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Harnack's inequality.

Schwarz reflection principle.

conformal mappings.

P8M, P8 pVA

Complex

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Liouville's theorem.

Hurwitz's theorem.

Schwarz Lemma

Maximum Modulus theorem.

Riemann Mapping theorem.

Cauchy's theorem - representation -
power series (convex case).

Morera's theorem.

Open mapping theorem.

Rouché's theorem

Laurent series.

Formula for zeros - argument principle.

Functions with natural boundaries -

Monodromy theorem.

zeros of holomorphic functions.

Picard's theorem.

Poisson formula.

open mapping / closed graph

Brouwer / Schauder fixed point theorems.

Real

Hahn decomposition

Fatou's lemma / Bounded convergence

Principle of uniform boundedness

Baire Category

Dini's Theorem

Ascoli Selection Principle

Inverse / implicit function theorem.

Measure spaces.

Extension ideas - defn of Lebesgue meas.

- product measures.

Fubini / Tonelli.

Littlewood's principles.

Jensen's inequality.

Differentiation - BV-AC.

L^p spaces - Hölder

Minkowski

Riesz-Fischer.

Riesz Representation.

Hahn Banach.

ODE

Gronwall's inequality.

Picard iteration

Poincaré Bendixson.

Boundary layer example.

Regular singular points - ODE's in the complex plane.

Linear equations - variation of parameters, fundamental solutions, equation for Wronskian.

Euler - Runge - Kutta.

Stability by looking at linearized problem + direct method.

Boundary value problems - Green's functions

Continuation.

Continuous dependence on parameters

(linear equations) $\Rightarrow$ perturbation methods

Floquet multipliers - look at stability of periodic orbits.

Wave equation - method of descent
Duhamel's principle.

Green's fn for Laplacian in the plane.
Single, double layer.

Mean value theorem $\Rightarrow$ Max principle.

Harnack's inequality $\Rightarrow$

Harmonic fn's bounded above are constant.

Weierstrass' Theorem.

Dirichlet problem:

1. Variational
2. Schwarz alternating.
3. Perron's subharmonic fn method.

PDE

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First order - single equation
2 space variables - uniqueness of symm. hyperbolic systems. + existence.

Characteristic conditions

Defn of well-posed problem.

Defn of elliptic equations, hyperbolic.

cone condition $\rightarrow$ Sobolev embedding $s > \frac{n}{2} \Rightarrow C_0 \subset H^s$

First order - continuous dependence on the initial data.

Abstract C-K (linear).

Holmgren uniqueness (without proof).

C-K classical approach.

Characteristics - can't determine normal derivatives.

- can't determine normal derivatives.

- (weak solutions) allowable discontinuities.

- relation to plane waves.

classification 2D 2 equations -

Hilbert Space

Riesz Representation theorem.

Polarization / Parallelogram.

Projection theorem.

Hellé's selection principle.

Lax - Milgram Lemma.

Hellinger - Toeplitz

Fredholm alternative

Spectral theory - compact s.a.

- bounded s.a.

- unitary.

AP functions.

10/2/2014

Fluids

Linearized surface waves - infinite depth
shallow water.

Helmholtz

Kelvin

Bernoulli

Complex Flow

Acoustic Waves.

Uniqueness for Euler equations.

Derivation of equations.

Potential Flow: Rankine vortex

Max principles

existence - uniqueness of
potential flows.

Blasius' theorem

D'Alembert's paradox

Hodograph plane - free streamlines.

Gas dynamics - Riemann invariants

simple waves

R-H conditions - Hugoniot line

RP for gas dynamics

CFL condition

Hydrostatics - fluids at rest
 Continuum mechanics
 Kinematics
 Dynamics
 Viscosity
 Poiseuille flow
 Couette flow
 Boundary layer
 Prandtl boundary layer equations
 Matching Prandtl & Euler
 Separation
 Prandtl semi-infinite plate boundary layer
 Particle methods
 Chorin's algorithm
 Wedge flow
 Stokes flow - & Stokes paradox
 Oseen flow

Gas dynamics - Random choice method
 Oleinik's condition.
 Transport theorems.
 3D - get rid of pressure in incompressible
 Flow by projection.
 Justification of incompressibility assumption.
 Couette, Poiseuille flow.
 Boundary layer - infinite flat plate.
 Prandtl Boundary layer equations.
 Matching Prandtl & Euler.
 Separation.
 Prandtl semi-infinite plate boundary
 layer.
 Particle methods.
 Chorin's algorithm.
 Wedge flow
 Stokes flow - & Stokes paradox.
 Oseen flow.

Numerical methods -

point vortex.

vortex blob + 3D.

Random vortex method.

Convergence of vortex blob method.

Vortex-in-cell method.

Greengard-Rokhlin fast poisson solver.

Vorticity-stream formulation.

Vorticity boundary conditions :

Projection

Anderson's + Example.

Conservation law :

weak solution.

travelling waves.

Entropy condition.

general RP for system.

existence for scalar law.

Vortex sheets:

Hilbert transform.

Linearized instability:

Kelvin-Helmholtz

Rayleigh - Taylor

Birkhoff - Rott.

development of singularities.

analytic existence.

Do conservation of
energy & Lagrangian
formulation.

Orals - Fluids

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① Acoustic waves 1D.

(i) $\rho_t + (\rho u)_x = 0$ conservation of mass.

derivation $\frac{d}{dt} \int_x^y \rho(s, t) ds = 0$ x, y move with the flow

$$\Rightarrow \int_x^y \rho_t + \rho u \Big|_x^y = 0$$

$$\Rightarrow \int_x^y [\rho_t + (\rho u)_x] = 0.$$

(ii) $\rho u_t + \rho u u_x + \frac{\partial p}{\partial x} = 0$ conservation of momentum.

derivation $\frac{d}{dt} \int_x^y \rho u = -p \Big|_x^y$

$\Rightarrow (\rho u)_t + (\rho u^2)_x + p_x = 0$ ← conservation form.

$\Rightarrow \rho u_t + u(\rho_t + \rho u_x + \rho_x u) + \rho u u_{xx} + p_{xx} = 0$ → zero by (i).

(iii) $p(\rho)$ isentropic.

Normally $p(\rho, S)$ but assuming S is constant.

Do general transport theorem. $\rightarrow$

(iv) let $p = p_0 + \tilde{p}$; $p = p_0 + \tilde{p}$; $u = \tilde{u}$.

Linearize $\Rightarrow$

$$\tilde{p}_t + p_0 \tilde{u}_x = 0 \quad < \frac{\partial}{\partial x} \times c^2$$

$$p_0 \tilde{u}_t + \frac{\partial p}{\partial p}(p_0) \tilde{p}_x = 0 \quad < \frac{\partial}{\partial t}$$

$c^2 < \text{true for physical reasons.}$

$$\Rightarrow \tilde{u}_{tt} - c^2 \tilde{u}_{xx} = 0 \quad \text{Wave equation.}$$

② Kelvin's Law.

(i) Transport Theorem for curves:

$$\frac{d}{dt} \int_{C_t} u \cdot ds = \int_{C_t} \frac{Du}{Dt} ds.$$

$\phi(x(s), t)$ describes C_t , a simple closed curve that follows the fluid.

$$\begin{aligned} \frac{d}{dt} \int_{C_t} u \cdot ds &= \frac{d}{dt} \int_0^1 u(\phi(x(s), t), t) \frac{\partial}{\partial s} \phi(x(s), t) ds \\ &= \int_0^1 \frac{Du}{Dt} \frac{\partial \phi}{\partial s} ds + \int_0^1 u \cdot \frac{\partial}{\partial s} u ds \quad \text{integrates to zero.} \end{aligned}$$

Why can closed loops never develop crossings? →

(ii) so $\frac{d}{dt} \int_{C_t} u \cdot ds = \frac{1}{\rho_0} \int_{C_t} -\nabla(p + \frac{1}{2}u^2) \cdot ds = 0$

Since the loop is closed (Stokes theorem).

works as long as $\nabla \times \frac{Du}{Dt} = 0$.

- ie if p isentropic, $\frac{\nabla p}{\rho} = \nabla \int \frac{dp}{\rho}$.

Applications:

- to Lagrange's Thm. If a flow starts out irrotational then it remains irrotational and can be described by a potential flow.

Notes: • This works for all simply closed curves immersed in the fluid (ie that have a surface S in the fluid with the curve as boundary).

• Lagrange's Theorem - any flow beginning as a potential flow remains

so always.

③ Potential Flow (Bernoulli's law).

Euler flow - incompressible, inviscid.

$$S_t + (S \cdot \nabla) S = -\frac{1}{\rho} \nabla p + \nabla \Omega \quad \leftarrow \begin{array}{l} \text{external} \\ \text{forces} \end{array}$$

$\nabla \cdot S = 0$ incompressible.

$$\text{Let } \psi = \int_0^x S \cdot d\mathbf{l} \Rightarrow S = \nabla \psi.$$

ψ is uniquely determined if the fluid is irrotational (Stokes law). In 2D, need circulation around bodies zero as well to determine single valued ψ .

Now $(S \cdot \nabla) S = \frac{1}{2} \nabla (|\nabla \psi|^2)$ compute backwards to show equality.

continuity $\Rightarrow \Delta \psi = 0$. or $\int \frac{dp}{\rho}$ in isentropic case.

$$\text{Euler} \Rightarrow \nabla \cdot \left(\psi_t + \frac{1}{2} |\nabla \psi|^2 + \frac{p}{\rho} - \Omega \right) = 0.$$

$$\text{so } \psi_t + \frac{1}{2} |\nabla \psi|^2 + \frac{p}{\rho} - \Omega = 0.$$

$\uparrow$
absorb $c(t)$ into the pressure.

$$u_t - w \times u + \nabla \left(\frac{q^2}{2} + i \right) = T \text{grad } S.$$

$$i = \int \frac{c^2(p)}{\rho} dp \quad \text{for isentropic case.}$$

alternate form of Euler

$$u_t - w \times u + \nabla \left(\frac{q^2}{2} + i \right) = 0 \quad \leftarrow T \text{grad } S.$$

if $w=0$ OK. Otherwise if time independent then $\frac{q^2}{2} + i$ is constant along vortex and streamlines.

