Orals' Review

Read Caflisch's article - on Vortex Dynamics.
Read Smoller Ch. 15 on shock waves.

Jan. 22, 1989. Talk with Hou.

- review Stang's trick.
- review Kato's 3D estimates - stability.
- review Hou's paper.
- review Caldéron - Zygmund lemma
- review Anderson's ideas.
- review Young's Theorem.

① (ODE) Gronwall's inequality.

Suppose $u(t)$ continuous and non-negative in $[0,T]$ and

$$u(t) \le C_1 + C_2 \int_0^t u(s)\,ds. \qquad C_1, C_2 > 0.$$

then $u(t) \le C_1 e^{C_2 t}$.

Set $V(t) = C_1 + C_2 \int_0^t u(s)\,ds.$

$$\dot V(t) = C_2\, u \le C_2 V.$$

or $(V e^{-C_2 t})^{\cdot} \le 0$ integrate $\Rightarrow$

$$V e^{-C_2 t} \le V(0) = C_1.$$

$\Rightarrow$ $u \le V \le C_1 e^{C_2 t}$.

Applications:

(i) to uniqueness and continuous dependence on initial data of ODE with Lipschitz condition.

(ii) Global existence for linear equations.

can get better estimates. →

② (ODE) Picard iteration.

Suppose $f(x,t)$ is C^1, then the problem

$$\dot{x} = f(x,t) \qquad x(0) = x_0$$

has a unique solution in some interval $[0, t_0)$.

Consider the scheme

$$x_n(t) = \int_0^t f(x_{n-1}(t), t)\, dt \quad + x_0.$$

with $x_0(t) = x_0$. Suppose $\frac{\partial f}{\partial x}$ is bounded by M, then choose $t_0 = 1/2M$.

$x_n(t)$ well defined and differentiable.

Let $e_n = \sup\limits_{0 \le t \le t_0} |x_n(t) - x_{n-1}(t)|$

Now $|x_n(t) - x_{n-1}(t)| \le \int_0^t [f(x_{n-1}(t), t) - f(x_{n-2}, t)]$

$\le M t\, e_{n-1}$.

$\Rightarrow \quad e_n \le \frac{1}{2} e_{n-1}$.

$\therefore$ x_n converges uniformly to a continuous fn x (telescoping series).

Since convergence is uniform, can pass limit under the integral so $x(t)$ satisfies integral equation $\Rightarrow$ sol$\underline{n}$.

Get uniqueness from Gronwall.

If f continuous and bounded by M, the iterates are uniformly bounded by M if $t_0 = 1/M$ (take $x_0 = 0$). Since $|\dot{x}_n| \le M$, fns are equicontinuous also - choose a uniformly convergent subsequence.

③ (ODE) Poincaré-Bendixson.

$f \in C^1(\mathbb{R}^2 \to \mathbb{R}^2)$. Let y be a semi-orbit for $y' = f(y)$ that is contained in a compact subset D^*. If D^* contains no singular points of f, then the limit point set L is a closed orbit.

properties of L:

(i) $L \subset D^*$ and L is not empty since D^* is compact.

review why this cannot →
happen.

(ii) if L consists of a single point ξ then $f(\xi)=0$

(iii) L is the union of orbits.

(iv) L is closed / compact.

(v) L is connected.

Notes: (1) to show that an orbit remains in D^*, show that the field f always points to the interior of D^* (such a D^* cannot be simply connected).

A transversal is a straight line segment with the direction field nowhere tangent. The crossing points on a transversal form a monotone sequence (Jordan curve theorem).

Suppose η is a limit point of a semi-orbit in D^*, take $T \subset D^*$ to be a transversal containing η. η is the monotone limit of crossing points of γ and T.

Important observation : η is the only point of L on T.

Start at η - must cross T (by continuity) so η must return to itself - periodic orbit - single orbit by connectivity.

<u>Extension</u> - if $y(t)$ tends asymptotically to a periodic orbit L, then L has one - sided orbital stability.

④ (ODE) example of boundary layer.

$$\frac{dy}{dx} = 0 \qquad y(1) = 1 \quad \Rightarrow \quad y = 1.$$

Add viscous term

$$\varepsilon \frac{d^2y}{dx^2} + \frac{dy}{dx} = 0 \qquad y(0) = 0 \ , \quad y(1) = 1$$

$$\Rightarrow \quad y = (1 - e^{-1/\varepsilon})^{-1} \underbrace{(1 - e^{-x/\varepsilon})}_{\text{away from zero, this has a small effect.}}$$

a better example is $y' = 1 + y^2$ →
always goes to ∞ in finite time.

⑤ (ODE) miscellaneous examples.

(i) linear equations 1st order, 1 variable.

$$y' + p(x)y = g(x) \qquad a(x) = e^{\int^x p(s)ds}$$

$$\Rightarrow (ay)' = ag$$

or $y = \frac{1}{a(x)}\left[\int^x a(s)g(s)ds + C\right]$.

