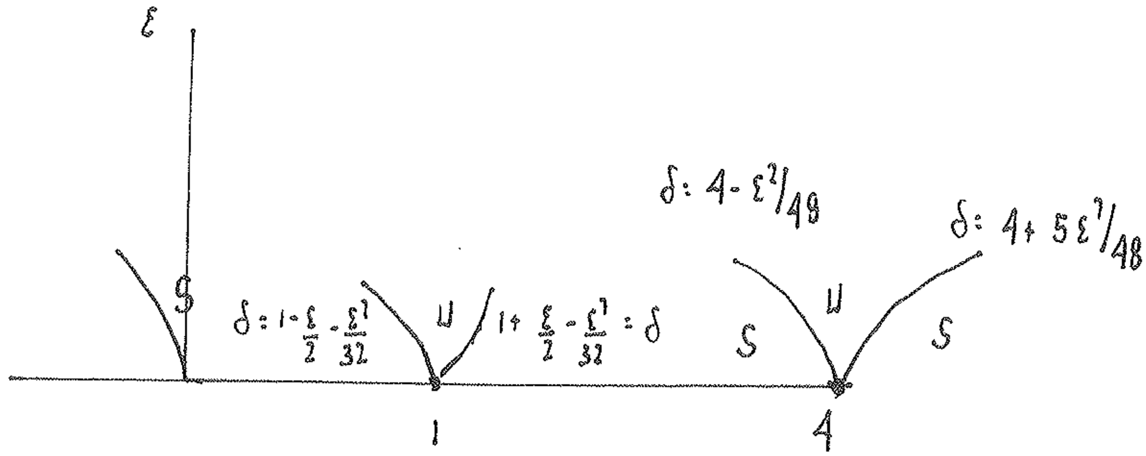
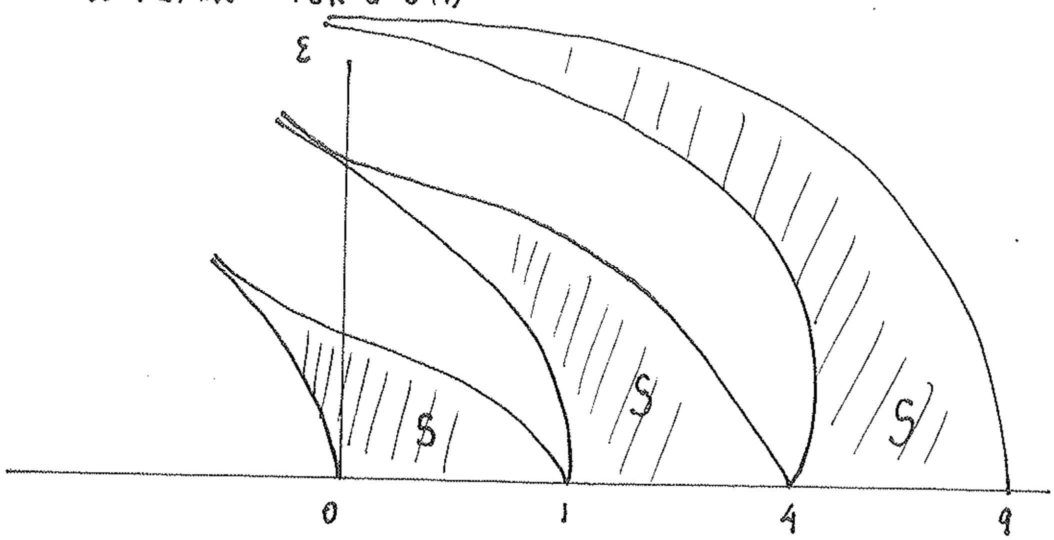
LOCAL PICTURE



GLOBAL PICTURE

COMPUTING FOR $\varepsilon = O(1)$



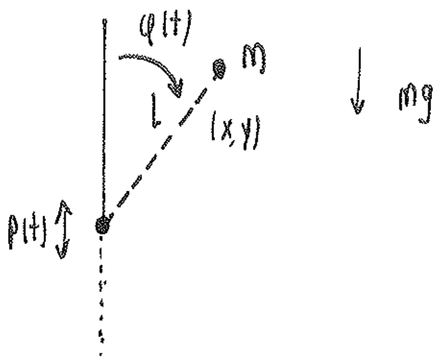
- RESONANCE TONGUES PINCH OFF AS $\varepsilon \to \infty$.

$$x'' + [\delta + \varepsilon \cos(2t)]x = 0.$$

<u>KEY</u> we can be stable if $\delta < 0$, PROVIDED THAT ε IS SUFFICIENTLY TUNED.

APPLICATION I (STABILIZATION OF INVERTED PENDULUM)

FOR A MASS m ON AN ARM OF LENGTH L THE PIVOT POINT P IS ASSUMED TO MOVE ALONG y-AXIS AND IS SUBJECT TO A VERTICAL OSCILLATION $p(t)$, WHICH PROVIDES AN ACCELERATION $r(t) = p''(t)$.



WE HAVE TO DERIVE THE ODE FOR $\varphi(t)$.

WE USE LAGRANGIAN MECHANICS:

$$\mathcal{L} \equiv \frac{1}{2} m v^2 - mgy = K.E. - P.E.$$

AND $$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\varphi}}\right) = \frac{\partial \mathcal{L}}{\partial \varphi}.$$ (LEAST ACTION PRINCIPLE).