**examine weaker forms of this $\rightarrow$
along streamlines.** (see above).

Need a better explanation for wing lift. →

Recall, f must satisfy $\int_{\partial\Omega} f = 0$. →

Do some work on this. →
Also, look at examples of 3D potential flow.

Consider also compressible but isentropic flow. →

along streamlines.

For steady flow $\psi_t = 0$ and

$$\frac{|u|^2}{2} + \frac{p}{\rho} + gz = 0.$$

Applications:

- pressure gauges in torpedos to calculate depth
- calculate pressure after potential flow is found
- explanation of wing lift: velocity increases, pressure decreases.

Notes • Potential flows in a bounded domain ($\frac{\partial\psi}{\partial n} = f$ on boundary) are uniquely determined.

• For exterior domain, need velocity prescribed at ∞ .

④ Complex Flow (incompressible, irrotational).

Define $\psi = \int_0^x u \cdot dl \Rightarrow u = \nabla\psi$.

uniquely determined by Stokes' law.

Simply
connected
domain.

$$\psi = \int_0^x \underline{u \cdot n}$$

Get more of a clue on this →
Why conformal (?)

Boundary condition $\psi = \text{constant}$.

Define $\psi = \int_0^x u dy - v dx \Rightarrow u = \psi_y, v = -\psi_x$.
uniquely defined by incompressibility
(by divergence theorem).

Note: differences in ψ between two points
is the flux through a curve joining
them.

$\zeta = \psi + i\psi$ satisfies C-R so analytic.

ψ constant along streamlines

Also, $\frac{d\zeta}{dz} = u - iv$.

Can calculate the pressure from Bernoulli's
law.

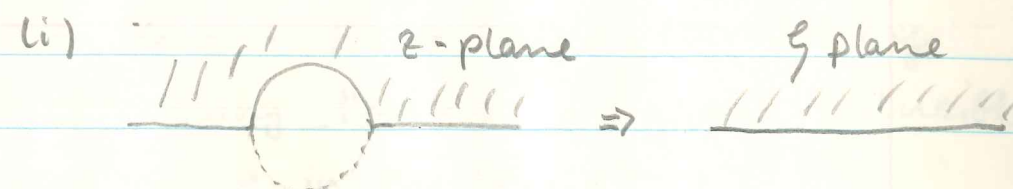
$\zeta(z)$ is a conformal mapping - taking
streamlines into straight lines.

$\zeta(z) = z + \dots$ when flow is in the
horizontal direction at ∞ .

this matches general formula
of Shimbrot. $\rightarrow$

Consider cylinder with circulation $\rightarrow$
around it + stagnation points.

⑤ Examples of complex Flow:

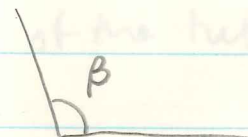


$$\zeta = z + \frac{1}{z} \Rightarrow u - iv = 1 - \frac{1}{z^2}$$

$$= 1 - \frac{1}{r^2} e^{-2i\theta}$$

$$\text{so } u = 1 - \frac{1}{r^2} \cos 2\theta$$

$$v = -\frac{1}{r^2} \sin 2\theta$$

(ii)  $\zeta = z^{\pi/\beta}$

$$W = \frac{d\zeta}{dz} = \frac{\pi}{\beta} z^{\pi/\beta - 1}$$

if $\beta < \pi$ $q = |W| \rightarrow 0$ at 0
otherwise $q \rightarrow \infty$.

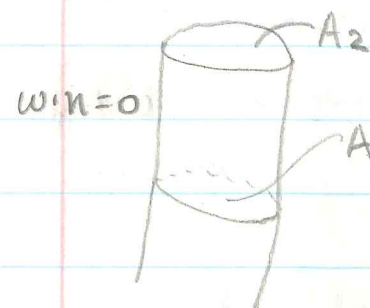
(iii) $\zeta = \log(z - z_0)$ source.

(iv) $\zeta = \log\left(\frac{z - z_1}{z - z_2}\right)$ source at z_1 ,
sink at z_2 .

(v) $\zeta = i\Gamma \log(z)$ point vortex.

⑥ Helmholtz's theorem:

vortex line - vortex tube (formed by lines passing through a given reducible closed curve in the fluid).

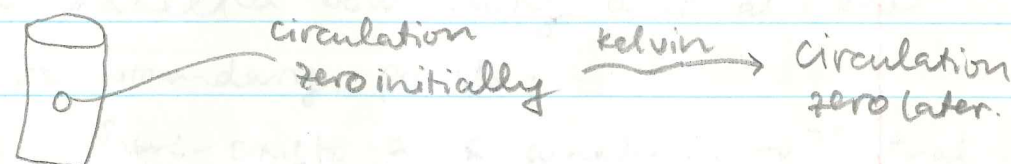


$$\text{Strength of the tube} = \int_A w \cdot n \, dA = \int_C \overset{\text{Stokes}}{u} \cdot dx$$

independent of surface since

$$\int_{A_2} w \cdot n \, dA - \int_A w \cdot n \, dA = \int \overset{\substack{\uparrow \\ \text{divergence} \\ \text{theorem}}}{\nabla \cdot w} \, dV = 0$$

Helmholtz - A vortex tube moves with the fluid and its strength remains constant.



material +
vortex tube
coincide initially.

Therefore at a later time, $w \cdot n = 0$ on the surface of the material tube so it must be a vortex tube.

Invariance of circulation of curves around the tube $\Rightarrow$ tube has constant strength.

Shrink tube to zero cross-section (or consider tube intersections) $\Rightarrow$ vortex lines move with the fluid.

Note A vortex tube cannot end in the interior of the fluid.

⑦ Blasius' Theorem.

(i). Let V be an exterior domain with a connected boundary and at least two boundary points. Let

There exists a k , analytic on V , that maps V conformally onto the exterior of the unit disk K with simple pole at ∞

with $k'(\infty) > 0$.

A complex potential for V is

velocity
at ∞ .

$$\rightarrow \frac{U - iV}{k'(\infty)} \left[k(z) + \frac{1}{k(z)} \right] - \frac{\gamma}{2\pi i} \log k(z).$$

Note • Can consider a weak potential, where $\gamma(z) \rightarrow \text{constant}$ as $z \rightarrow \infty$ on ∂V .

• ∂V gets mapped to a line segment on the real axis.

(ii) Force on an object in the flow

$$F = - \oint p \underline{n} \, ds \quad \text{note } \underline{n} \, ds = -i dz$$

rotate by $-\pi/2$

$$F = i \oint p \, dz$$

$$F = \frac{i\rho}{2} \oint q^2 \, dz \quad \left(\text{since } \frac{p}{\rho} + \frac{q^2}{2} = 0 \right)$$

Bernoulli.

$$\text{so } \bar{F} = \frac{i\rho}{2} \oint q^2 \, d\bar{z}$$

$$\bar{F} = \frac{i\rho}{2} \oint w^2 \, d\bar{z}.$$

$$\begin{aligned} \text{use } w &= q e^{-i\theta} \\ q^2 &= w^2 e^{+2i\theta} \\ d\bar{z} &= ds e^{-i\theta} \\ dz &= ds e^{+i\theta} \end{aligned}$$

check residue formulas. $\rightarrow$

There is a subtlety when boundary $\rightarrow$
of object is not smooth.

Need to derive this... $\rightarrow$

(iii) Now evaluate integral in (ii) using form of W from (i) - evaluate as a pole at ∞ (take $V=0$).

$$k = k'(\infty)z + \frac{C_1}{z} + \frac{C_2}{z^2} + \dots$$

$$W = U - \frac{\gamma}{2\pi i} \frac{1}{z} + \dots$$

$$\therefore W^2 = -\frac{2\gamma U}{2\pi i} \cdot \frac{1}{z} + \dots$$

Evaluating (ii) by residues gives

$$D - iL = -i\gamma XU$$

$$\text{so } L = \gamma XU \text{ and } D = 0.$$

⑧ 2D Vortex methods (No boundary).

$\leftarrow$ - Lagrangian formulation.

(i) Helmholtz's equation

$$\frac{Dw}{Dt} = w \cdot \nabla u \quad \Rightarrow \quad \frac{Dw}{Dt} = 0 \text{ in 2D.}$$

Review how to show that u $\rightarrow$
is smooth and $\nabla \times u = \omega$.

with $\frac{D\omega}{Dt}$, this is the $\nearrow$
vorticity, streamfunction formulation
of the Euler equations.

Needs $\frac{\partial J}{\partial t} = J \operatorname{div} u$ proof later $\rightarrow$
in Transport theorem.

Review how to get Jacobian in $\rightarrow$
change of variables for integral.

$$(ii) \quad \nabla \cdot u = 0 \Rightarrow u = \nabla \times \psi = \left(\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x} \right).$$

$$\text{compute } \Delta \psi = -\omega$$

$$\therefore u(x) = \int K(x-x') \omega(x') dx'$$

$$\text{where } K = \nabla \times G = \frac{1}{2\pi|x|^2} (-x_2, x_1).$$

$\uparrow$
 $\frac{1}{2\pi} \log r$ Green's fn for Laplacian.

Lagrangian flow map

$$x(\alpha, 0) = \alpha$$

$$\frac{dx}{dt}(\alpha, t) = u(x(\alpha, t), t).$$

$$\omega(x(\alpha, t), t) = \omega_0(\alpha) \text{ by } \frac{D\omega}{Dt} = 0.$$

$$dx = d\alpha \text{ (by incompressibility).}$$

$$\Rightarrow \frac{dx}{dt}(\alpha, t) = \int K(x(\alpha, t) - x(\alpha', t)) \omega_0(\alpha') d\alpha'.$$

$$\text{initial conditions } x(\alpha, 0) = \alpha.$$

(iii) Discretize $\Lambda = \{\alpha_i = h \cdot i\}$ suppose w_0 has compact support. $\tilde{x}_i(0) = \alpha_i$

$$\frac{d}{dt} \tilde{x}_i(t) = \sum_{\substack{j \in \Lambda \\ j \neq i}} K(\tilde{x}_i(t) - \tilde{x}_j(t)) w_j h^2$$

Point vortex method

(iv) Replace K by $K * f_\delta$ to get vortex blob method.