(ii) example for non-uniqueness

$$y' = y^{1/2}$$

solutions $y = 0$ as well as $y = x^2/4$
(can switch from first to second at an arbitrary x).

(iii) example for lack of global existence

$$y' = y^2 \quad y(0) = 1$$

$$\Rightarrow y = \frac{1}{1-x}$$

(iv) exact equations

homogenous equations $\frac{dy}{dx} = F\left(\frac{y}{x}\right)$
$\Rightarrow$ separable.

(v) (biological model)

$$\frac{dA}{dt} = \underset{\substack{\uparrow \\ \text{growth}}}{\varepsilon A} \; \underset{\substack{\uparrow \\ \text{retardation.}}}{-\sigma A^2}$$

$A = \varepsilon/\sigma$ is the saturation level - globally attracting.

(vi) nonlinear pendulum

$$\ddot{\theta} + \sin\theta = 0.$$

linear stability analysis provides no information (eigenvalues pure imaginary). use Liapunov's 2nd (direct) method with

$$Q(\theta, \dot{\theta}) = \underbrace{(1-\cos\theta)}_{\text{looks like } \theta^2/2 \text{ near origin.}} + \dot{\theta}^2/2.$$

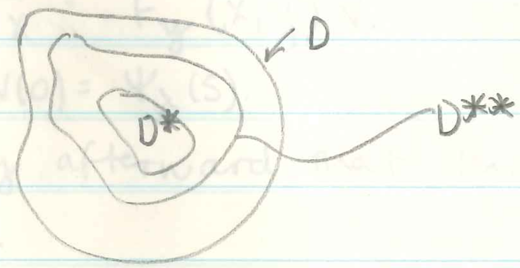
(vii) Another example of the direct method:

$$y'' + y + y^3 = 0$$

($(y')^2 + y^2 + \frac{1}{2}y^4$ is conserved on trajectories).

⑥ (ODE) continuation theorem.

Let D^* be a closed and bounded subset of D (the domain of definition of f). If the trajectory $z = (x, z(x))$ defined on $\alpha < x < \beta$ (β is the maximum of the intervals over which the trajectory is defined) and $\beta < \infty$, then z lies outside of D^* for all x sufficiently close to β.



choose α small enough so the trajectory can't leave D^{**} in this time if it starts in D^*. Then we can extend the solution past β if the trajectory is in D^* for $\beta - \alpha < x < \beta$.

⑦ (ODE) Continuous dependence on parameters.

$$\frac{dy}{dx} = f(x,y)$$
$$y(0) = \psi(\lambda).$$

Write $y(x;\lambda)$. Suppose y is C^1 in λ.
Write $v = \frac{\partial y}{\partial \lambda}(x;s)$. Then formally,

$$\frac{dv}{dx} = f_y(x,y)\,v. \qquad \text{- linear equation for } v.$$
$$v(0) = \psi_\lambda(s).$$

Verify afterward that this v really is $\partial y/\partial \lambda$.

<u>Notes</u> (i) can convert parameters in f to initial data - OK.
(ii) Linear equation for v - easy to solve $\Rightarrow$ perturbation method.

⑧ (ODE) linear equations

$\dot{y} = A(t)\, y.$

(i) Y fundamental solution matrix

$W = \det(Y)$. $\quad W' = (\operatorname{tr} A)\, W$

(justify the above formula):

$Y(x+h) = (I + hA)\, Y(x) + o(h^2).$

$\therefore \; W(x+h) = [1 + h(\operatorname{tr} A) + o(h^2)]\, W(x) + o(h^{2N}).$

(ii) variation of parameters for

$\dot{y} = A(t)\, y + b(x).$

$y(x) = Y(x)\, c(x)$ since Y is invertible

now $Y' = AY \Rightarrow c' = Y^{-1}(b)$

or $c(x) = c(0) + \int_0^x Y^{-1}(s)\, b(s)\, ds.$

(iii) Jordan normal form $\Rightarrow$ fundamental solution for constant A.

9 (ODE) Boundary value problems.

(*) $Ly = (ay')' + by = f$ ($= \lambda y$ for eigenvalue problem).
with $y(0) = y(1) = 0$.

(i) If we had a green's $\underline{fn}$, $K(x,\xi)$
- $LK = 0 \qquad x \neq \xi$
- $K(0,\xi) = K(1,\xi) = 0$
- K continuous across $x = \xi$
- $(aK')\Big|_{\xi^-}^{\xi^+} = 1$ across $x = \xi$.

then $y(x) = \int_0^1 K(x,\xi) f(\xi)\, d\xi$ would be a solution of (*).

(ii) A Green's fn for this problem is

$$K(x,\xi) = \begin{cases} \dfrac{z_2(\xi)\, z_1(x)}{a(\xi)\, W(\xi)} & x < \xi \\[2ex] \dfrac{z_1(\xi)\, z_2(x)}{a(\xi)\, W(\xi)} & x > \xi. \end{cases}$$

(iii) Now ⊛ becomes $y = KF$, and we can use Hilbert space theory. K is compact and self adjoint. Range of K is dense in L^2 so can expand fn's in its eigenvalues.