WE CALCULATE

$$x = L \sin\varphi \qquad \dot{x} = L \cos\varphi\, \dot{\varphi}$$

$$y = L\cos\varphi + p(t) \qquad \dot{y} = -L\sin\varphi\, \dot{\varphi} + \dot{p}.$$

THUS

$$\frac{1}{2} m v^2 = \frac{1}{2} m\left[(L^2\cos^2\varphi + L^2\sin^2\varphi)\dot{\varphi}^2 - 2L\sin\varphi\,\dot{\varphi}\dot{p} + \dot{p}^2\right]$$

$$= \frac{1}{2} m L^2 \dot{\varphi}^2 - Lm\sin\varphi\,\dot{\varphi}\dot{p} + \frac{1}{2} m \dot{p}^2.$$

THE LAGRANGIAN IS

$$\mathcal{L} = \frac{1}{2} m L^2 \dot{\varphi}^2 - Lm\sin\varphi\,\dot{\varphi}\dot{p} + \frac{1}{2} m\dot{p}^2 - mg\left[L\cos\varphi + p(t)\right].$$

NOW

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\varphi}}\right) = \frac{d}{dt}\left[\frac{1}{2} m L^2 \dot{\varphi} - Lm\sin\varphi\,\dot{p}\right] = mL^2\ddot{\varphi} - Lm\cos\varphi\,\dot{\varphi}\dot{p} - Lm\sin\varphi\,\ddot{p}$$

$$\left(\frac{\partial \mathcal{L}}{\partial \varphi}\right) = -Lm\cos\varphi\,\dot{\varphi}\dot{p} + mgL\sin\varphi.$$

EQUATING, WE OBTAIN

$$mL^2\ddot{\varphi} - Lm\cos\varphi\,\dot{\varphi}\dot{p} - Lm\sin\varphi\,\ddot{p} = -Lm\cos\varphi\,\dot{\varphi}\dot{p} + mgL\sin\varphi.$$

THIS YIELDS THAT

$$\ddot{\varphi} - \frac{1}{L}\ddot{p}\sin\varphi = \frac{g}{L}\sin\varphi$$

OR $$\ddot{\varphi} + \frac{1}{L}\left(-g - \ddot{p}\right)\sin\varphi = 0.$$

NOW LET $p(t) = + A\cos(wt)$. $\ddot{p} = -Aw^2\cos(wt)$.

THIS YIELDS $$\ddot{\varphi} + \left[-\frac{g}{L} + \frac{Aw^2}{L}\cos(wt)\right]\sin\varphi = 0.$$

NOW LET $2\tau = wt$ SO $d\varphi/_{dt} = d\varphi/_{d\tau}\,\frac{w}{2}$.

THIS YIELDS $$\frac{d^2\varphi}{d\tau^2} + \left[-\frac{4g}{w^2L} + \frac{4A}{L}\cos(2\tau)\right]\sin\varphi = 0.$$

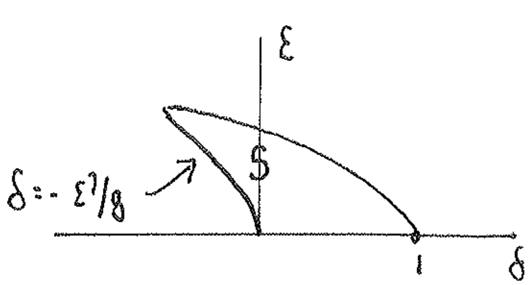
NOW LET'S ASSUME $A/_L \ll 1$ AND $w^2 \gg g/_L$ HIGH FREQUENCY.

LABEL $$\delta = -\frac{4g}{w^2L} \qquad \varepsilon = \frac{4A}{L}.$$

SO $$\frac{d^2\varphi}{d\tau^2} + \left(\delta + \varepsilon\cos(2\tau)\right)\sin\varphi = 0.$$

NOW FOR φ SMALL WE PUT $\sin\varphi \sim \varphi$. THIS GIVES LINEARIZED PROBLEM

$$\frac{d^2\varphi}{d\tau^2} + \left(\delta + \varepsilon\cos(2\tau)\right)\varphi = 0.$$



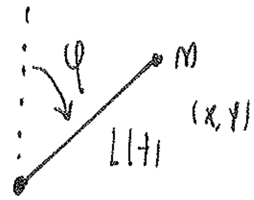
WE HAVE STABILITY IF $-\frac{1}{8}\varepsilon^2 < \delta$.

THIS YIELDS $$-\frac{1}{8}\left(\frac{16A^2}{L^2}\right) < -\frac{4g}{w^2L}.$$

$$A^2w^2 > 2gL.$$

THUS FOR $A/L \ll 1$, $\omega \gg \sqrt{g/L}$, WE HAVE STABILITY OF INVERTED PENDULUM WHEN $A\omega > \sqrt{2gL}$.

<u>APPLICATION II</u> WE NOW CONSIDER A PENDULUM OF LENGTH $L(t)$ THAT VARIES IN TIME.



TO DERIVE EQUATION OF MOTION FIND THE LAGRANGIAN:

$$x = L \sin\varphi \qquad y = L\cos\varphi$$

$$\dot{x} = \dot{L}\sin\varphi + L\dot{\varphi}\cos\varphi \qquad \dot{y} = \dot{L}\cos\varphi - L\dot{\varphi}\sin\varphi.$$

SO $$\frac{1}{2} m v^2 = \frac{1}{2} m\left[\dot{L}^2(\sin^2\varphi + \cos^2\varphi) + L^2\dot{\varphi}^2(\cos^2\varphi + \sin^2\varphi)\right]$$

$$= \frac{1}{2} m\left[\dot{L}^2 + L^2\dot{\varphi}^2\right]$$

SO $$\mathcal{L} = \frac{1}{2} m v^2 - mgy = \frac{1}{2} m\left[\dot{L}^2 + L^2\dot{\varphi}^2\right] - mgL\cos\varphi.$$

SO $$\frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = mL^2\dot{\varphi} \qquad \frac{\partial \mathcal{L}}{\partial \varphi} = mgL\sin\varphi.$$

SO $$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\varphi}}\right) - \frac{\partial \mathcal{L}}{\partial \varphi} = 0 \longrightarrow \frac{d}{dt}\left(L^2\dot{\varphi}\right) - gL\sin\varphi = 0.$$

NOW $$L^2\ddot{\varphi} + 2L\dot{L}\dot{\varphi} = gL\sin\varphi \longrightarrow L\ddot{\varphi} + 2\dot{L}\dot{\varphi} = g\sin\varphi.$$