Advantages:

- doesn't introduce artificial viscosity
- $V = \int w$ so good approximation of w gives an even better approx of V .
- For many flows the vorticity is the most important feature of the fluid motion (vortices tend to concentrate in areas of activity).

(v) Random vortex method (split step)
Add random walk - Gaussian $\sqrt{\nu \delta t}$.

vortex stretching

→

⑨ 3D vortex method (No boundary).

(i) $w(\alpha, t) = \nabla_{\alpha} x \cdot w_0(\alpha, 0)$

derivation:

$$\frac{\partial x}{\partial \alpha} \cdot \frac{\partial \alpha}{\partial x} = I \quad \frac{d}{dt} x(\alpha, t) = u.$$

$$\frac{d}{dt} \left(\frac{\partial x}{\partial \alpha} \right) \frac{\partial x}{\partial \alpha} + \underbrace{\frac{\partial x}{\partial \alpha} \frac{d}{dt} \left(\frac{\partial x}{\partial \alpha} \right)}_{\frac{\partial x}{\partial \alpha} \cdot \frac{\partial u}{\partial \alpha} = \nabla u} = 0$$

$$\therefore \frac{d}{dt} \left(w \cdot \frac{\partial x}{\partial \alpha} \right) = \frac{dw}{dt} \cdot \frac{\partial x}{\partial \alpha} - w \cdot (\nabla u \frac{\partial x}{\partial \alpha})$$

↓ Helmholtz

$$= (w \cdot \nabla u) \frac{\partial x}{\partial \alpha} - w \cdot \nabla u \frac{\partial x}{\partial \alpha} = 0.$$

$\therefore w \cdot \frac{\partial x}{\partial \alpha}$ constant

(ii) $K(x-x') = \nabla x \cdot \frac{1}{|x-x'|} \xrightarrow{1/4\pi}$

$$K(x-x') = -\frac{1}{4\pi|x-x'|^3} \begin{pmatrix} 0 & x_3 - x_2 \\ -x_3 & 0 & x_1 \\ x_2 & -x_1 & 0 \end{pmatrix}$$

discrete approximation of (i) $\rightarrow$

(iii) As in (8),

$$\frac{d\tilde{x}_i}{dt} = \sum_{j \neq i} K_{ij} (\tilde{x}_i(t) - \tilde{x}_j(t)) \tilde{w}_j(t).$$

$$\text{compute } \tilde{w}_j(t) = \nabla_{\alpha}^h \tilde{x}(\alpha_j, t) \cdot w_0(\alpha_j).$$

(iv) vortex filaments.

Suppose $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and that the material lines corresponding to α_2, α_3 constant correspond initially to vortex lines.

$$\therefore w = c(\alpha) \frac{\partial x}{\partial \alpha_1} \text{ initially.}$$

But since $w \frac{\partial x}{\partial \alpha_1}$ is constant, this relationship persists.

(10) Existence theorems:

(i) 2D Euler flow. need smooth initial data.

Kato smooth, classical solution for all time (allows boundaries).

Yudovich - weak solution (suitable for vortex patches).

look at the proof of this theorem. $\rightarrow$

Even weaker solutions by DiPerna & Majda - measure valued (these are not necessarily unique).

(ii) 3D. Kato proves local existence, if $u_0 \in H^s$ with $s \geq 3$ then the Euler equations have a solu in $H^s [0, T]$ with $T(\|u_0\|_s)$.

↑
with some time smoothness.

(iii) Some work in existence of periodic solutions with periodic forcing (N-S).

(iv) Existence of weak solutions to N-S equations in 3D - Shinbrot. Solutions may suffer from lack of uniqueness and jump decreases in the energy.

② Stability + Consistency $\Rightarrow$ convergence.

Model equation $\dot{x} = g(x)$ ODE.

x_n exact solution

$$\tilde{x}_{n+1} = \tilde{x}_n + \Delta t g(\tilde{x}_n).$$

error

$$e_n = |x_n - \tilde{x}_n| \leq e_n^c + e_n^s \quad \begin{array}{l} \text{consistency} \\ \text{stability} \end{array}$$

$$e_n^c = |x_n - x_n^*| \quad \begin{array}{l} x_n^* = \\ x_{n-1} + \Delta t g(x_{n-1}) \end{array}$$

consistency at each step.

$$e_n^s = |x_n^* - \tilde{x}_n| \quad \text{growth of consistency error.}$$

$$\begin{aligned} \text{(i)} \quad e_n^c &= |x_n - x_{n-1} - \Delta t g(x_{n-1})| \\ &= \left| \int_{(n-1)\Delta t}^{n\Delta t} [g(x(t)) - g(x((n-1)\Delta t))] dt \right| \end{aligned}$$

$$\leq C \Delta t^2$$

$$\begin{aligned} \text{(ii)} \quad e_n^s &= |x_{n-1} + \Delta t g(x_{n-1}) - x_{n-1} - \Delta t g(\tilde{x}_{n-1})| \\ &\leq (1 + C \Delta t) e_{n-1} \end{aligned}$$

Stability of numerical procedure.

c is related to the lipshitz constant for g .

$o(\Delta t)$ would be sufficient.

(iii) combine $e_n \leq C(\Delta t)^2 + (1+C\Delta t)e_{n-1}$ $e_0 = 0$.
 $S_n = (1+C\Delta t)^{-n} e_n$ $S_0 = 0$.

$\Rightarrow S_n - S_{n-1} \leq C(\Delta t)^2 (1+C\Delta t)^{-n}$
 sum (telescoping) $\Rightarrow$

$$S_n \leq (\Delta t) (1 - (1+C\Delta t)^{-n}).$$

or $e_n \leq (\Delta t) [(1+C\Delta t)^n - 1]$.

take $n = T/\Delta t$. In the range $t \in [0, T]$,

$$e_n \leq \Delta t [e^{CT} - 1].$$

② Chorin's Algorithm.

(i) Prandtl Boundary layer equations.

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\partial_x P + \frac{1}{R} \partial_y^2 u.$$

$$\frac{\partial p}{\partial y} = 0 \quad \text{Get } p \text{ from outer flow.}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \begin{array}{l} \text{get } v \text{ from } u \\ u = v = 0 \text{ on } x\text{-axis} \\ u \text{ given at } y = \infty. \end{array}$$

derivation relies on the fact that $\rightarrow$
 ν small, but $\frac{\partial}{\partial y}$ large.

discuss derivation $\rightarrow$

$$w = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

$$w = -\frac{\partial u}{\partial y} \text{ in Prandtl.}$$

$$\frac{Dw}{Dt} = \frac{1}{R} \frac{\partial^2 w}{\partial y^2}$$

Prandtl equation
for vorticity.

$$u = \int_0^y w dy \quad u(\infty) \text{ given} \Rightarrow \text{implicit condition on boundary values for vorticity.}$$

$$v = -\int_0^y \frac{\partial u}{\partial x} dy.$$

(ii) Algorithm:

(A) Approximate overall flow at $t=0$ by a collection of N vortex sheets of finite width h , extending from $x_i - h/2$ to $x_i + h/2$ with y coordinate y_i and with strength w_i .

(B) Compute velocity as follows:

$$u = u(\infty) - \int_{-\infty}^y w dy$$

$$\Rightarrow u(x_i, y_i) = u(\infty) - \sum_j w_j$$

sum over the sheets above that overlap the centre.

there is a convenient discrete $\rightarrow$
formula for this.

reflection or $2u_e$??

$$V = -\frac{\partial}{\partial x} \int_0^y u(x, s, t) ds.$$

$$\Rightarrow V = -\frac{1}{h} \left(\int_0^y u(x+h/2, s, t) ds - \int_0^y u(x-h/2, s, t) ds \right)$$

Now transport the vortex sheets with this velocity.

③ Add vortex sheets on the boundary with intensity $2u_e$.

④ Vortex sheets undergo a gaussian random walk with variance $\gamma \Delta t$ - with reflection.

⑬ Proof of convergence of vortex blob method.

(i) Choose f smooth, vanishing moments, decay at ∞ .

$$(ii) V[x]_i := \sum_{j \in \Lambda^h} K \delta(x(\alpha_i) - x(\alpha_j)) w_j h^2.$$

$$2D \text{ vortex blob method } \frac{d}{dt} \tilde{x}_i(t) = V[\tilde{x}]_i.$$

$$\text{Velocity error } \underset{\substack{\text{L}^2\text{-norm} \\ \downarrow}}{\|\dot{\tilde{x}} - V[\tilde{x}(t)]\|} \leq \|e_c\| + \|e_s\|$$

$$e_c = |\tilde{x} - V[x]| \quad \text{consistency}$$

$$e_s = |V[x] - V[\tilde{x}]| \quad \text{stability error.}$$

(would be zero if approximate positions were the same as exact).

$$e_c \leq e_d + e_m \quad \begin{array}{l} \nearrow \text{moment error} \\ \searrow \end{array}$$

$$\sum K_\delta - \int K \quad \text{discretisation error.}$$

Moment lemma

$$\|e_m(t)\| \leq C_m \delta^p \quad \begin{array}{l} \nearrow \text{number of vanishing} \\ \searrow \text{moments.} \end{array}$$

Discretization lemma

$$\|e_d(t)\|_h \leq C_d \left(\frac{h}{\delta}\right)^L \delta \quad \begin{array}{l} \text{uses smoothness} \\ \text{of } F. \end{array}$$

Stability lemma

$$\text{if } \|\tilde{x}(t) - x(t)\|_h \leq h\delta \quad \text{for all } t \in [0, T] \quad \text{then}$$

$$\|V[\tilde{x}(t)] - V[x(t)]\| \leq C_s \|\tilde{x}(t) - x(t)\|$$

discretization error is of
order h^L , but constant goes like $\frac{1}{\delta^L}$.
so need $\delta \gg h$.

for h/δ small, get convergence in position and velocity.

- can extend to 3D, random vortex blob.

