)}, \tag{3.220}$$

$$0 = -\delta_2 b - \frac{b}{16(-n+1)} - \frac{b}{16(n+1)}, \tag{3.221}$$

which can be rewritten as

$$0 = -a\left(\delta_2 - \frac{1}{8(n^2-1)}\right), \tag{3.222}$$

$$0 = -b\left(\delta_2 - \frac{1}{8(n^2-1)}\right). \tag{3.223}$$

As a result, in order to avoid a nonzero x_0 (i.e., making sure that we don't simultaneously have $a = 0$ and $b = 0$), we must have

$$\delta_2 = \frac{1}{8(n^2-1)}. \tag{3.224}$$

Case n = 2

In the case $n = 2$, eliminating the secular terms in equation 3.219 tells us that

$$0 = -\delta_2 a + \frac{a}{8} - \frac{a}{48}, \tag{3.225}$$

$$0 = -\delta_2 b - 0 - \frac{b}{48}, \tag{3.226}$$

which become

$$0 = -a\left(\delta_2 - \frac{5}{48}\right), \tag{3.227}$$

$$0 = -b\left(\delta_2 + \frac{1}{48}\right). \tag{3.228}$$

As a result, for $n = 2$ we must have either

$$a = 0, \qquad \delta_2 = -\frac{1}{48} \tag{3.229}$$

or

$$b = 0, \qquad \delta_2 = \frac{5}{48}. \tag{3.230}$$

As a result, for $n = 2$, we either have

$$\delta = 4 - \epsilon^2 \frac{1}{48} + O\left(\epsilon^3\right) \tag{3.231}$$

or

$$\delta = 4 + \epsilon^2 \frac{5}{48} + O\left(\epsilon^3\right). \tag{3.232}$$

We also have either

$$x = b \sin(2t) + \epsilon \frac{b}{24} \sin(4t) + O\left(\epsilon^2\right) \tag{3.233}$$

or

$$x = a \cos(2t) + \epsilon \left(-\frac{a}{8} + \frac{a}{24} \cos(4t)\right) + O\left(\epsilon^2\right), \tag{3.234}$$

which both have period π, as expected.

Case n = 1

In the case $n = 1$, eliminating the secular terms in equation 3.211 tells us that

$$0 = -\delta_1 a - \frac{a}{2}, \tag{3.235}$$

$$0 = -\delta_1 b + \frac{b}{2} \tag{3.236}$$

and so we must either have

$$\delta_1 = -\frac{1}{2}, \qquad b = 0 \tag{3.237}$$

or

$$\delta_1 = \frac{1}{2}, \qquad a = 0. \tag{3.238}$$

In either of these cases, equation 3.211 becomes

$$x_1'' + x_1 = -\frac{a}{2} \cos(3t) - \frac{b}{2} \sin(3t). \tag{3.239}$$

As before, we let

$$x_1 = \sum_{i=1}^{\infty} c_i \sin(it) + d_i \cos(it) \tag{3.240}$$

and find that

$$c_3 = \frac{b}{16}, \qquad d_3 = \frac{a}{16}. \tag{3.241}$$

As a result,

$$x_1 = \frac{b}{16} \sin(3t) + \frac{a}{16} \cos(3t). \tag{3.242}$$

Then equation 3.207 becomes

$$
\begin{aligned}
x_2'' + x_2 = & -\delta_1 \left(\frac{b}{16} \sin(3t) + \frac{a}{16} \cos(3t) \right) \\
& - \delta_2 (a \cos(t) + b \sin(t)) \\
& - \left(\frac{b}{16} \sin(3t) + \frac{a}{16} \cos(3t) \right) \cos(2t) \qquad (3.243) \\
= & -\delta_1 \left(\frac{b}{16} \sin(3t) + \frac{a}{16} \cos(3t) \right) - \delta_2 a \cos(t) - \delta_2 b \sin(t) \\
& - \frac{b}{32} \sin(t) - \frac{b}{32} \sin(5t) - \frac{a}{32} \cos(t) - \frac{a}{32} \cos(5t) . \qquad (3.244)
\end{aligned}
$$

In order to eliminate the secular terms, we must have

$$0 = -a \left(\delta_2 + \frac{1}{32} \right) \qquad (3.245)$$

$$0 = -b \left(\delta_2 + \frac{1}{32} \right) . \qquad (3.246)$$

As a result, $\delta_2 = -1/32$, so that either

$$\delta = 1 - \epsilon \frac{1}{2} - \epsilon^2 \frac{1}{32} + O\left(\epsilon^3\right) \qquad (3.247)$$

or

$$\delta = 1 + \epsilon \frac{1}{2} - \epsilon^2 \frac{1}{32} + O\left(\epsilon^3\right) . \qquad (3.248)$$

We also have either

$$x = a \cos(t) + \epsilon \frac{a}{16} \cos(3t) + O\left(\epsilon^2\right) \qquad (3.249)$$

or

$$x = b \sin(t) + \epsilon \frac{b}{16} \sin(3t) + O\left(\epsilon^2\right) , \qquad (3.250)$$

which are periodic with period 2π, as expected.