NOW WRITE THIS AS $$\frac{d}{dt}\left(L\dot{\varphi}\right) + \dot{L}\dot{\varphi} = g\sin\varphi.$$

LET $\phi = L(\varphi - \pi)$ THEN $\dot{\phi} = L\dot{\varphi} + \dot{L}(\varphi - \pi)$ SUBSTITUTE TO OBTAIN

$$\frac{d}{dt}\left(\dot{\phi} - \dot{L}(\varphi - \pi)\right) + \dot{L}\dot{\varphi} = g\sin\varphi \longrightarrow \ddot{\phi} - \ddot{L}(\varphi - \pi) = g\sin\varphi.$$

BUT $\varphi - \pi = \phi/L$ SO $$\ddot{\phi} - \phi\,\ddot{L}/L = g\sin\left(\phi/L + \pi\right).$$

THIS GIVES THE ODE

$$\ddot{\phi} - \phi \ddot{L}/L = g \sin\left(\pi + \phi/L\right)$$

WHERE $\varphi = \pi + \phi/L$.

WE NOW ASSUME THAT WE ARE STARTING NEAR $\varphi \doteq \pi$ SO THAT $\phi \ll 1$. WE LINEARIZE TO OBTAIN

$$\ddot{\phi} - \phi \frac{\ddot{L}}{L} = g\left[\sin(\pi) + \cos(\pi)\, \phi/L + \cdots\right]$$

WHICH YIELDS $\ddot{\phi} + \left[\frac{g - \ddot{L}}{L}\right]\phi = 0$ AS THE LINEARIZED EQUATION.

WE NOW PUT $L = L_0\left[1 + \Delta \cos(wt)\right]$ WITH $\Delta > 0$ SMALL.

WE CALCULATE FOR Δ SMALL THAT

$$\frac{g - \ddot{L}}{L} \doteq \frac{g + L_0 \Delta w^2 \cos(wt)}{L_0\left(1 + \Delta \cos(wt)\right)} \sim \frac{\left(g + L_0 \Delta w^2 \cos(wt)\right)}{L_0}\left(1 + \Delta\cos(wt)\right)^{-1}$$

$$\doteq \left(\frac{g}{L_0} + \Delta w^2 \cos(wt)\right)\left(1 - \Delta \cos(wt)\right) + \cdots$$

LET $\Omega_0 \doteq \sqrt{g/L_0}$. THEN FOR $\Delta \ll 1$,

$$\frac{g - \ddot{L}}{L} \doteq \left(\Omega_0^2 + \Delta\left(w^2 - \Omega_0^2\right)\cos(wt)\right).$$

THIS YIELDS,

$$\ddot{\phi} + \left[\Omega_0^2 + \Delta\left(w^2 - \Omega_0^2\right)\cos(wt)\right]\phi = 0$$

NOW CHANGE VARIABLES. WE LET $2\tau = wt$ $\quad d\phi/dt = d\phi/d\tau \; w/2$.

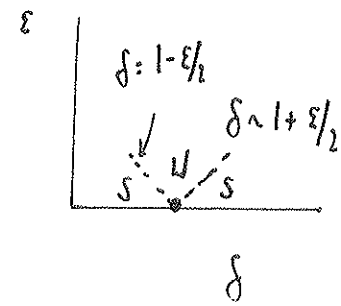
SO $$\frac{d^2\phi}{d\tau^2} + \left[\frac{4\Omega_0^2}{w^2} + \frac{4\Lambda}{w^2}\left(w^2 - \Omega_0^2\right)\cos(2\tau)\right]\phi = 0.$$

DEFINE $$\delta \equiv \frac{4\Omega_0^2}{w^2}, \qquad \varepsilon = \frac{4\Lambda}{w^2}\left(w^2 - \Omega_0^2\right) \equiv 4\Lambda\left(1 - \frac{\Omega_0^2}{w^2}\right)$$

WE ASSUME $w = O(1)$, $\Lambda \ll 1$ SO THAT $\varepsilon \ll 1$.

$$\frac{d^2\phi}{d\tau^2} + \left[\delta + \varepsilon\cos(2\tau)\right]\phi = 0.$$

NOW RECALL STABILITY CHART OF MATHIEU'S EQUATION. WE WANT TO FIND $\delta > 0$ SO THAT $\phi = 0$ IS UNSTABLE; I.E. THE OSCILLATION INCREASES



FOR ε SMALL, NEED $$1 - \varepsilon/2 < \delta < 1 + \varepsilon/2$$

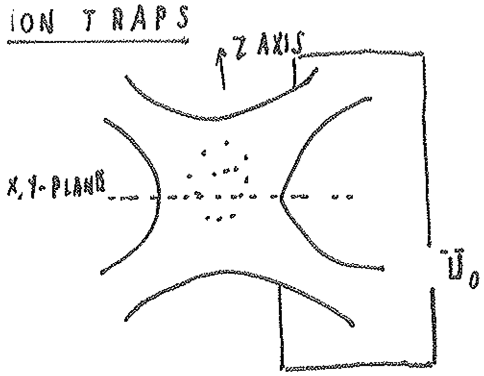
SO THAT $$\frac{4\Omega_0^2}{w^2} \to 1 \quad \text{i.e.} \quad w = 2\Omega_0.$$

THUS THE PERIOD OF HOW L CHANGES IN TIME SHOULD BE 1/2 OF THE PERIOD OF THE LINEARIZED PENDULUM.

I.E.

OR TO MAKE FREQUENCY OF LENGTH VARIATION ABOUT TWICE THAT OF THE INTERNAL FREQUENCY Ω_0. THIS WILL ALLOW ESCAPE.