(14) Hou's point vortex convergence method.

x - point vortex positions v - velocities
 y - exact positions u - velocities.

$$(i) \|x(t) - y(t)\| \leq C(T, p) h^2$$

$$\|v(t) - u(t)\| \leq C(T, p) h^2$$

need
 $4 < p < \infty \rightarrow$

(ii) suppose $\|x(t) - y(t)\| \leq h^{2-1/4}$ for $0 \leq t \leq T$
 then there exists an $A(T, p)$ so that

$$\|v(t) - u(t)\| \leq A(T, p) \|x(t) - y(t)\| + A(T, p) h^2$$

Note: (ii) $\rightarrow$ (i).

(15) Uniqueness of Euler flow in bounded domain.

Suppose two solutions, let s & p denote the differences, then

$$s_t + (s_1 \cdot \nabla) s + (s \cdot \nabla) s_2 = -\nabla p / \rho. \quad (*)$$

$$\nabla \cdot s = 0$$

$$n \cdot s = 0 \text{ on } \partial V.$$

$$s = 0 \text{ for } t = 0.$$

multiply $(*)$ by s and integrate

$$\frac{d}{dt} \int_V |s|^2 + \int s \cdot \nabla s_2 \cdot s = -\frac{1}{\rho} \int s \cdot \nabla p - \frac{1}{2} s_1 \cdot \nabla |s|^2$$

$$\text{RHS} = -\int \nabla \cdot \left(\frac{1}{\rho} p s + \frac{1}{2} |s|^2 s_1 \right) = -\int_{\partial V} n \cdot \left(\frac{1}{\rho} p s + \frac{1}{2} |s|^2 s_1 \right)$$

$$= 0.$$

$$\therefore \frac{d}{dt} \int_V |s|^2 \leq \sup_V |\nabla s_2| \int_V |s|^2 dq.$$

This gives a uniqueness & stability result for the Euler equations.

⑩ (Projection idea).

Any vector field w on D can be !
decomposed in the form

$$w = u + \text{grad } p \quad (*)$$

where $\underline{w} = \underline{u} + \nabla p$, and u is in the
class $A = \{u: \nabla \cdot u = 0 \text{ \& } u \cdot n = 0 \text{ on } \partial D\}$.

(i) $A \perp$ to gradients

$$\int u \cdot \nabla p = \int \text{div}(pu) \, dv \\ = \int p \, u \cdot n = 0$$

$\Rightarrow$ uniqueness of decomposition $(*)$

(ii) Take divergence of $(*)$.

$$\text{div } w = \Delta p, \quad w \cdot n = \frac{\partial p}{\partial n}.$$

Let p solve the Neumann problem.

also, $w - \text{grad } p$ is divergence free

with $u \cdot n = 0$ on boundary.

(iii) P projection onto A (onto divergence
free part with $u \cdot n = 0$).

Apply P to N-S

$$\frac{\partial u}{\partial t} = P \left(-u \cdot \nabla u + \frac{1}{R} \Delta u \right).$$

recover pressure $\nabla p = (I - P) (-u \cdot \nabla u + \frac{1}{R} \Delta u)$

(17) Proof of Kato's 2D Theorem (sketch).

with

$$\varphi(x, 0)$$

$$= \nabla \times u_0$$

Start with arbitrary scalar $\varphi(x, t)$ Δ_0^{-1}
Determine tangential flow $v = \text{rot } G\varphi$

Then solve $D_t w = 0$ with $w(\cdot, 0) = \nabla \times u_0$
using this flow. Denote the process
as $w = F[\varphi]$ on a class $\mathcal{S}$.

Choose $\mathcal{S}$ so that F maps $\mathcal{S}$ into itself
and is a convex, compact set - apply
Schauder's fixed point theorem.

$\mathcal{S}$ is a class of Hölder continuous fn's
with bounded Hölder norms $\Rightarrow$ compact
in uniform topology.

Note: $\nabla \times (u_t + u \cdot \nabla u) = 0$ (gives equation for vorticity which is satisfied) $\Rightarrow$

$$u_t + u \cdot \nabla u = \nabla p \quad \text{for some } p \checkmark.$$

⑮ Young's Theorem - Let K be a measurable fn on $\mathbb{R}^n \times \mathbb{R}^n$ such that

$$\int |K(x, y)| dy \leq C \quad \& \quad \int |K(x, y)| dx \leq C \quad \forall x, y.$$

If $TF(x) = \int K(x, y) F(y) dy$, then

$$\|TF\|_p \leq C \|F\|_p.$$

PF $|TF(x)| \leq \int |K|^{1/q} |K|^{1/p} |F(y)| dy$

$$\leq \underbrace{\left\{ \int |K| \right\}^{1/q}}_{\leq C^{1/q}} \left\{ \int |K| |F(y)|^p dy \right\}^{1/p}$$

$$\begin{aligned} \therefore \int |TF(x)|^p dx &\leq C^{p/q} \iint |K(x, y)| |F(y)|^p dy dx \\ &\leq C^{1+p/q} \int |F(y)|^p dy \\ &= C^{1+p/q} \|F\|_p^p. \end{aligned}$$

Application: convolution of $g \in L^1$ and $h \in L^p$ is in L^p and

$$\|g * h\|_p \leq \|g\|_1 \|h\|_p.$$

does not work for $p=1!$ $\rightarrow$

This is wrong! $\rightarrow$

works for ∇K where

$$\nabla K(\alpha x) = \frac{1}{\alpha^2} \nabla K(x) \checkmark$$

①9 Calderon-Zygmund.

Integral operators with kernel K that satisfy $\hat{K}(\alpha \underline{k}) = \hat{K}(\underline{k})$ are bounded on L^p $1 < p < \infty$

Application - to B-S kernel. In this case, $K(\alpha x) = \frac{1}{\alpha^2} K(x)$ which gives the necessary condition on $\hat{K}$.

$$\begin{aligned} \text{proof in } \mathbb{R}^2: \|K * f\|_2 &= \|\widehat{K * f}\| \\ &= \|\hat{K} \hat{f}\| \leq \underset{\substack{\uparrow \\ \text{bounded}}}{C} \|\hat{f}\| = C \|f\|. \end{aligned}$$

(works on each component).

②0 Vortex-stream function methods. (Israeli - Orszag).

Consider Stokes flow for simplicity

$$\nabla \cdot u = 0 \quad u_t = \nu \Delta u - \nabla p / \rho$$

or $w_t = \nu \Delta w$ and $\Delta \psi = -w$ with $\psi = 0$ and $\frac{\partial \psi}{\partial n} = 0$ on the boundary.

The idea of the method:

interior $w^{n+1} = w^n + \nu \Delta t \nabla^2 w^n$,
 with $\psi = 0$ on $\rightarrow$ solve $\nabla^2 \psi^{n+1} = w^{n+1}$,
 boundary Now $w_B = (\nabla^2 \psi)_B = \frac{\partial^2 \psi}{\partial y^2}$ (flat plate case)
 $\uparrow$ boundary $\uparrow$
 needs elements outside computational domain, but we can use $\frac{\partial \psi}{\partial y} = 0$.

(21) (Projection method).

$$\frac{Dw}{Dt} = \nu \Delta w$$

with $\Delta \psi = -w$ and $u = (-\frac{\partial \psi}{\partial y}, \frac{\partial \psi}{\partial x})$.

$\psi = 0, \frac{\partial \psi}{\partial n} = 0$ on boundary.

$\uparrow$

Solve for ψ with this condition, in order for the other to be true, it is necessary & sufficient that

$$\int_{\Omega} w(x, t) h(x) dx = 0 \quad \forall \text{ harmonic } h.$$

to prove, start with $0 = \int \Delta h \psi \rightarrow$

can use this to get extra equations for the w_0 .

(22) Anderson's method

$$\psi(x) = - \int_{\Omega} G(x,s) w(s,t) ds$$

↑
for Δ with Dirichlet b.c.'s.

Need $\frac{\partial \psi}{\partial n} = 0 \Rightarrow$

$$\frac{\partial}{\partial n} \int_{\Omega} G(x,s) w(s,t) ds = 0 \quad x \in \partial\Omega.$$

Starts off zero - take time derivative.
(use PDE). $\Rightarrow$

$$\frac{\partial}{\partial n} \int_{\Omega} G(x,s) [\nabla \Delta w - U \cdot \nabla w] ds = 0.$$

↑
by parts.

$$\begin{aligned} \Rightarrow \frac{\partial w}{\partial n} + \frac{\partial}{\partial n} \int_{\partial\Omega} \frac{\partial G}{\partial n}(x,s) ds \\ = \frac{1}{\nu} \frac{\partial}{\partial n} \int_{\partial\Omega} G(x,s) [U \cdot \nabla w(s,t)] ds. \end{aligned}$$

use discrete equations so that
discrete version remains zero -
better for steady state
calculations.

Example: (Prandtl flow)

$$u = u_0 + \int_y^\infty w(x, s, t) ds \quad (w = -\frac{\partial u}{\partial y}).$$

$$v = -\frac{\partial}{\partial x} \int_0^y u(x, s, t) ds.$$

$u=0$ at $y=0$ not automatically satisfied,

need
$$\int_0^\infty w(x, s, t) ds = -u_0.$$

Assume valid, take time derivative, use PDE as before $\Rightarrow$

$$\int_0^\infty (\nu \frac{\partial^2 w}{\partial y^2} - u \cdot \nabla w) = 0$$

$$\text{or } \frac{\partial w}{\partial y} \Big|_{y=0} = -\frac{1}{\nu} \int_0^\infty (u \cdot \nabla w) ds.$$

$\uparrow$
use to get the boundary values for w .

review formal arguments
For higher order schemes.