(iv) extra properties from ODE

- eigenvalues tend to $+\infty$ or $-\infty$ only
- eigenvalues are simple.

⑩ (ODE) Stability.

$y' = Ay$. if $\mathrm{re}(\lambda) < 0$ for all eigenvalues λ of A then there exists a real +ve definite quadratic form $Q(y)$ so that

$$\frac{dQ}{dx}(y) \leq -\delta\, Q(y)$$

for all y. (this gives asymptotic stability).

(i) linear theory.

want $Q(y) = y^T \overset{\text{-ve definite}}{B} y$

check out Jordan normal forms. →

so $\frac{dQ}{dx} = y^T(A^TB + BA)y = q(y)$

If T is a matrix that takes A to real Jordan normal form, ie $\underset{\substack{\uparrow \\ \text{Jordan form}}}{\tilde{A}} = T^{-1}AT$,

then take $B = (T^{-1})^T(T^{-1})$.

In this case $g(y) = (T^{-1}y)^T \underbrace{(\tilde{A}^T + \tilde{A})}_{\text{easy to make this -ve definite - make off diagonal elements small.}} (T^{-1}y)$.

$\therefore$ $q(y) \leq -K\|y\|^2 \leq -\delta Q(y)$, for some constants K and δ.

(ii) This persists into the non linear theory : $y' = Ay + g(y)$, where $g(y) = O(y^2)$. In this case

$$\frac{dQ}{dx} = y^T(A^TB + BA)y + \underbrace{O(y^3)}_{\text{for } y \text{ small enough, can absorb this term.}}$$

(11) (ODE) Floquet multipliers.

(i) $\dot{y} = A(t)y$ A periodic in t.

If $Y(t)$ is a fundamental matrix then so is $Y(t+p)$.

$\therefore$ there exists an Ω so that

$Y(t+np) = Y(t)\Omega^n$ (all Ω's are similar). $\Omega = Y^{-1}(0)Y(p)$.

ω - eigenvalue of Ω - Floquet multiplier.

If all ω's have $|\omega|<1$ (or all ω with $|\omega|=1$ possess a full set of eigenvectors) $\Rightarrow$ solutions are bounded.

(ii) $W(t+p) = W(t)\det\Omega$.

Use this to examine Hill's equation:

$$\ddot{z} + \underbrace{f(t)}_{\text{period } 1} z = 0$$

$$\begin{pmatrix} z \\ \dot{z} \end{pmatrix}^{\cdot} = \underbrace{\begin{pmatrix} 0 & 1 \\ -f(t) & 0 \end{pmatrix}}_{A} \begin{pmatrix} z \\ \dot{z} \end{pmatrix}$$

since $W(p) = W(0)\, e^{\int_0^p \operatorname{tr} A \, ds}$,

$\det \Omega = 1$.

ln's of matrices - addressed →
later.

Therefore $\omega_1 \omega_2 = 1$ and we have three cases

1. $\omega_1 \neq \omega_2$ and both real $\Rightarrow$ there are unbounded solutions (one $\omega_i > 1$).
2. $\omega_1 \neq \omega_2$ and $\omega_1 = \overline{\omega_2}$. In this case, $|\omega_1| = |\omega_2| = 1$ - have full set of e.values $\Rightarrow$ bounded solutions.
3. $\omega_1 = \omega_2 = \pm 1$ have unbounded solutions unless $\Omega = \pm I$.

(iii). Find C such that $\Omega = e^{pC}$. Let $P(t) = Y(t)\, e^{-tC}$ (P is periodic) and let $y(t) = P(t)\, z(t). \Rightarrow$

$$\dot{z} = Cz$$

(an equation with constant coefficients). since $\log \Omega = C$, C has eigenvalues with real part < 0 iff Ω has eigenvalues ω with $|\omega| < 1$.

Can extend to nonlinear effects, too.

review this a little more closely. →

(iv). Consider $\eta' = f(t,\eta)$ - f periodic with period 1 in t and suppose $\xi(t)$ is a particular periodic soln.

soln F depending on initial condition ξ

Let $Y(t) = F_\xi(t, \xi(0))$ - variation of F about $\xi(0)$.

Y solves $\dfrac{dY}{dt} = A(t)\,Y$ where
$A(t) = \underbrace{F_\eta(t, \xi(t))}_{p\text{-periodic}}$.

Consider Ω corresponding to this soln and stability considerations.

(v) autonomous systems always have $\omega = 1$ an eigenvalue of Ω. If this is the only one, the solution is orbitally stable.

(12) (Complex) Maximum modulus theorem.
If f is analytic on D and continuous on $\bar{D}$ and if $|f(z)| \leq M$ on ∂D then $|f(z)| < M$ at all interior points of D unless f is constant.