ION TRAPS



$$\Phi = \frac{\Phi_0}{(r_0^2 + 2z_0^2)} (r^2 - 2z^2) \qquad r^2 = x^2 + y^2.$$

$$\Delta \Phi = 0.$$

$$\Phi_0 = U + V \cos(wt).$$

NOW THE MOTION OF CHARGED PARTICLE OF MASS m IS

$$m\, d^2x/dt^2 = -e\, \Phi_x$$

$$m\, d^2z/dt^2 = -e\, \Phi_z$$

THIS YIELDS

$$x'' + \frac{2e}{m(r_0^2 + 2z_0^2)} \left[U + V \cos wt \right] x = 0$$

$$z'' - \frac{4e}{m[r_0^2 + 2z_0^2]} \left[U + V \cos(wt) \right] z = 0.$$

NOW LET $d_0 = r_0^2 + 2z_0^2, \quad 2\tau = wt.$

THEN $$\ddot{x} + \frac{8e}{mw^2 d_0} (U + V \cos 2\tau)\, x = 0$$

$$\ddot{z} - \frac{16e}{mw^2 d_0} (U + V \cos(2\tau))\, z = 0$$

THUS $$\ddot{\eta} + [\delta + \varepsilon \cos(2\tau)]\, \eta = 0$$

WE WANT TO ENSURE THAT x, z BOUNDED $\forall$ TIME τ.

$$\delta = \frac{8e}{mw^2 d_0} U \qquad \varepsilon = \frac{8eV}{mw^2 d_0} \qquad x \text{ EQ'N.}$$

$$\delta = -\frac{16eU}{mw^2 d_0} \qquad \varepsilon = -16eV/mw^2 d_0 \qquad z\text{-EQ'N.}$$

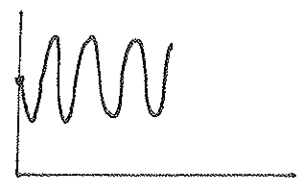
FLOQUET THEORY DISPERSAL

POPULATION LIVES IN TWO PATCHES WITH DENSITIES X_1 AND X_2. IN EACH PATCH POPULATION GROWS AND SHRINKS AT RATE $r_1(t) = \sin(2\pi t)$ AND $r_2 = -\sin(2\pi t)$. SO GROW/DECAY HAS RATE OUT OF PHASE WITH SAME PERIOD. NOW ASSUME SOME MIXING SO THAT

$$X_1' = r_1(t) X_1 + d(X_2 - X_1)$$

$$X_2' = r_2(t) X_2 + d(X_1 - X_2)$$

$X_1(0) = X_2(0) = 1$



IF $d = 0$ WE EXPECT NO GROWTH OR DECLINE

WE GET $X_1(t) = X_1(0) \exp\left[-\frac{1}{2\pi} \left(\cos(2\pi t) - 1 \right) \right]$ OSCILLATION.

$X_2(t) = X_2(0) \exp\left[+\frac{1}{2\pi} \left(\cos(2\pi t) - 1 \right) \right]$.

WHICH IS PERIODIC.

NOW IF $d \to \infty$ WE WOULD EXPECT $X_1' + X_2' \sim 0$ SO $X_1 + X_2 = C$.

THUS $X_1' + X_2' = r_1 X_1 + r_2 X_2 = r_1 (X_1 - X_2)$ SO $r_1(X_1 - X_2) = 0$ OR $X_1 \simeq X_2$.

SO $X_1' \sim r_1 X_1 + d(X_2 - X_1)$

THUS WE WOULD EXPECT $X_1 = X_2 \approx 1 \ \forall t > 0$.

NOW WHAT ABOUT d FINITE?

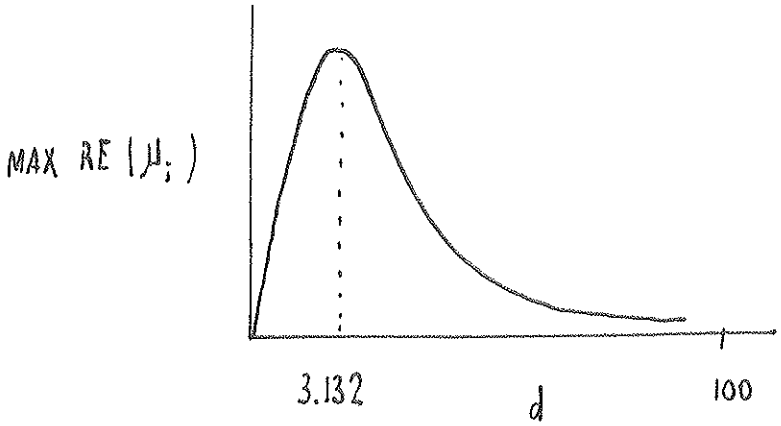
WE WRITE $\begin{pmatrix} X_1' \\ X_2' \end{pmatrix} = A \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$ $\quad A = \begin{pmatrix} r_1(t) - d & d \\ d & r_2(t) - d \end{pmatrix}$

$\operatorname{trace} A = -2d.$

LET $B = \tilde{X}(1)$ BE FUNDAMENTAL MATRIX EVALUATED AT $t = T = 1$.

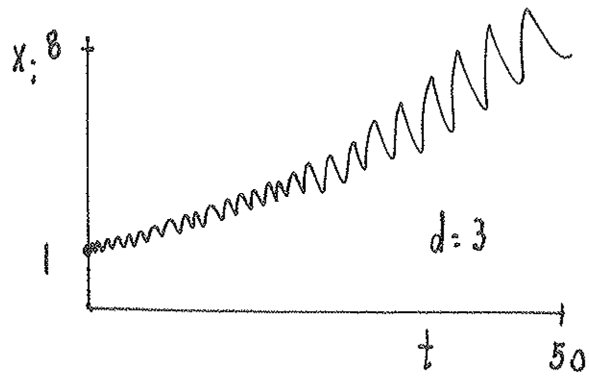
THEN $\rho_1 \rho_2 = \exp[-2d] < 1.$

let $e^{\mu_i} = \rho_i$ so $\mu_i = \log \rho_i$.