$\rightarrow$

(23) Transport theorem

$$(i) \frac{dJ}{dt} = J \operatorname{div} u \quad (J = \left| \frac{\partial x}{\partial \alpha} \right|).$$

$$\frac{dJ}{dt} = \begin{vmatrix} \frac{\partial}{\partial \alpha_1} u & \frac{\partial x}{\partial \alpha_2} & \frac{\partial x}{\partial \alpha_3} \\ \frac{\partial}{\partial \alpha_2} u & \frac{\partial y}{\partial \alpha_2} & \vdots \\ \frac{\partial}{\partial \alpha_3} u & \frac{\partial z}{\partial \alpha_2} & \vdots \end{vmatrix} + \dots$$

$$\frac{\partial u}{\partial \alpha_i} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial \alpha_i} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial \alpha_i} + \dots$$

l. dep. to other rows,
no contribution

$$\text{so } \frac{dJ}{dt} = \frac{\partial u}{\partial x} J + \frac{\partial u}{\partial y} J + \frac{\partial u}{\partial z} J = J \operatorname{div} u.$$

(ii) Flow incompressible $\Rightarrow J$ constant

(iii) Transport theorem:

$$\frac{d}{dt} \int_{W_t} \rho f \, dv = \int_{W_t} \rho \frac{Df}{Dt} \, dv.$$

to show, compute as follows:

$$\begin{aligned} \frac{d}{dt} \int_{W_t} \rho f \, d\mathbf{x} &= \frac{d}{dt} \int_{W_0} \rho f J \, d\mathbf{x} \\ &= \int \left(\frac{D\rho}{Dt} f + \rho \frac{Df}{Dt} \right) J + \rho f J \operatorname{div} \mathbf{u} \\ &\quad \underbrace{\hspace{10em}}_{=0 \text{ by continuity equation.}} \end{aligned}$$

$$= \int_{W_t} \rho \frac{Df}{Dt} \, d\mathbf{x}.$$

(24) Energy estimate:

$$S_t + S \cdot \nabla S + \nabla p = \nu \Delta S \quad S=0 \text{ on } \partial\Omega.$$

multiply by S & integrate

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int |S|^2 + \nu \int |\nabla S|^2 &\quad \frac{1}{2} S \cdot \nabla |S|^2 = \nabla \cdot (S |S|^2) = 0 \\ &= - \int S \cdot \nabla p - \int S \cdot (S \cdot \nabla S) \\ &\quad \underbrace{\hspace{1em}}_{\int \nabla \cdot (pS)} \text{ by incompressibility} \\ &= \int_{\partial\Omega} p S \cdot \underline{n} = 0 \end{aligned}$$

$$\therefore \frac{1}{2} \frac{d}{dt} \int |S|^2 + \nu \int |\nabla S|^2 = 0.$$

for most known solutions, $\rightarrow$

$$u \cdot \nabla u = 0.$$

Couette flow is really $p=0$, $u(y)=y$
($\psi = y^2/2$).

(25) Couette Flow.

$$\begin{array}{c} 1 \text{ --- } v, u=0 \\ u(y) \\ v=0. \\ 0 \text{ --- } v, u=0 \end{array}$$

> can get from one to the other from momentum equation

momentum equation for $v \Rightarrow p_y = 0$. so $p(x)$.

Now $\nabla u''(y) = p'(x) = K$, so

$$p = Kx$$

$$u = -\frac{K}{2\nu} y(1-y).$$

(26) Steady Flow in a pipe.

$$v, w=0 ; u(y, z).$$

(continuity satisfied) and $p(x)$ as in (25)

$$u_{yy} + u_{zz} = \frac{1}{\nu} p_x = K.$$

or $\Delta' u = K$ Poisson's equation with constant RHS.

circular pipe, $u(r) \Rightarrow$

$$\Delta' u = u_{rr} + \frac{1}{r} u_r = K$$

time dependent!

→

boundary layer of thickness $\sqrt{\nu t}$ →

solution is $u = \alpha + \beta \log r + \frac{kr^2}{4}$.

$$\Rightarrow u = \frac{k}{4} (r^2 - R^2).$$

(27) Flow past an impulsively started flat plate.

$V=0 \Rightarrow u(y)$ as before.

$u \cdot \nabla u$ term vanishes. assume $p=0$.

$$\Rightarrow \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial y^2}.$$

velocity (assume plate) at ∞ $\frac{y}{\sqrt{t}}$

Similarity, let $u = V F(\eta)$

equation becomes $-\frac{1}{2} \frac{y}{t^{3/2}} F' = \frac{1}{t} F''$.

or $F'' + \frac{1}{2} \eta F' = 0$. Boundary conditions $F(\infty) = 1$ $F(0) = 0$.

$$F' = ce^{-\eta^2/4}$$

$$\therefore F = \frac{1}{\sqrt{\pi}} \int_0^\eta e^{-s^2/4} ds.$$

②⑧ solution to boundary layer past a semi-infinite plate.

- steady flow

- external flow is $Ux^{\hat{1}} \Rightarrow p = 0$.

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \partial_y^2 u.$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \Rightarrow \text{introduce stream}$$

f_{η} ,

$$\eta = \frac{y}{\sqrt{x}}.$$

- correct scaling $\psi(x, y) = \sqrt{x} f(\eta)$

gives equation $-2f''f = f'''$ (*) (momentum with boundary conditions

$$f(0) = 0$$

$$f'(0) = 0$$

$$f''(0) = 1 \text{ by some similarity argument.}$$

$$(*) \text{ gives } \frac{f'''}{f''} + 2f = 0$$

how do you get this
scaling? $\rightarrow$

where do these come
from? $\rightarrow$

or $F'' = e^{-2 \int_0^h f(\xi) d\xi}$

$F'' = e^{-\int_0^h (h-\xi)^2 F'' d\xi}$ integrate by parts using boundary values.

solve by iteration...

(29) Piston problem. Isentropic flow.

$$u_t + u u_x + p_x / \rho = 0 \quad p' = c^2$$

$$p_t + u p_x + \rho u_x = 0$$

or $\begin{pmatrix} p \\ u \end{pmatrix}_t + \underbrace{\begin{pmatrix} u & p \\ \frac{c^2}{\rho} & u \end{pmatrix}}_A \begin{pmatrix} p \\ u \end{pmatrix}_x = 0.$

A's eigenvalues (wave speeds) are

$$u \pm c.$$

$\Gamma^\pm$ are the characteristic curves

- they satisfy $\frac{dx^\pm}{dt} = u \pm c.$

Along these curves, the quantities

$$C_{\pm} = u \pm \int \frac{C(s)}{s} ds$$

are constant (check that $\nabla_u C_{\pm}$ are the eigenvectors of A).

Higher order systems don't necessarily have riemann invariants.

Simple wave is a region where one of C^+ or C^- are constant.

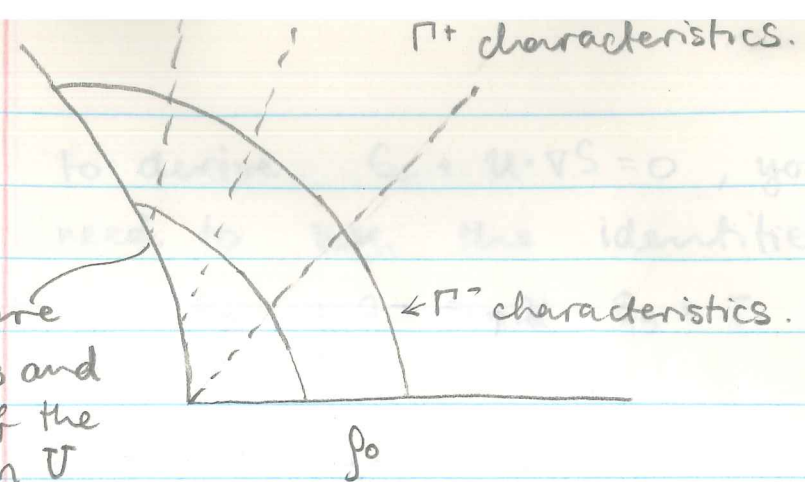
In a C^+ simple wave, the Γ^- characteristics are straight lines. (Note that a state next to a constant state must be a simple wave and the curve bounding a constant state must be a straight line).

Consider now the flow generated by a piston withdrawing from a tube filled initially with gas at rest with density ρ_0 .

and both u and p are constant. $\rightarrow$

draw the picture of
shock formation. →

gas here
has p_B and
speed of the
piston U



Also $U - \int_{p_B}^{p_0} \frac{c(s)}{s} ds = - \int_{p_0}^{p_B} \frac{c(s)}{s} ds$. (information carried on Γ^- characteristic)

Since U is negative, $p_0 > p_B$ hence this is called a rarefaction wave. Also, since $c'(p) > 0$ (genuine nonlinearity condition), the speed of the Γ^+ characteristics is decreasing through the fan.

(30) Equations of gas dynamics with derivation of equation of entropy.

$$p_t + (pu)_x = 0$$

$$u_t + uu_x + \frac{p_x}{\rho} = 0. \quad e = \frac{1}{2} \rho u^2 + p\varepsilon$$

$$\frac{\partial e}{\partial t} + ((e+p)u)_x = 0$$

$$\epsilon_r = -p.$$

→

$$\Delta = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial v}{\partial r} \right)$$

$$+ \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial v}{\partial \theta} \right)$$

$$+ \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 v}{\partial \phi^2}.$$

$$\nabla \psi = \left(\frac{\partial v}{\partial r}, \frac{1}{r} \frac{\partial v}{\partial \theta}, \frac{1}{r \sin \theta} \frac{\partial v}{\partial \phi} \right)$$

How does this
work?? ✓ OK.

→

to derive $S_t + u \cdot \nabla S = 0$, you
need to use the identities

$$\epsilon_p = \frac{p}{\rho^2} \quad \text{and} \quad \epsilon_s = T.$$

and the chain rule.

③ 3D- Potential Flow.

(i) $\psi = Ux$ uniform flow.

$\psi = U r \cos \theta$ in spherical coordinates.

← verify this integration
(ii) Combine (i) with $\psi = \frac{\cos \theta}{r^2}$
to get

$$\psi = U \left(r + \frac{1}{2r^2} \right) \cos \theta.$$

Flow past a sphere.

(iii) d'Alembert's paradox. In steady
potential flow, the net force on a bounded
body is zero.

verify for a sphere ...

$$\underline{F} = - \int_{\partial B} p \underline{n} dA$$

$$= - \int_{\partial B} (p \underline{n} + \rho (\underline{u} \cdot \underline{n}) \underline{u}) dA$$

$$= - \int_{\partial R} (p \underline{n} + \rho (\underline{u} \cdot \underline{n}) \underline{u}) dA$$

↑
large sphere

$$\text{Since } \nabla p + \rho \underline{u} \cdot \nabla \underline{u} = 0.$$

$$\text{now } \underline{u} = U + O(r^{-3})$$

$$\text{and } p = p_0 + O(r^{-3})$$

$$\therefore \underline{F} = 0 + O(R^{-1}) \rightarrow 0 \text{ as } R \rightarrow \infty.$$

↑
cancellation

not representable by potential
since $u_y \neq v_x$.