(i) if $f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$ then

$$\sum_{n=0}^{\infty} |c_n|^2 r^{2n} = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(a+re^{i\theta})|^2 d\theta.$$

↑
parseval's formula.

(ii) Suppose f achieves a maximum value at a in the interior of D. Choose a disk of radius r inside D. Then

$$\sum_{n=0}^{\infty} |c_n|^2 r^{2n} \leq |f(a)|^2 = |c_0|^2$$

So $c_1 = c_2 = \dots = 0$ so $f(z) = f(a)$ in the disk and $f(z) = f(a)$ in D by continuation.

(iii) $|f(z)|$ can have no minimum in D other than zero (apply to $1/f$).

(iv) A real-valued fn u in a region cannot have a maximum at an interior point of the region.

Suppose $f(z) = u+iv$ is analytic. $|e^{f(z)}| = e^u$ can have no maximum.

(13) (Complex) Liouville's theorem -

Every bounded entire fn is constant.

Use estimates from (12) to get $c_n = 0$ for all $n > 0 \Rightarrow f$ is constant.

Extension - any entire fn with a bound $|f(re^{i\theta})| \leq Mr^n$ is a polynomial of degree $\leq n$.

Application: fundamental theorem of calculus.

can extend using mappings →
of circles onto themselves.

(14) (Complex) Schwarz lemma. Suppose f regular on $|z|<1$ and $|f(z)| \leq 1$, and $f(0)=0$.

Then $|f(z)| \leq |z|$ (1)

and $|f'(0)| \leq 1$ (2).

if equality holds in (1) for $|z| \neq 0$ or equality holds in (2) then $f(z) = \lambda z$.

(i) consider $g(z) = \begin{cases} f(z)/z & z \neq 0 \\ f'(0). & z = 0 \end{cases}$

now $|g(z)| \leq 1 \Rightarrow$ (1) & (2) by maximum principle. If $|g(z)| = 1$ for some z, then g is constant λ $(|\lambda| = 1) \Rightarrow f(z) = \lambda z$.

(ii) Application to proof of uniqueness of mapping in Riemann mapping theorem.

Suppose there were 2 such mappings f_1 & f_2. Set $f = f_1 \circ f_2^{-1}$ & $g = f_2 \circ f_1^{-1}$.

$g'(0) = f_2'(f_1^{-1}(0)) / f_1'(f_1^{-1}(0)) > 0.$ →

true if f_n converge uniformly →
on the boundary, say.
uniform convergence ⇒ f is analytic
(Morera's theorem).

$f(0) = 0$; $g(0) = 0$.

Schwarz ⇒ $|f(z)| \le |z|$; $|g(z)| \le |z|$.
$|f(g(w))| = |w| \ge |g(w)|$.
∴ $|g(w)| = |w|$ ⇒ $g(z) = e^{iw} z$, but
normalization ⇒ $g(z) = z$. Similarly
$f(z) = z$ ⇒ $f_1 = f_2$.

(15) (Complex) Hurwitz' Theorem.

If $f_n(z)$ is a sequence of non-zero analytic functions on a region Ω that converge to $f(z)$ uniformly on every compact subset of Ω, then $f(z)$ is either identically zero or never zero in Ω.

(i) Assume $f(z)$ is not identically zero, so zeros are isolated. ∃ $r > 0$ so $f(z) \ne 0$ for $0 < |z - z_0| \le r$. Let m be the minimum of $|f(z)|$ on the circle. Choose N large enough so

used in the Riemann mapping $\rightarrow$
theorem.

that $|f_n(z) - f(z)| < m$ on circle.
Rouché $\Rightarrow$ f_N & f have the same number of zeros. Since f_N is never zero #

(ii) $\{f_n\}$ a sequence of univalent analytic f_n's that converge to f uniformly on compact subsets of Ω, then f is either univalent or constant.

(iii) $f_n \overset{\text{uniformly on compact subsets}}{\rightrightarrows} f$ and f is not identically zero, then $f(z_0)=0$ iff z_0 is a limit point of the set of zeros of the functions $f_n(z)$.
(consider $f_n(z) = 1/n \Rightarrow f$ could be identically zero).

(16) (Complex) Riemann mapping theorem.
Given any simply connected domain Ω of the complex plane which is not the whole complex plane, and a point $z_0 \in \Omega$, there exists a unique

if g 1-1 continuous from a locally → compact space Ω, then g^{-1} is continuous on $g(\Omega)$.

analytic fn $f(z)$ in Ω, normalized by the conditions $f(z_0)=0$, $f'(z_0)>0$ such that $w=f(z)$ defines a one-to-one mapping of Ω onto the disk $|w|<1$.

Notes: (1) Since f is 1-1 it is of necessity conformal.

(2) Must exclude the whole plane (Liouville's theorem).