NUMERICAL COMPUTATIONS YIELD THAT MAX RE(μ_i) HAS A MAXIMUM AT SOME d FINITE

WE OBTAIN A SOLUTION OF THE FORM



C. KLAUSMEIR FLOQUET THEORY: A USEFUL TOOL FOR UNDERSTANDING NONEQUILIBRIUM DYNAMICS.

THEOR. ECOLOGY 2008 (1), pp. 153-161.

IMPLICATIONS FOR NONLINEAR SYSTEMS

CONSIDER $\quad \underline{x}' = \underline{f}(\underline{x}) \qquad \underline{x}$ IN $\mathbb{R}^2$.

SUPPOSE $\exists\ \underline{x} = \underline{\phi}(t)$ WITH $\underline{\phi}(t+T) = \underline{\phi}(t)$.

IF WE LINEARIZE AROUND $\underline{\phi}(t)$ WE OBTAIN

$$(1) \qquad \underline{v}' = A\,\underline{v} \qquad\qquad A = \begin{pmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 \end{pmatrix}_{\underline{x} = \underline{\phi}(t)}$$

SO THAT $A(t+T) = A(t)$. WE WOULD LIKE TO ANALYZE THE STABILITY OF $\underline{\phi}(t)$.

LET $\quad B = \Psi(T)$ BE THE PRINCIPAL FUNDAMENTAL MATRIX AT TIME T.

WE HAVE IN TERMS OF EIGENVALUES OF B THAT

$$\text{trace } B = \rho_1 \rho_2 = \exp\left(\int_0^T \text{trace } A \, ds \right) = \exp\left(\int_0^T \nabla \cdot \underline{f} \,\Big|_{\underline{x} = \underline{\phi}} \, ds \right).$$

THUS

$$\rho_1 \rho_2 = \exp\left(\int_0^T (\nabla \cdot \underline{f}) \Big|_{\underline{x} = \underline{\phi}} \, ds \right).$$

WE NOW SHOW THAT $\exists$ EIGENVALUE OF B WITH $\rho_1 = 1$.

<u>PROOF</u> WE HAVE $\quad \underline{\phi}' = \underline{f}[\underline{\phi}(t)] \quad$ WITH $\quad \underline{\phi}(t+T) = \underline{\phi}(t)$.

WE DIFFERENTIATE WRT t TO GET

$$(\underline{\phi}')' = A(t)\,[\underline{\phi}'(t)] \qquad \text{SO} \qquad \underline{v}(t) = \underline{\phi}'(t)$$

SATISFIES (1). BY FLOQUET, $\quad \underline{v}(t+T) = \rho\, \underline{v}(t) \quad$ SO $\quad \rho = 1 \quad$ SINCE $\underline{\phi}'(t+T) = \underline{\phi}'(t)$.

THUS $\quad \rho_1 = 1 \quad$ AND $\quad \rho_2 = \exp\left(\int_0^T (\nabla \cdot \underline{f}) \Big|_{\underline{x} = \underline{\phi}} \, ds \right)$.

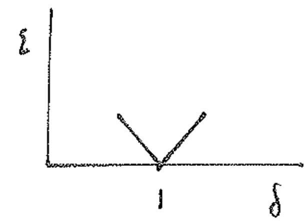
THUS IF $\quad (\nabla \cdot \underline{f}) \Big|_{\underline{x} = \underline{\phi}} < 0 \ \longrightarrow\ \rho_2 < 1 \quad$ AND WE HAVE LINEAR STABILITY.

EFFECT OF SMALL NONLINEAR TERMS IN PARAMETRIC RESONANCE

WE FIRST INTRODUCE THE IDEA OF MULTI-TIME SCALE ANALYSIS FOR MATHIEU'S EQUATION.

$$x'' + [\delta + \varepsilon \cos 2t]\, x = 0$$

$$x(0) = a, \quad x'(0) = b.$$



WE ASSUME THAT WE ARE NEAR THE FIRST INSTABILITY MODE AND SO WE LET $\delta = 1 + \varepsilon \delta_1$

WHERE δ_1 IS A PARAMETER. THEN THE FLOQUET EXPONENTS ARE $O(\varepsilon)$ WHICH SUGGESTS THAT THERE ARE TWO TIME-SCALES $t = O(1)$ AND $t = O(\varepsilon)$. AS SUCH, WE WILL LOOK FOR A MULTI-SCALE SOLUTION IN THE FORM

$$x = X[T, \tau] \quad \text{WITH} \quad T = t \text{ AND } \tau = \varepsilon t.$$

WE SUBSTITUTE TO OBTAIN

$$(X_{TT} + 2\varepsilon X_{T\tau} + \varepsilon^2 X_{\tau\tau}) + (1 + \varepsilon\delta_1 + \varepsilon \cos(2T))\, X = 0.$$

WE HAVE $X = a$ AND $X_T + \varepsilon X_\tau = b$ AT $t = 0$.

WE EXPAND $X = X_0 + \varepsilon X_1 + \dots$ TO OBTAIN,

$$O(1) \left\{ \begin{array}{l} X_{0TT} + X_0 = 0 \\ X_0 = a, \quad X_{0T} = b \quad \text{AT } T = 0 \end{array} \right. \qquad X_0 = A(\tau)\cos T + B(\tau)\sin T$$

$$O(\varepsilon) \Big\} \quad X_{1TT} + X_1 = -2X_{0T\tau} - \delta_1 X_0 - \cos(2T)\, X_0$$

SO

$$X_{1TT} + X_1 = 2A' \sin T - 2B' \cos T - \delta_1 A \cos T - \delta_1 B \sin T - A \cos T \cos 2T - B \sin T \cos(2T).$$

BY USING TRIG IDENTITIES:

$$X_{1TT} + X_1 = \left(-\delta_1 A - 2B' - A/2\right)\cos T + \left(2A' - \delta_1 B + B/2\right)\sin T$$
$$- A/2 \cos(3T) - B/2 \sin(3T)$$

WE WANT A SOLUTION THAT IS BOUNDED ON TIME-SCALE.