→

(iv) Rankine vortex

part 1 →

$$u = -\omega y$$

$$v = \omega x$$

$$w = 0$$

} uniform angular
velocity

define pressure by

$$p = p_0 + \frac{1}{2} \omega^2 (x^2 + y^2). \quad \checkmark$$

take $\Omega = -gz$. At free surface,

$$p = 0 \text{ (say)} \Rightarrow gz = \frac{1}{2} \omega^2 (x^2 + y^2)$$

- paraboloid of revolution.

part 2 → point vortex

$$u = -\frac{\mu y}{r^2}$$

$$v = \frac{\mu x}{r^2}$$

$$\boxed{\psi = -\mu \theta}$$

get pressure from Bernoulli

$$p = -\rho gz - \frac{1}{2} \mu^2 / r^2 + \text{constant}$$

so surface is $gz = \text{constant} - \mu^2 / 2r^2$

Combine the solutions of parts 1 & 2 to make the velocity continuous $\Rightarrow$ Rankine vortex.

③② Goodman-Hou-Lowengrub. Convergence of 2D point vortex method.

(i) Continuous equations

$$\frac{dx(\alpha, t)}{dt} = V(\alpha, t) \quad ; \quad x(\alpha, 0) = \alpha$$

$$V(\alpha, t) = \int K(x(\alpha, t) - x(\alpha', t)) \omega_0(\alpha') d\alpha'$$

$$\text{where } K(x) = \frac{1}{2\pi} \frac{(-x_2, x_1)}{|x|^2}$$

(ii) discrete equations

$$v_j(t) = h^2 \sum_{k \neq j} K(x_j(t) - x_k(t)) \omega_k := \nabla[x]_j$$

$$\dot{x}_j = v_j(t).$$

(iii) Analysis of the error.

$$\begin{aligned} & \|v_j(t) - u(y_j(t), t)\| \\ & \leq \|v_j(t) - V[y]_j\| \quad \text{stability error.} \\ & \quad + \|u(y_j(t), t) - V[y]_j\| \quad \text{consistency error.} \end{aligned}$$

(iv) Analysis in the case where K is smooth

(consistency) This error is just due to the error in trapezoidal rule integration which is $O(h^N)$ for any N if K and w are arbitrarily smooth and decay fast enough.

This result is obtained from Poisson's identity as follows:

$$\begin{aligned} \text{trapezoidal rule approximation} & \rightarrow \frac{1}{N} \sum_{i=1}^N f\left(\frac{i}{N}\right) = \sum \hat{f}(iN) \\ & = \hat{f}(0) + \sum \hat{f}(iN) \end{aligned}$$

integral to be approximated

if $f \in C^m$ then these decay like $1/n^m$ bl.

$$= \int f(x) dx + \frac{1}{N^m} \sum \frac{c}{i^m}$$

the constant C is bounded by $\sup |\widehat{\partial^m f}| \leq \|\partial^m f\|_{L^1}$.

(stability) Assume that $\|x - y\| \leq h^{2-1/4}$ for $0 \leq t \leq T$.

$$\begin{aligned} & \textcircled{*} h^2 \sum_{k \neq j} (K(x_j - x_k) - K(y_j - y_k)) \omega_k \\ &= h^2 \sum \nabla K(y_j - y_k) [(x_j - y_j) - (x_k - y_k)] \omega_k \\ &+ h^2 \sum \nabla^2 K(z_j - z_k) \cdot [(x_j - y_j) - (x_k - y_k)]^2 \omega_k. \end{aligned}$$

Now $\|x - y\|_\infty \leq h^{2-1/4-2/p}$. So the distances between the approximate particle positions and the exact ones are small compared to h .

Note: z_j lies on the line segment connecting x_j to y_j - similarly for z_k .

why is this again?

$$h^2 |z_j|^p \leq \|z\|^p$$

$$\Rightarrow |z_j| \leq \|z\| h^{-2/p}$$

why $p > 4$?

so $\star$ can be split into 3 terms:

$$T_1 = h^2 \sum_k \nabla K(y_j - y_k) (x_j - y_j) w_k$$

$$T_2 = h^2 \sum_k \nabla K(y_j - y_k) (x_k - y_k) w_k$$

$$T_3 = h^2 \sum_k \nabla^2 K(z_j - z_k) [(x_j - y_j) - (x_k - y_k)] w_k$$

$$(T_3) \leq K \cdot \underbrace{h^{2-1/4-2/p}}_{\text{use bound on the}} \|x - y\|$$

$\|x - y\|_\infty$ for one term.

(T_1) This is multiplication by a diagonal matrix with diagonal entries

$$h^2 \sum \nabla K(y_j - y_k) w_k$$

which are bounded in the smooth case. $\therefore \|T_1\| \leq C \|x - y\|$.

(T_2) Need to show that the

$$\text{matrix } m_{jk} = \begin{cases} h^2 \nabla K(y_j - y_k) & j \neq k \\ 0 & j = k \end{cases}$$

details?



63.

as an operator
on l_p

satisfies $\|M\|_{l_p} \leq C$.

But the entries are smaller than C/N . Use Young's inequality...

(combine) We have shown that

if $\|x - y\| \leq h^{2-1/4}$ for $0 \leq t \leq T$

then there exists an $A(T, p)$ so that

$$\|v(t) - u(t)\| \leq A(T, p) \|x - y\| + A(T, p) h^2$$

($A(T, p)$ nondecreasing) for $0 \leq t \leq T$.

This gives the existence of $C(T, p)$

so that for $0 < t \leq T$,

$$\|x(t) - y(t)\| \leq C(T, p) h^2$$

$$\|v(t) - u(t)\| \leq C(T, p) h^2.$$

(V) When K is the singular kernel, the following modifications have to be made:

(consistency) the most singular part of the kernel is odd and so

does not contribute to the integral.

(Stability) The case T_3 is changed as follows:

$$\begin{aligned} \nabla^2 K(z_j - z_k) &\leq \frac{C}{|z_j - z_k|^3} \leq \frac{C}{|y_j - y_k + o(h)|^3} \\ &\leq \frac{C}{|z_j - z_k|^2} \end{aligned}$$

Approximate the sum $\sum_k |\nabla^2 K(z_j - z_k)|$

by an integral (over $z' > \alpha h$) to show that this sum is bounded by C/h .

By Young's inequality (row & column sums are $\leq C/h$) $\Rightarrow$

$$\|T_3\| \leq \frac{C}{h} \cdot h^{2-1/4-2/p} \|x-y\|$$

bounded!

The cases T_2 and T_3 - relate to continuous integrals and then bound (bound on ∇K as operator comes from the Calderón Zygmund lemma).

Check out the use of the
Calderón-Zygmund lemma. $\rightarrow$

(vi) More explicit derivation for consistency with singular kernel.
Consider at time zero:

$$K(-\xi) w_0(\xi) = \frac{(-\xi, \xi)}{|\xi|^2} (w_0(0) + \nabla w_0(0) \cdot \xi + \dots)$$

$$\text{Now } \int K(-\xi) w_0(\xi) = h^2 \sum_{k \neq 0} K(-\xi_k) w_k$$

divide into boxes centered at k .

$$\sum_{k \neq 0} \int K(-\xi) w_0(\xi)$$

(ignoring $k=0$ gives an error of $O(h^2)$)

Now approximate the error in the other boxes using the mid point rule:

$$\therefore \left| \sum_k \int K(-\xi) w_0(\xi) - h^2 \sum_{k \neq 0} K(-\xi_k) w_k \right|$$

$$\leq h^2 \sum_k |\nabla^2 K(\theta_k)| h^2$$

but the most singular part contributes nothing, so $\nabla^2 K(\xi) = O(1/|\xi|^2)$, not

$$1/|y|^3 \text{ so}$$

$$\sum |\nabla^2 K(\theta_k)| \sim \int_h^R \frac{1}{r^2} r dr \sim \log h.$$

$\therefore \|T_3\| \leq K h^2 \log h$ - not quite as good as h^2 , but good enough to get the result to work.

(33) Stokes' Flow and Stokes' paradox.

$$\nabla^2 s = -\frac{1}{\mu} \nabla p.$$

$$\nabla \cdot s = 0.$$

Can compute drag past a sphere using these equations.

Stoke's paradox - there is no solution to flow past a cylinder in 2D.

Note Oseen Flow - write $s = U + \sigma$ and keep the term $U \cdot \nabla \sigma$ in the convection part.

more generally,

$$\psi_j(\underline{x}) = -\Gamma_j G(\underline{x}, \underline{x}_j)$$

where G is the Green's function for the Poisson equation, with boundary conditions $\psi = 0$.

67.
(34) Point vortices.

$$\frac{dx_j}{dt} = -\frac{1}{2\pi} \sum_{i \neq j} \frac{\Gamma_i (y_j - y_i)}{r_{ij}^2}$$

$$\frac{dy_j}{dt} = +\frac{1}{2\pi} \sum_{i \neq j} \frac{\Gamma_i (x_j - x_i)}{r_{ij}^2}$$

↑

They form a Hamiltonian system.

Could scale the variables or write as

$$H = -\frac{1}{4\pi} \sum_{i \neq j} \Gamma_i \Gamma_j \log |\underline{x}_i - \underline{x}_j|.$$

then $\Gamma_j \frac{dx_j}{dt} = \frac{\partial H}{\partial y_j}$

$$\Gamma_j \frac{dy_j}{dt} = -\frac{\partial H}{\partial x_j}$$

⇒ H is conserved under motion of the point vortices. (H is the energy defect).

The error is only $O(\Delta x) = O(\Delta t)$ in the region where the flow is smooth. Godunov method treats discontinuities in the flow more realistically.