$\mathcal{F}$ - family of analytic fn's on Ω such that

(1) f univalent

(2) f maps Ω into the open unit disk

(3) $f(z_0)=0$, $f'(z_0)>0$.

(i) $\mathcal{F}$ is non-empty. Choose $a \notin \Omega$. One can define $g(z) = \sqrt{z-a}$ on Ω (univalent) $\Rightarrow$ $g(\Omega)$ is open, so let $w_0 = g(z_0)$. $\{w : |w-w_0| < \rho\} \subset g(\Omega)$.

if $w \in g(\Omega)$ then $-w \notin g(\Omega)$

so

$$|g(z) + w_0| \geq \rho \quad \text{for all } z \in \Omega.$$

$\therefore\ \psi(z) = \rho/2 \left(g(z) + w_0\right)^{-1}$ analytic, univalent, $|\psi(z)| \leq 1$

⋮

(ii) $\{f_n\}$ is a sequence that maximizes $f'(z_0)$. Convergent subsequence (Ascoli) that is analytic, attains maximum, univalent (Hurwitz).

(iii) If f does not map onto the open unit disk, can increase $f'(z_0)$ #.

(iv) Uniqueness by Schwarz lemma.

(v). Caratheodory - extension to a homeomorphism from $\bar{\Omega}$ to $\bar{U}$ in some cases.

17 (Real) Baire category theorem.

A set E is nowhere dense if $\sim \bar{E}$ is dense (ie $\bar{E}$ contains no spheroid or $E = \sim$ dense open set).

X complete metric space.

(1) If $\{O_n\}$ is a countable collection of dense open sets then $\bigcap O_n$ is not empty.

take $E_n = \sim O_n$

(2) A complete metric space is not the union of a countable collection of nowhere dense sets $\{E_n\}$.

proof; Pick $x_1 \in O_1$ and S_1 a sphere of radius r_1 centered at x_1 and contained in O_1. O_2 dense $\Rightarrow$ there is a point in $O_2 \cap S_1$. O_2 open $\Rightarrow$ spheroid S_2 centered at x_2 and contained in O_2. Take $r_2 < r_1/2$ and small enough so that $\overline{S_2} \subset S_1$. Continue. $\langle x_n \rangle$ Cauchy, completeness $\Rightarrow \underline{x_n \to x}$. $x_n \in S_{N+1}$ for $n > N$, so $x \in \overline{S_{N+1}} \subset S_N \subset O_N$. Hence $x \in \bigcap O_n$.

(18) (Real) Principle of uniform boundedness.

If $\Phi_1, \ldots$ are bounded linear functionals on a Banach space B such that $\{\Phi_k(h)\}$ is bounded for each $h \in H$, then $\|\Phi_k\| \le M$.

- Let $B_n = \{h \in H : |\Phi_k(h)| \le n \quad \forall k\}$.
$H = \bigcup B_n$ and by Baire Category, $\exists B_N$ with non-empty interior - let B_N contain $\{h : |h - h_0| < \varepsilon\}$.

$$\therefore |\Phi_k(f)| = \frac{\|f\|}{\varepsilon} \Phi_k\left(\frac{\varepsilon f}{\|f\|}\right)$$

$$= \frac{\|f\|}{\varepsilon}\left[\Phi_k\left(\frac{\varepsilon f}{\|f\|} - h_0\right) + \Phi_k(h_0)\right].$$

$$\le \frac{\|f\|}{\varepsilon}(2N)$$

$\Rightarrow \|\Phi_k\| \le 2N/\varepsilon$.

Application: weakly convergent subsequences must be bounded.

Variant: F is a family of real-valued continuous functions on a complete metric space X and suppose that for each $x \in X$ there is a number M_x such that $|f(x)| \leq M_x$ for all $f \in F$. Then there is a non-empty open set $O \subset X$ and a constant M such that $|f(x)| \leq M$ for all $f \in F$ and all $x \in O$.

(19) (Real) Ascoli selection principle.
F uniformly bounded, equicontinuous family of fn's defined on a compact set. Then from every sequence $\{f_n(x)\}$ chosen from F it is possible to select a uniformly convergent subsequence.

<u>Note</u>: if $f'(x) \leq c$ for all $f \in F$ and $x \in S$ then F is equicontinuous.

Choose $\{f_{nn}\}$ sequence that converges at each point in a countable dense

need DF to be continuous in a →
neighborhood of x_0.

subset S^+.

(ε-δ) Choose a finite subset S^* of S^+ so that each point of S^+ is no further than δ from at least one point of S^*.

Choose N large enough so that for $n,m > N$, $|f_{mm}(y) - f_{nn}(y)| < \varepsilon$ at each point y of S^*. For each point x of S, choose $y \in S^*$ so that $|x-y| < \delta$. Then for $n, m > N$

$$|f_{nn}(x) - f_{mm}(x)| \le |f_{nn}(x) - f_{nn}(y)| + |f_{nn}(y) - f_{mm}(y)| + |f_{mm}(y) - f_{mm}(x)| < 3\varepsilon.$$

20 (Real) Inverse & Implicit fn theorems.