TO PREVENT "SECULAR TERMS" WE SET

$$-\delta_1 A - 2B' - A/2 = 0, \qquad 2A' - \delta_1 B + B/2 = 0.$$

THUS

$$(**)\left\{\begin{aligned} &\begin{pmatrix} A' \\ B' \end{pmatrix} = \begin{pmatrix} 0 & -1/4 + \delta_1/2 \\ -1/4 - \delta_1/2 & 0 \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} \\ &\begin{pmatrix} A(0) \\ B(0) \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \quad \text{AT} \quad T = 0. \end{aligned}\right.$$

THEN OUR LEADING ORDER APPROXIMATION IS

$$X \sim X_0 \equiv A(\tau)\cos T + B(\tau)\sin T.$$

LET

$$\mathcal{A} \equiv \begin{pmatrix} 0 & \delta_1/2 - 1/4 \\ -1/4 - \delta_1/2 & 0 \end{pmatrix} \qquad \begin{aligned} &\text{trace } \mathcal{A} = 0 \\ &\det \mathcal{A} \equiv (\delta_1/2 - 1/4)(\delta_1/2 + 1/4) = \delta_1^2/4 - 1/16. \end{aligned}$$

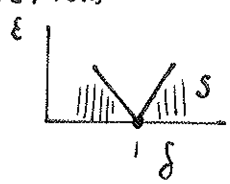
- NOW FOR STABILITY WE NEED $\text{trace } \mathcal{A} \leq 0$ $\det \mathcal{A} \geq 0. \Rightarrow |\delta_1| \geq 1/2.$

THEN LET σ_1, σ_2 BE eigenVALUES OF $\mathcal{A}$.

WE WOULD THEN HAVE $\sigma^2 + \det \mathcal{A} = 0$ $\quad \sigma = \pm i\,[+\det \mathcal{A}]^{1/2}.$

THUS IN $|\delta_1| > 1/2$ WE HAVE QUASI-PERIODIC SOLUTIONS

i.e. $\delta = 1 + \varepsilon \delta_1$ WITH $|\delta_1| > 1/2$



- WE HAVE INSTABILITY ON A LONG-TIME SCALE, i.e. EXPONENTIAL GROWTH IF $|\delta_1| < 1/2$ AS EXPECTED SINCE $\det \mathcal{A} < 0$.

NOW WE CONSIDER THE EFFECT OF NONLINEARITY.

CONSIDER THE NONLINEAR PROBLEM

$$x'' + [\delta + \varepsilon \cos(2t)] \sin x = 0.$$

WE EXPAND FOR $x \ll 1$, $\sin x = x - x^3/6 + \dots$ SO THAT

$$x'' + [\delta + \varepsilon \cos(2t)](x - x^3/6 + \dots) = 0.$$

WE THEN PUT $x = \sqrt{\varepsilon}\, y$ SO THAT

$$y'' + [\delta + \varepsilon \cos(2t)](y - \varepsilon y^3/6 + \dots) = 0.$$

THIS YIELDS THE APPROXIMATE MODEL

$$y'' + (\delta + \varepsilon \cos(2t))\, y - \frac{\delta}{6} \varepsilon y^3 + O(\varepsilon^2) = 0.$$

THIS MOTIVATES STUDYING THE PERTURBED PROBLEM

(1) $\quad x'' + [\delta + \varepsilon \cos(2t)]\, x + \varepsilon \alpha x^3 = 0 \quad$ WITH α A PARAMETER

WE WILL CONSIDER THE REGION NEAR $\delta = 1$ BY WRITING

$$\delta = 1 + \varepsilon \delta_1$$

FOR SOME $\delta_1 = O(1)$.

WE NOW INTRODUCE $T = t$, $\tau = \varepsilon t$ AND EXPAND $x = X(T, \tau) = X_0(T,\tau) + \varepsilon X_1(T,\tau) + \dots$

WE HAVE $\quad (X_{TT} + 2\varepsilon X_{T\tau} + \varepsilon^2 X_{\tau\tau}) + (1 + \varepsilon\delta_1 + \varepsilon \cos(2T))\, X + \varepsilon \alpha X^3 = 0.$

WE HAVE

$$X_{0TT} + X_0 = 0$$

$$X_{1TT} + X_1 = -2X_{0T\tau} - \delta_1 X_0 - \cos(2T)\, X_0 - \alpha X_0^3.$$

WE OBTAIN $X_0 = A(\tau)\cos T + B(\tau)\sin T.$

WE CALCULATE $X_{0T\tau} = -A'\sin T + B'\cos T$

SO
$$X_{1TT} + X_1 = 2A'\sin T - 2B'\cos T - \delta_1 A\cos T - \delta_1 B\sin T$$
$$- \cos(2T)\left[A\cos T + B\sin T\right] - \alpha\left[A\cos T + B\sin T\right]^3$$
$$= 2A'\sin T - 2B'\cos T - \delta_1 A\cos T - \delta_1 B\sin T$$
$$-\frac{A}{2}\left[\cos(3T) + \cos T\right] + \frac{B}{2}\left[\sin T - \sin 3T\right]$$
$$-\alpha\left[A^3\cos^3 T + B^3\sin^3 T + 3A^2B\cos^2 T\sin T + 3AB^2\cos T\sin^2 T\right]$$

$$X_{1TT} + X_1 = \sin T\left[2A' - \delta_1 B + B/2\right] + \cos T\left[-2B' - \delta_1 A - A/2\right]$$
$$-A/2\cos(3T) - B/2\sin(3T)$$
$$-\alpha A^3\left[\frac{3}{4}\cos T + \frac{1}{4}\cos 3T\right] - \alpha B^3\left[\frac{3}{4}\sin T - \frac{1}{4}\sin 3T\right]$$
$$-3\alpha A^2B\left[\frac{1}{4}\sin T + \frac{1}{4}\sin 3T\right] - 3\alpha AB^2\left[\frac{1}{4}\cos T - \frac{1}{4}\cos 3T\right]$$