(ii) Lax-Wendroff

(one-step)

from Taylor series...

$$U_j^{n+1} = U_j^n + \Delta t \left(\frac{\partial U}{\partial t} \right)_j^n + \frac{\Delta t^2}{2} \left(\frac{\partial^2 U}{\partial t^2} \right)_j^n + \dots$$

$$\frac{\partial^2 U}{\partial t^2} = - \frac{\partial}{\partial t} \frac{\partial F}{\partial x} = - \frac{\partial}{\partial x} \left(A \frac{\partial U}{\partial t} \right) = \frac{\partial}{\partial x} \left(A \frac{\partial F}{\partial x} \right)$$

$$U_j^{n+1} = U_j^n - \frac{1}{2} \frac{\Delta t}{\Delta x} \left(\overset{\uparrow \nabla_x F}{F_{j+1}^n} - F_{j-1}^n \right) + \frac{1}{2} \left(\frac{\Delta t}{\Delta x} \right)^2 \left[A_{j+1/2}^n (F_{j+1}^n - F_j^n) - A_{j-1/2}^n (F_j^n - F_{j-1}^n) \right]$$

where $A_{j+1/2}^n$ denotes $A(1/2 U_{j+1}^n + 1/2 U_j^n)$.
Has second order accuracy.

advantage is that you don't have to compute $\nabla_u F$ →

ie. $(|u| + c) \frac{\Delta t}{\Delta x} < 1$ →

(two step)

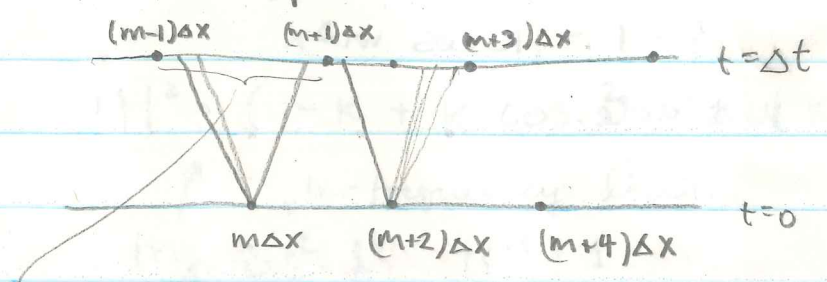
provisional $U_{j+1/2}^{n+1/2} = \frac{1}{2}(U_{j+1}^n + U_j^n) - \frac{\Delta t}{2\Delta x}(F_{j+1}^n - F_j^n)$
only first order accurate (error term $(\Delta t)^2$).
but the following equation is second order accurate owing to cancellation:

$$U_j^{n+1} = U_j^n - \frac{\Delta t}{\Delta x} (F_{j+1/2}^{n+1/2} - F_{j-1/2}^{n+1/2})$$

Get linear stability from amplification matrix using the hyperbolicity of the equations with $\Delta t/\Delta x$ small enough.

Dispersive and dissipative at 4th order.
Get oscillations at a shock.

(iii) Glimm scheme: If $TV(u_0)$ is sufficiently small, the problem has a solution.



pick random point in this interval.

Single shock - on average the speed is correct and the total variation is preserved.

(36) Upwinding finite difference scheme.

For the equation

$$u_t + u_x = 0.$$

Should use

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x} (u_i^n - u_{i-1}^n). \quad (*)$$

Decompose u into Fourier components

$$u_j^n = \sum e^{imj\Delta x}$$

This gives

$$\xi = 1 - K(1 - e^{-im\Delta x})$$

$$|\xi|^2 = (1 - K + K \cos^2) + (K \sin^2)$$

↑ low frequency limit.

$$\text{try } \cos^2 = 1 \quad |\xi|^2 = 1. \quad \text{need } K < 1.$$

$$\sin^2 = 1 \quad |\xi|^2 = (1 - K)^2 + K^2 = 1 - 2K + 2K^2$$

Centered difference gives

$$\xi = 1 - K \cos m\Delta x$$

unstable.

other way is $K < 0 \rightarrow$ unstable.

$$\zeta(x(t), t) = y(t).$$

take time derivative...

$$\frac{\partial \zeta}{\partial t} + u \frac{\partial \zeta}{\partial x} = v.$$

why can the constants be
chosen to be equal?



(37) Kelvin Helmholtz instability.

$z = \zeta(x, t)$ is the interface between two regions of irrotational fluid.

$$\underline{u} = \nabla \phi = \begin{cases} \nabla \phi_2 & y > \zeta \\ \nabla \phi_1 & y < \zeta. \end{cases}$$

$$\Delta \phi_j = 0$$

$u_j \rightarrow U_j$ (constant velocity at ∞) as $|y| \rightarrow \infty$
Bernoulli + ρ continuous across the interface $\Rightarrow$

$$C_1 - \frac{\partial \phi_1}{\partial t} - \frac{1}{2} |\nabla \phi_1|^2 = C_2$$

Kinematic condition (A particle on the interface stays on the interface) $\Rightarrow$

$$\frac{\partial \zeta}{\partial t} = \frac{\partial \phi_j}{\partial y} - \frac{\partial \phi_j}{\partial x} \frac{\partial \zeta}{\partial x} \quad \text{at } y = \zeta.$$

Linearize $\phi_j = U_j x + \tilde{\phi}_j$ ζ small.

(i) $\Delta \phi_j = 0$

(ii) $\nabla \tilde{\phi}_j \rightarrow 0$ as $|z| \rightarrow \infty$.

(iii) $\frac{\partial \tilde{\phi}_j}{\partial y} = \frac{\partial \zeta}{\partial t} + U_j \frac{\partial \zeta}{\partial x}$

(iv) $U_1 \frac{\partial \tilde{\phi}_1}{\partial x} + \frac{\partial \tilde{\phi}_1}{\partial t} = U_2 \frac{\partial \tilde{\phi}_2}{\partial x} + \frac{\partial \tilde{\phi}_2}{\partial t}$

Consider solutions of the form

$$\begin{pmatrix} \bar{\phi}_1 \\ \bar{\phi}_2 \\ \bar{\eta} \end{pmatrix} = e^{ikx + st} \begin{pmatrix} \bar{\phi}_1(y) \\ \bar{\phi}_2(y) \\ \bar{\eta} \end{pmatrix}$$

$$(i) \Rightarrow -k^2 \bar{\phi}_j + \frac{d^2 \bar{\phi}_j}{dy^2} = 0$$

$$(ii) \Rightarrow \begin{aligned} \bar{\phi}_1 &= A_1 e^{ky} & y < 0. \\ \bar{\phi}_2 &= A_2 e^{-ky} & y > 0. \end{aligned}$$

$$(iii) \Rightarrow A_1 = \frac{s + ikU_1}{k} \bar{\eta} \quad ; \quad A_2 = -\left(\frac{s + ikU_2}{k}\right) \bar{\eta}$$

$$(iv) \Rightarrow (s + ikU_1)^2 \bar{\eta} = -(s + ikU_2)^2 \bar{\eta}.$$

$$\text{or } s + ikU_1 = \pm i(s + ikU_2).$$

take + sign as a representative variable

$$s = \frac{-ikU_2 - kU_1}{(1-i)}$$

$$\text{Re } s = -\frac{k}{2} (U_2 - U_1) \rightarrow \text{unstable.}$$

$$\begin{aligned}
 \text{kernel } & \frac{1}{2\pi i} \frac{1}{z-z'} \\
 &= \frac{1}{2\pi i} \frac{(z-z')}{|z|^2} \\
 &= \frac{1}{2\pi i} \frac{(x-x') - i(y-y')}{|z|^2} \\
 &= \frac{1}{2\pi} \frac{-(y-y') - i(x-x')}{|z|^2} = \overline{K}
 \end{aligned}$$

→

(38) Vortex sheets & Birkhoff - Rott equation.

Recall that $(u-iv)(z) = \frac{1}{2\pi i} \int \frac{w(z')}{z-z'} dx'$

Suppose that the vorticity is concentrated on a curve with strength $\sigma(s)$. Then formally,

$$(u-iv)(z) = \frac{1}{2\pi i} \int \frac{\sigma(s)}{z-z(s)} ds.$$

Let $\gamma = \int \sigma(t) dt$ so that the above formula can be written:

$$(u-iv)(z) = \frac{1}{2\pi i} \int \frac{d\gamma}{z-z(\gamma)}.$$

In 2D, vorticity values mark material points $\Rightarrow \gamma$ is a Lagrangian variable.

Principle: The velocity of a fluid point on a vortex sheet is the average of the limiting velocities on either side of the sheet.

Birkhoff - Rott.

→

75.

γ Lagrangian variable $\Rightarrow$ velocity on the sheet is

$$\frac{\partial}{\partial t} \overline{z(\gamma, t)} = \frac{1}{2\pi i} \text{PV} \int_{-\infty}^{\infty} \frac{d\gamma'}{z(\gamma, t) - z(\gamma', t)}$$

The right hand side comes from the "principle" and the following result

$$\lim_{z \rightarrow z_0^+} \int_{-\infty}^{\infty} \frac{d\gamma}{z - z(\gamma)} = \text{PV} \int_{-\infty}^{\infty} \frac{d\gamma}{z_0 - z(\gamma)} + \frac{\pi i}{z'(0)}$$

The last term cancels with a similar result for the limiting value from below the sheet.

Suppose $z(\gamma) = \gamma + S(\gamma)$. Extend the B-R equation to complex γ as

follows:

Define $S^*(\gamma) = \overline{S(\overline{\gamma})}$ analytic if S is analytic and equal to $\overline{S}$ when γ is real. Now

$$\frac{\partial}{\partial t} S^*(\gamma, t) = -\frac{1}{2\pi i} \text{PV} \int_{-\infty}^{\infty} (\eta + S(\gamma + \eta) - S(\gamma))^{-1} d\eta.$$

Suppose $S(x, 0)$ is initially analytic in a strip $|\operatorname{Im} z| < \rho$ and

$$\|S\|_{\rho} = \int_{-\infty}^{\infty} e^{\rho|k|} |\hat{f}(k)| dk < \infty.$$

(this fact can be used to show that S defined by

$$S(z) = \int_{-\infty}^{\infty} e^{-ikz} \hat{S}(k) dk$$

is analytic in the strip).