Suppose $f(x_0) = y_0$ and Df exists and is continuous in a neighborhood of x_0, with $Df(x_0)$ nonsingular. Then f maps a neighborhood of x_0 1-1 and onto a neighborhood of y_0. The inverse is continuously differentiable.

(i) Statement of fixed point theorem:

Let M be a complete metric space. Suppose $F: M \to M$ and assume that there is a constant λ $(0 \le \lambda < 1)$ such that for all $x, y \in M$,

$$d(F(x), F(y)) \le \lambda d(x, y).$$

then F has a unique fixed point $x_0 \in M$.

(ii) Assume WLOG $x_0 = 0$, $f(x_0) = 0$, $Df(x_0) = I$.

Let $g(x) = x - f(x)$ so $Dg(0) = 0$. Choose $r > 0$ so that

$$\|x\| \le r \Rightarrow \|Dg(x)\| \le 1/2.$$

Mean value theorem $\Rightarrow$ $\|x\| \le r \Rightarrow \|g(x)\| \le r/2$

for $y \in \overline{B_{r/2}}(0)$ let $g_y(x) = y + x - f(x)$.

if $x_1, x_2 \in \overline{B_r}(0)$ then

$$\|g_y(x_i)\| \le \|y\| + \|g(x_i)\| \le r.$$

& $\|g_y(x_1) - g_y(x_2)\| \le \frac{1}{2}\|x_1 - x_2\|$.

what about continuity & differentiability of the map. →

do a direct proof. →

$\therefore$ $g_y(x)$ has a unique fixed point x in $\overline{B_r}(0)$. This point is the unique solution of $f(x) = y$.

(iii) Implicit fn theorem.

$f(x_0, y_0) = 0 \qquad f : X \times Y \to Z$

$D_2 : Y \to Z$ is an isomorphism.

Then in a neighborhood of (x_0, y_0), the solutions of $f(x,y) = 0$ lie on a differentiable curve $(x, y(x))$.

consider $\Phi : \underset{\text{neighborhood of } x_0}{U} \times \underset{\text{of } y_0}{V} \to X \times Z \; ; \; (x,y) \to (x, f(x,y))$.

$$D\Phi = \begin{pmatrix} I & 0 \\ D_1 f & D_2 f \end{pmatrix}$$

invertible with $(D\Phi)^{-1} = \begin{pmatrix} I & 0 \\ -[D_2 f]^{-1} D_1 f & [D_2 f]^{-1} \end{pmatrix}$

Φ has a unique local inverse, $(x, w) \to (x, g(x,w))$.

if $w = 0$, let $\tilde{f}(x) = g(x,0)$ and

$f(x, \tilde{f}(x)) = f(x, g(x,0)) = 0$.

what about continuity & differentiability of the map. →

do a direct proof. →

$\therefore$ $g_y(x)$ has a unique fixed point x in $\overline{B_r}(0)$. This point is the unique solution of $f(x)=y$.

(iii) Implicit fns theorem.

$f(x_0, y_0) = 0 \qquad f: X \times Y \to Z$

$D_2 : Y \to Z$ is an isomorphism.

Then in a neighborhood of (x_0, y_0), the solutions of $f(x,y)=0$ lie on a differentiable curve $(x, y(x))$.

consider $\Phi : \underbrace{U}_{\text{neighborhood of } x_0} \times \underbrace{V}_{\text{of } y_0.} \to X \times Z \; ; \; (x,y) \to (x, f(x,y))$.

$$D\Phi = \begin{pmatrix} I & 0 \\ D_1 f & D_2 f \end{pmatrix}$$

invertible with $(D\Phi)^{-1} = \begin{pmatrix} I & 0 \\ -[D_2 f]^{-1} D_1 f & [D_2 f]^{-1} \end{pmatrix}$

Φ has a unique local inverse, $(x,w) \to (x, g(x,w))$.

if $w=0$, let $\tilde{f}(x) = g(x,0)$ and $f(x, \tilde{f}(x)) = f(x, g(x,0)) = 0$.

review degree theory. →

(21) (Real) Brouwer & Schauder fixed point theorems

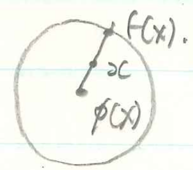
(i) Let D be the open unit ball in $\mathbb{R}^n$. Then there is no continuous mapping $f: \bar{D} \to \partial D$ such that $f|_{\partial D}$ is the identity. Otherwise

$$d(f, D, 0) = d(I, D, 0) = 1 \neq 0$$

Therefore, $0 \in f(D)$ #.

(ii) Brouwer's fixed point theorem. Let D be any set homeomorphic to an open ball and let $\phi: \bar{D} \to \bar{D}$ be continuous. Then ϕ has a fixed point in $\bar{D}$, ie. there is an $x \in \bar{D}$ with $\phi(x) = x$.