Case n = 0

In the case $n = 0$, we get

$$x_0 = a + bt. \qquad (3.251)$$

Now we expect a periodic solution, so $b = 0$. As a result, equation 3.206 becomes

$$x_1'' = -\delta_1 a - a \cos(2t) . \qquad (3.252)$$

In analogy with before, when we eliminated secular terms, we must have $\delta_1 = 0$. As a result, we have

$$x_1'' = -a \cos(2t) \qquad (3.253)$$

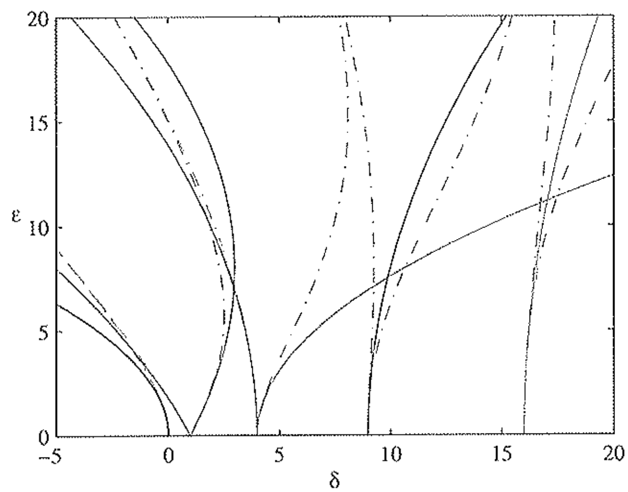

Figure 3.4: The quadratic approximations to the boundary between stability and instability of the Mathieu equation, in comparison with the approximation from §3.3.1, with $n = 20$.

so that

$$x_1 = \frac{a}{4}\cos(2t) \tag{3.254}$$

and equation 3.207 becomes

$$x_2'' = -\delta_2 a - \frac{a}{4}\cos(2t)\cos(2t) \tag{3.255}$$

$$= -\delta_2 a - \frac{a}{8} - \frac{a}{8}\cos(4t) \tag{3.256}$$

so that we must have

$$0 = -a\left(\delta_2 + \frac{1}{8}\right). \tag{3.257}$$

so that $\delta_2 = -1/8$ and

$$\delta = 0 - \epsilon^2\frac{1}{8} \tag{3.258}$$

with

$$x = a + \epsilon\frac{a}{4}\cos(2t) + O\left(\epsilon^2\right), \tag{3.259}$$

which is again periodic with period π, as expected.

These approximations to $\delta(\epsilon)$ for ϵ small are compared to the approximation in the previous section (which is valid for both small and large ϵ) in figure 3.4.

3.3.3 Damped Case

We follow Richards [24]. Our equation is

$$x'' + kx' + (\delta + \epsilon \cos(2t))\, x = 0. \tag{3.260}$$

If we let

$$y(t) = e^{\frac{k}{2}t} x(t), \tag{3.261}$$

we obtain that

$$y'' + (a + \epsilon \cos(2t))\, y = 0 \tag{3.262}$$

where

$$a = \delta - \frac{k^2}{4}. \tag{3.263}$$

Now equation 3.260 isn't of the form of equation 3.124 (§3.2.3), but equation 3.262 is. As a result, we know that the solution to equation 3.262 is of the form

$$y(t) = e^{\mu_1 t} p_1(t) + e^{\mu_2 t} p_2(t) \tag{3.264}$$

where μ_1 and μ_2 satisfy

$$e^{\mu\pi} = \rho = \phi \pm \sqrt{\phi^2 - 1} \tag{3.265}$$

where ϕ is half of the trace of $\mathbf{B}$ for $y(t)$ above when we use the initial conditions $\mathbf{X}(0) = \mathbf{I}$. As a result, the largest μ (the one most likely to cause instability) satisfies

$$e^{\mu\pi} = \rho = \phi + \sqrt{\phi^2 - 1} \tag{3.266}$$

so that

$$\mu\pi = \ln\left(\phi + \sqrt{\phi^2 - 1}\right) \tag{3.267}$$

$$\mu\pi = \cosh^{-1}(\phi) \tag{3.268}$$

$$\mu = \frac{\cosh^{-1}(\phi)}{\pi}. \tag{3.269}$$

Now in order for $x(t)$ to be stable, we must have

$$0 \geq \mathrm{Re}\left(\mu - \frac{k}{2}\right) \tag{3.270}$$

$$\frac{k}{2} \geq \mathrm{Re}(\mu) \tag{3.271}$$

with μ as above. This can be used to numerically determine the stability of the damped equation. The result for $k = 0.2$ is shown in figure 3.5.

3.3.4 Damped Case with ϵ small

Consider the damped Mathieu equation

$$x'' + kx' + (\delta + \epsilon \cos(2t))\, x = 0. \tag{3.272}$$

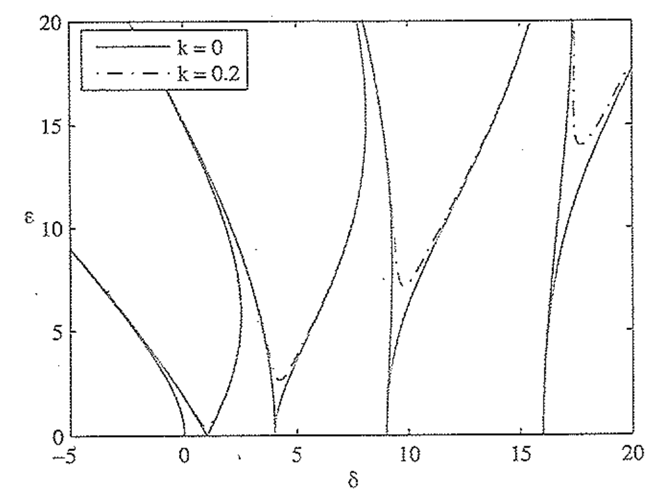

Figure 3.5: The border of the region of stability of the Mathieu equation, in the damped case.