This norm satisfies

$$\|S\|_{\rho} < \|S\|_{\rho'} \quad \text{if } \rho < \rho'$$

$$\text{and } \|S_z\|_{\rho} < \frac{\rho}{\rho' - \rho} \|S\|_{\rho'}$$

It can be shown that the B-R equation satisfies the conditions of the abstract C-K theorem $\Rightarrow$ the existence of a solution $S(x, t)$ in the spaces $\rho(t) \leq \rho_0 - at$.

return after examining the
abstract C-K theorem. $\rightarrow$

(39) Derivation of equations (see also # 23).

Flow map $x(X, t)$. $J = \left| \frac{\partial x}{\partial X} \right|$.

(i) Conservation of momentum:

$$M = \iiint_{V(t)} \rho dx = \iiint_{V(0)} \rho J dX \text{ constant.}$$

$$\Rightarrow \frac{d}{dt}(\rho J) = 0 \quad \text{or} \quad \rho(X, t) J = \rho(X, 0).$$

$$\therefore \frac{d\rho}{dt} J + \frac{dJ}{dt} \rho = 0$$

$$\Rightarrow \rho_t + u \cdot \nabla \rho + \rho \nabla \cdot u = 0 \quad \text{using (23)}$$

(ii) $T(x, n)$ = force / unit area acting on a hyperplane $\perp$ to n through x .

Balance of forces:

$$\frac{d}{dt} \iiint_{V(t)} \rho u = \iiint_{V(t)} \rho f + \iint_{\partial V(t)} T(x; n) dA$$

$T_i(n)$ must be of the form $T_{ij}(x) n_j$ (use the tetrahedral argument).

Balance of angular momentum $\Rightarrow$

$$T_{ij} = T_{ji}.$$

use $T = -pI$ (no shearing stresses in fluids).

$$\rho(u_t + u \cdot \nabla u) + \nabla p = \rho f$$

(iii) add in viscosity by assuming

$$T = -pI + \underline{\underline{\sigma}}(\nabla u) \leftarrow \text{linear function of } \nabla u.$$

- $\underline{\underline{\sigma}}$ invariant under rigid body rotation and translation:

$$\underline{\underline{\sigma}}(U \cdot \nabla u \cdot U^{-1}) = U \cdot \underline{\underline{\sigma}}(\nabla u) \cdot U^{-1}.$$

U orthogonal.

So $\underline{\underline{\sigma}}$ can only depend on $\underline{\underline{D}}$ -

can simultaneously diagonalize and get

$$\underline{\underline{\sigma}} = \lambda (\text{div } u) \underline{\underline{I}} + 2\mu \underline{\underline{D}}$$

$\uparrow$
trace of $\underline{\underline{D}}$

$\Rightarrow \nabla \cdot \underline{\underline{\sigma}} = \mu \Delta u$ in the divergence free case.

R is 10 times bigger in water! $\rightarrow$
 (viscous forces small compared
 to inertial forces).

$\frac{1}{R}$ measures how important viscous
 forces are compared to inertial forces.

④① Derivation of boundary layer equations.

Scale $y = \gamma^\alpha h$; $v = \gamma^\beta w$.

Continuity equation $\Rightarrow \alpha = \beta$.

To keep both terms in the momentum equation, must take $\alpha = 1/2$.

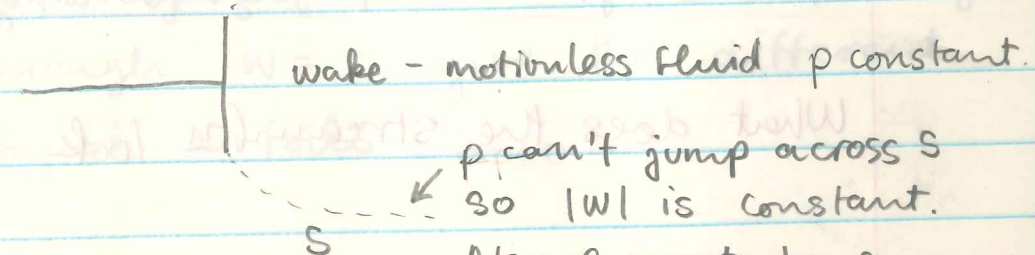
④① Cavity Flows.

(i) Flow past a flat plate

$$f(z) = \sqrt{z^2 + 1}$$

(velocity is 1 at ∞). Maps exterior of slit to the whole plane (except the interval $[-1, 1]$ on the real axis).

(ii) Suppose we want to describe flow past a flat plate with a wake.



Also, S must be a

Streamline.

$\rightarrow$ page 81

with $k'(\infty)$ real $\rightarrow$
 k has a simple pole at ∞

$z = j(w)$ $j'(w) = 0$ when $w = \pm c$ $\rightarrow$
 gives sharp edge to wing.

What does the streamfunction look like? $\rightarrow$

④2 Joukowski Airfoils

(i) Recall - if we have a body V and k maps the exterior of V onto the exterior of the unit disk k , then the velocity potential is

$$\phi(z) = \frac{U}{k'(\infty)} \left(k(z) + \frac{1}{k(z)} \right) - \frac{\gamma}{2\pi i} \log k(z).$$

Let $j(w) = k^{-1}(w)$ - this maps the exterior of the unit disk onto the exterior of V .

$$j(w) = w + a_0 + \frac{a_1}{w} + \frac{a_2}{w^2} + \dots$$

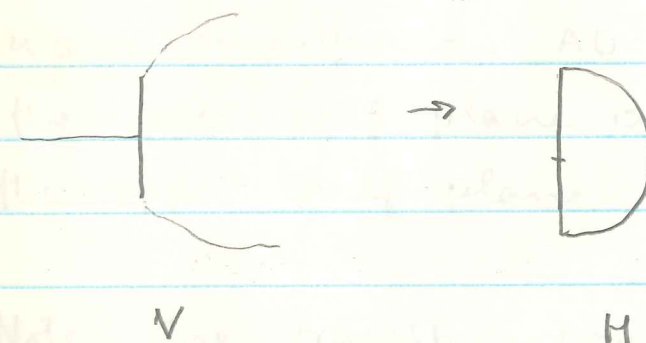
Joukowski transformation is $j(w) = w + \frac{c^2}{w}$.
 The images of different circles passing through $w = c$ give the different shaped airfoils.

Choose circulation so that stagnation point coincides with the trailing edge.

Choose $|w|$ on S so that it equals the value upstream.

w -plane is called the hodograph plane.
 V -region in physical (z) space
 H -region in hodograph (w) space.

Assume the map $w: V \rightarrow H$ is 1-1!



ζ maps V onto the entire plane, slit along the half-line $\text{Re}(\zeta) > 0$.

Let $S = \sqrt{\zeta}$ (takes V to the upper half plane).

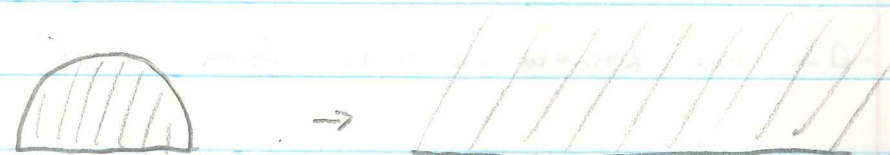
Also $t = iw + \frac{1}{iw}$ maps onto the upper half plane.

So s and t must be related by

$$s = \frac{At+B}{Ct+D}$$

A, B, C, D real (this is the most general conformal mapping from the upper half plane onto itself). There is a sign condition on $AD - BC$ to ensure the upper half plane is mapped to the upper half plane.

Note The transformation $z \mapsto z + \frac{1}{z}$ maps the half circle onto the upper half plane:



Now have found $\xi(w)$. To solve for $w(z)$ proceed as follows:

$$w = \frac{d\xi}{dz} = \xi'(w) \cdot \frac{dw}{dz}$$

integrate to get $z(w)$.

(43) Lagrangian formulation of fluid dynamic equations:

Let h be a Lagrangean density variable in 1-D

$$h = \int_{x(0,t)}^{x(h,t)} \rho dx.$$

The Lagrangean equations of motion

$$\text{are } \begin{cases} v_t - u_h = 0 \\ u_t + p_h = 0 \end{cases} \quad \left. \begin{array}{l} \text{p system when } S \\ \text{constant} \Rightarrow p(v). \end{array} \right\}$$

$$E_t + (uP)_h = 0 \quad \text{or } S_t = 0.$$

$$e + \frac{u^2}{2}$$

(44) Linearized surface waves in 2D.

$$y - \eta(x,t) = 0$$

Kinematic condition on the interface
(a particle on the surface stays on the surface).

$$\Rightarrow v - h_x u - h_t = 0$$

$$\text{or } h_t = v - h_x u.$$

Bernoulli (with $p=0$ on the surface) $\Rightarrow$

$$\psi_t + \frac{g^2}{2} + gh = 0.$$

Linearize about $\psi=0, h=0$ (evaluate conditions on the surface at $y=0$).

$$\text{Bernoulli} \Rightarrow \psi_t + gh = 0 \quad (*)$$

$$\text{Kinematic} \Rightarrow \psi_y = h_t.$$

combine $\psi_{tt} + g\psi_y = 0$. Determine h later from $(*)$

Look for solutions of the form $\psi = e^{i\sigma t} \Phi(x, y)$, with $\Delta\Phi = 0$, and $\Phi_y - \sigma^2/g \Phi = 0$ on top.

If Φ is harmonic then so is $\psi = \Phi_y - \sigma^2/g \Phi$. $\psi = 0$ on top. If ψ bounded as $y \rightarrow \infty$, $\psi \equiv 0$.

Therefore

$$\Phi_y = -\sigma^2/g \Phi.$$

R-T instability
 g negative $\Rightarrow$
 σ^2 negative
 exponential growth
 in time.

$$\Phi = e^{-\sigma^2/g y} g(x)$$

$$\Delta \Phi = 0 \Rightarrow \Phi = e^{-\sigma^2/g y} e^{i\sigma^2/g x}$$

$$\eta = -\frac{1}{g} \Phi_t = -\frac{i\sigma}{g} e^{i(\sigma t \pm \frac{\sigma^2 x}{g})}.$$

$$\Rightarrow \omega = \sqrt{k g}. \quad c = \frac{\omega}{k} = \sqrt{g/k} \quad \text{longer wavelength waves move faster.}$$

$$\text{In shallow water } c \propto \sqrt{gh}.$$