SO THAT
$$X_{1TT} + X_1 = \sin T\left[2A' - \delta_1 B + B/2 - \frac{3\alpha B^3}{4} - \frac{3A^2B\alpha}{4}\right]$$
$$+ \cos T\left[-2B' - \delta_1 A - \frac{A}{2} - \frac{3\alpha A^3}{4} - \frac{3\alpha AB^2}{4}\right] + \text{HIGHER HARMONICS.}$$

TO SUPPRESS RESONANCE WE NEED THAT $A = A(\tau)$, $B = B(\tau)$ SATISFY

$$(*)\quad \left\{\begin{aligned} 2A' &= (\delta_1 - 1/2)B + \frac{3\alpha B}{4}(B^2 + A^2) \\ 2B' &= -(\delta_1 + 1/2)A - \frac{3\alpha A}{4}(A^2 + B^2) \end{aligned}\right.$$

NOW WE TRANSFORM TO POLAR COORDINATES.

LET $A = R\cos\varphi$, $B = R\sin\varphi$. $\varphi = \tan^{-1}(B/A)$ SO $\varphi' = \dfrac{1}{1+B^2/A^2}\left(B'/A - BA'/A^2\right)$

THEN $R^2 = A^2 + B^2$ AND $2RR' = 2AA' + 2BB'$. OR $\varphi' = \dfrac{AB' - BA'}{A^2+B^2}$.

MULTIPLYING THE FIRST OF (*) BY A AND THE SECOND BY B AND THEN ADDING GIVES,

$$2(AA' + BB') = 2RR' = -AB = -R^2\sin\varphi\cos\varphi.$$

THUS $$RR' = -R^2\sin(2\varphi)/4.$$

THIS YIELDS $$R' = -R\sin(2\varphi)/4.$$

NOW $$\varphi' = \frac{1}{A^2+B^2}\left(B'A - BA'\right) = \frac{1}{2[B^2+A^2]}\left[A\left(-(\delta_1+1/2)A - \frac{3\alpha A}{4}(A^2+B^2)\right) - B\left((\delta_1-1/2)B + \frac{3\alpha B}{4}(A^2+B^2)\right)\right]$$

OR $$\varphi' = \frac{1}{2(A^2+B^2)}\left[-\delta_1(A^2+B^2) - \frac{1}{2}(A^2-B^2) - \frac{3\alpha}{4}(A^2+B^2)^2\right]$$

BUT $A^2+B^2 = R^2$ AND $A^2 - B^2 = R^2[\cos^2\varphi - \sin^2\varphi] = R^2\cos(2\varphi)$.

SO $$\varphi' = \frac{1}{2R^2}\left[-\delta_1 R^2 - \frac{1}{2}R^2\cos(2\varphi) - \frac{3\alpha}{4}R^4\right].$$

THIS YIELDS THE COUPLED SYSTEM, REFERRED TO AS THE "SLOW-FLOW EQUATIONS"

$$(+)\left\{\begin{aligned} R' &= -R\sin(2\varphi)/4 \equiv f(R,\varphi) \\ \varphi' &= -\delta_1/2 - 1/4\cos(2\varphi) - \frac{3\alpha}{8}R^2 \equiv g(R,\varphi) \end{aligned}\right.$$

AS BEING EQUIVALENT TO (*).

THE SOLUTION TO LEADING ORDER IS $X_0 = A\cos T + B\sin T$

SO $X_0 = R\cos(T-\varphi)$. WITH $R = R(\tau)$, $\varphi = \varphi(\tau)$ AND $\tau = \varepsilon t$.

REMARKS

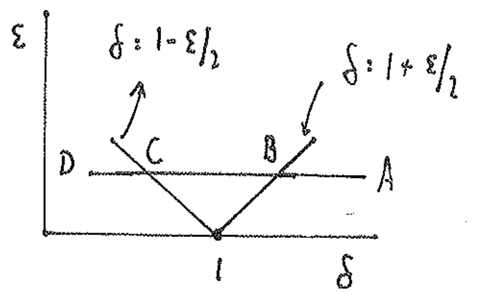
(i) THE SOLUTION $R=0$ IS DEGENERATE FOR (+). IT IS EASY TO SEE FROM (*) THAT $A=B=0$ IS ALWAYS A SOLUTION. THIS IS SIMPLY THE ZERO STATE OF THE LINEAR MATHIEU EQUATION WHICH IS UNSTABLE IF $|\delta_1|<1/2$ AND STABLE IF $|\delta_1|>1/2$. HENCE IN (+) WE NEED NOT CONSIDER $R=0$.

(ii) A SOLUTION WHERE R AND φ ARE CONSTANTS CORRESPONDS TO A PERIODIC SOLUTION IN X THAT HAS $1/2$ THE FREQUENCY (OR TWICE THE PERIOD) OF THE INTERNAL FORCING TERM $\cos t$.

(iii) A LIMIT CYCLE OR PERIODIC SOLUTION IN (A,B) PLANE $\longrightarrow$ QUASI-PERIODIC SOLUTION IN X.

<u>WE NOW ILLUSTRATE THE ANALYSIS FOR $\alpha>0$.</u>

WE THINK OF δ_1 AS A BIFURCATION PARAMETER AND WE CROSS THE WEDGE REGION IN THE (ε,δ) PLANE AS SHOWN:



FOR $\delta = 1+\varepsilon\delta_1$ WE ASK WHAT BIFURCATIONS CAN OCCUR AS δ_1 IS VARIED.