Assume $\bar{D}$ is the unit ball WLOG, and assume that ϕ has no fixed point. Construct the map $f: \bar{D} \to \partial D$ as follows:



This function f violates (i).

(iii) Schauder's fixed point theorem:
if Λ is a convex compact set of a Banach space B and T is a continuous mapping of Λ into itself. This map has a fixed point in Λ.

Construct S which maps Λ continuously to a finite dimensional closed convex subset of Λ with

$$\|S(x)-x\| < \varepsilon$$

ST has a fixed point y (Brouwer) and $\|T(y)-y\| < \varepsilon$. Use compactness...

(22) (Real) Dini's theorem.
Monotone convergence (on a compact set) of continuous fn's to a continuous fn is uniform.

(23) (Real)

if f is AC then $T_a^b(f) = \int_a^b |f'|$.

$$\int |f'| = \int |P' - N'|$$
$$\leq \int P' + N' \leq P_a^b + N_a^b = T_a^b.$$

$f(x) = \int^x f'^+ - \int^x f'^-$ since f is AC.

$$T_a^b(f) \leq T_a^b \int f'^+ + T_a^b \int f'^-.$$
$$\leq \int_a^b (f')^+ + \int_a^b (f')^-$$
$$= \int_a^b |f'|.$$

(24) (Real) Hahn decomposition theorem.

Let ν be a signed measure on the measurable space $(X, \mathcal{B})$. Then there is a positive set A and a negative set B such that $X = A \cup B$ and $A \cap B = \phi$.

Let $\lambda = \sup_{A \in \mathcal{B} \text{ positive}} \nu A$

the sup is attained by a +ve set A whose complement must be -ve. ✓

Can define $\nu^+(E) = \nu(E \cap A)$
$\nu^-(E) = \nu(E \cap B)$.

ν^+ and ν^- mutually singular - the Jordan decomposition of ν.

(25) (Real) Littlewood's principles.

(i) Every measurable set E is nearly a finite union of intervals:

given $\varepsilon > 0$ $\exists$ open set $O \supset E$ with $m(O \sim E) < \varepsilon$, and a closed set F with $m(E \sim F) < \varepsilon$.

mE finite $\Rightarrow$ given $\varepsilon > 0$ there is a finite union U of open intervals such that $m(U \Delta E) < \varepsilon$.

(ii) every measurable fn is nearly continuous.

(Lusin) Let f be a measurable real valued fn on $[a,b]$. Given $\delta > 0$

if $f_n \to f$ in L^p, then $\to$
there is a subsequence that
converges a.e.

there is a continuous fn φ on $[a,b]$
such that $m\{x : f(x) \neq \varphi(x)\} < \delta$.

Given $f \in L^p$ and $\varepsilon > 0$, there is a
continuous fn φ and a step fn ψ
such that
$\|f - \varphi\| < \varepsilon$ & $\|f - \psi\| < \varepsilon$.

(iii) Every convergent sequence of
measurable functions is nearly
uniformly convergent.
E finite measure : $f_n \to f$ a.e. on E.
Given $\varepsilon > 0$ and $\delta > 0$ there is a
measurable set $A \subset E$ with $mA < \delta$
and an integer N such that
for all $x \notin A$ and $n \geq N$,
$|f_n(x) - f(x)| < \varepsilon$.
(Egoroff) if $f_n \to f$ on a set of
finite measure then given $\eta > 0$ there
is a subset $A \subset E$ with $mA < \eta$ such
that f_n converges to f uniformly
on $E \sim A$.

(26) (Hilbert Space). Hahn Banach

Let M be a subspace of a Hilbert Space H and let $f: M \to \mathbb{C}$ be a linear functional on M with bound K. There is a ! extension of f to a continuous linear functional on H with the same bound K.

Note: Much harder proof in Banach space case.

Applications:

(i) Let x be an element in a normed vector space X. Then there is a bounded linear on X such that $f(x) = \|f\| \|x\|$.

(ii) If M is a linear subspace, y is in $\overline{M}$ iff there is no bounded linear f~~n~~al f on X such that $f(x) = 0$ for all $x \in M$ but $f(x_0) \neq 0$.

complete the proof. →

(27) (Real). Open Mapping / Closed Graph Theorems.

(i) If A is a bounded linear transformation of X <u>onto</u> Y then the image by A of the unit sphere in X contains a sphere about the origin in Y (A is an open mapping).

- let $S_0 = \{ x : \|x\| < 1 \}$.

  $X = \bigcup_{k=1}^{\infty} k S_0$; $Y = \bigcup_{k=1}^{\infty} k A(S_0)$ (A is onto).

Y complete $\Rightarrow$ $\overline{A(S_0)}$ contains a sphere...

(ii) A bounded, linear, from X <u>onto</u> Y, then $\|Ax\| \geq \delta \|x\|$ for some $\delta > 0$. In particular, if A is 1-1 then A^{-1} is bounded, (if A is 1-1 and range closed, there is a bounded inverse on the range of A).