THE EQUILIBRIA TO (+) ARE FOR $R\neq 0$

$$\sin(2\varphi_e)=0 \quad \text{AND} \quad R_e^2=\frac{8}{3\alpha}\left(-\delta_1/2-\cos(2\varphi_e)/4\right)=\frac{-4}{3\alpha}\left(\delta_1+\cos(2\varphi_e)/2\right)$$

THE JACOBIAN J OF (+) IS

$$J=\left.\begin{pmatrix} f_R & f_\varphi \\ g_R & g_\varphi \end{pmatrix}\right|_{R_e,\varphi_e}=\begin{pmatrix} -\sin(2\varphi_e)/4 & -\frac{R_e}{2}\cos(2\varphi_e) \\ -\frac{3\alpha}{4}R_e & \frac{1}{4}\sin(2\varphi_e) \end{pmatrix}$$

THUS

$$\det J=-\frac{1}{4}\sin^2(2\varphi_e)-\frac{3}{8}\alpha R_e^2\cos(2\varphi_e)=-\frac{3}{8}\alpha R_e^2\cos(2\varphi_e)$$

$$\operatorname{trace} J=0.$$

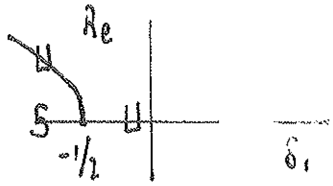
THUS IF $\det J_e > 0 \rightarrow$ STABILITY AS J_e HAS COMPLEX EIGENVALUES

$\det J_e < 0 \rightarrow$ SADDLE POINT AND INSTABILITY.

THE EQUILIBRIA ARE:

(I) $\varphi_e = 0, \pi$. THEN $R_{e_+}^2 = -\dfrac{4}{3\alpha}\left(\delta_1 + 1/2\right)$ SO THAT $\delta_1 + 1/2 < 0$ IS NEEDED.

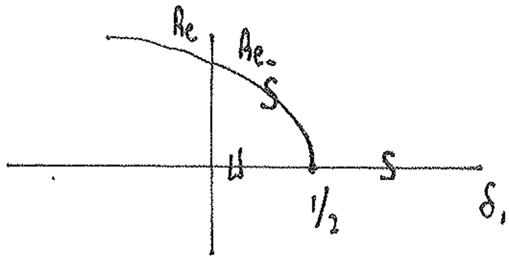
THESE EQUILIBRIA ARE IN REGION $\delta_1 < -1/2$ AND $\det J < 0 \rightarrow$ UNSTABLE.



SUBCRITICAL BIFURCATION.
LEFT TRANSITION

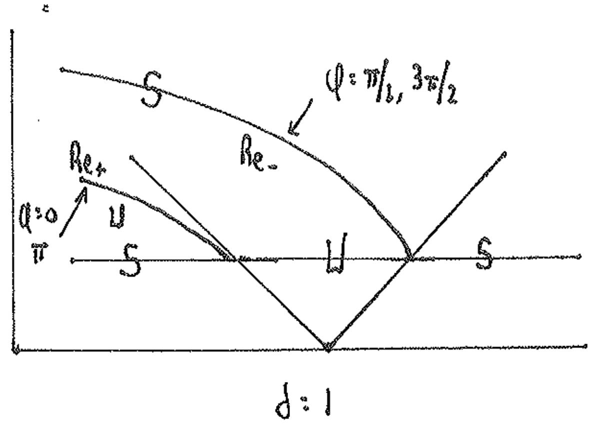
(II) $\varphi_e = \pi/2, 3\pi/2$ THEN $R_{e_-}^2 = -\dfrac{4}{3\alpha}\left(\delta_1 - 1/2\right)$ SO THAT $\delta_1 < 1/2$ IS NEEDED.

WE CALCULATE $\det J > 0$ SO THEY ARE STABLE.



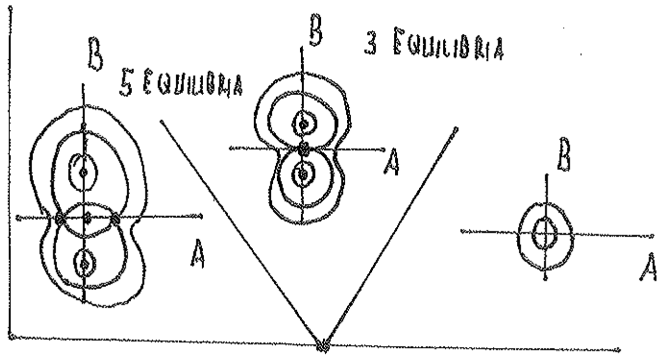
RIGHT TRANSITION

PUTTING THIS ALL TOGETHER WE HAVE



- THUS FOR $\delta_1 < -1/2$, $\exists$ FIVE STEADY-STATE SOLUTIONS TO (A, B) SYSTEM (*)
- FOR $-1/2 < \delta_1 < 1/2$, $\exists$ THREE STEADY-STATE SOLUTIONS TO (A, B) SYSTEM (*)

IN THE A, B PLANE WE HAVE



$\delta = 1$

<u>REMARKS</u>

(i) THE FIGURE ON PRECEEDING PAGE INDICATES HYSTERESIS AS δ_1 IS DECREASED BELOW $\delta_1 = 1/2$ AND THEN BELOW $\delta_1 = -1/2$. THE PATH IS NOT REVERSIBLE.

(ii) THUS AS δ_1 IS DECREASED BELOW $\delta_1 = 1/2$, THE ZERO SOLUTION LOSES ITS STABILITY TO A STABLE SUBHARMONIC THAT HAS HALF THE FREQUENCY OF THE INTERNALLY IMPOSED OSCILLATION.

(iii) AT THE LEFT TRANSITION AS δ_1 INCREASES ABOVE $\delta_1 = -1/2$ THERE WOULD BE A <u>JUMP</u> TO A STABLE SUBHARMONIC.

(iv) <u>KEY POINT</u>: IN WHAT WAS ONCE AN UNSTABLE REGION $|\delta_1| < 1/2$, THE NONLINEARITY LEADS TO A STABILIZED PATTERN.

HENCE AS δ_1 IS SWEPT BACK AND FORTH ACROSS $\delta_1 = \pm 1/2$ TRANSITIONS WE WOULD EXPECT HYSTERESIS.

IF $\alpha < 0$, THEN THE BEHAVIOR AT THE LEFT AND RIGHT TRANSITIONS IS SIMPLY SWAPPED.