(iii) Let X be a linear vector space complete in the two norms $\|\cdot\|$ and $|||\cdot|||$. If $\|x\| \leq c|||x|||$ $\forall x$ then the two norms are equivalent.

this means that the graph is closed. →

"moving bump" gives an example of where equality doesn't hold. →

(iv) A linear from X to Y. If whenever $x_n \to x$ and $Ax_n \to y$, $y = Ax$, then A is continuous.

- define $|||x||| = \|x\| + \|Ax\|$. X complete in this norm, so by (iii)

$$\|x\| + \|Ax\| \leq C'\|x\|$$

$$\Rightarrow \quad \|Ax\| \leq C'\|x\| \quad : A \text{ bounded}.$$

(28) (Real). Convergence theorems.

(i) If $f_n \to f$ pointwise on a set of finite measure and $\{f_n\}$ are bounded by M, then $\int f_n = \int f$ (use the fact that pointwise convergence of functions is almost uniform convergence).

(ii) (Fatou). If a sequence of non-negative functions $\{f_n\}$ converge pointwise to f, then

$$\int f \leq \liminf \int f_n.$$

- $\int f = \sup\limits_{\substack{h \text{ bounded} \\ \text{and } m(\text{support}) \text{ finite} \\ \text{and } h \leq f}} \int h.$

let $h_n = \min\{h(x), f_n(x)\}$.

application to completeness of L^p spaces (Riesz - Fischer). →

Bounded convergence theorem (i)

$\int h = \lim \int_E h_n \leq \liminf \int_E f_n$.

(iii) Monotone convergence theorem.
if $\{f_n\}$ are nonnegative and $f_n(x)$ is an increasing sequence and $f_n \to f$ pointwise, then

$$\int f = \lim \int f_n.$$

(iv) (Dominated convergence) Let g be integrable and let f_n be a sequence of measurable f_n's such that $|f_n| \leq g$ and $f_n(x) \to f(x)$. then

$$\int f = \lim \int f_n.$$

Application: $\hat{f} \in C(\mathbb{R})$ if $f \in L^1$.

29 (Real). E measurable if

$$m^*(A) = m^*(A \cap E) + m^*(A \cap \tilde{E}) \text{ for}$$

all sets A.

use n'th root test. →

means $\int_\alpha^\beta \frac{K(\eta(s))}{\eta - z} \eta'(s)\, ds.$ →

(30) (Complex). Cauchy's theorem.

(i) f is differentiable if the limit

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \quad \text{exists.}$$

if $f'(z_0)$ exists for every $z_0 \in \Omega$, we say that f is holomorphic in Ω.

if $f(z) = u + iv$, then $u_x = v_y$ and $u_y = -v_x$.

if f is representable by power series, then f is holomorphic and f' representable by power series. (in fact f has derivatives of all orders).

(ii) Suppose Γ is a simple closed curve and

(*). $f(z) = \int_\Gamma \frac{K(\eta)}{\eta - z} d\eta$ K continuous, say

then f is representable by a power

review those rules for →
interchanging summation, differentiation, integration, etc.

review derivation of this. →

series inside Γ. Use the fact that

$$\frac{1}{\eta - z} = \sum_{n=0}^{\infty} \frac{(z-a)^n}{(\eta - a)^{n+1}}$$

converges whenever $\frac{|z-a|}{|\eta - a|} < 1$

ie. whenever $|z-a|$ is smaller than the distance of a to Γ. Then can pass the integral into the sum in ⊛ and get

$$f(z) = \sum c_n (z-a)^n$$

where $$c_n = \int_\Gamma \frac{K(\eta)}{(\eta - a)^{n+1}}$$

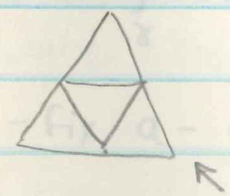
(iii) $\mathrm{Ind}_\gamma(z) = \frac{1}{2\pi i} \int_\gamma \frac{d\eta}{\eta - z} \qquad z \in \Omega = (\gamma^*)^c$

- integer valued
- constant in each component of Ω
- 0 in unbounded component.

Ω simply connected. $\rightarrow$

(iv). If $F \in H(\Omega)$ and F' is continuous in Ω, then $\int_\gamma F'(z)\,dz = 0$ for every closed path γ in Ω. (gives Cauchy's theorem for polynomials).

(V) If Δ is a closed triangle in a plane open set Ω; $p \in \Omega$, f is continuous on Ω, and $f \in H(\Omega - \{p\})$ then $\underbrace{\int_{\partial\Delta} f(z)\,dz}_{J} = 0$.



$$J = \sum_{j=1}^{4} \int_{\partial\Delta_j} f(z)\,dz$$

one of these is at least $|J|/4$ in absolute value.

$\therefore |J| \le 4^n \left| \int_{\partial\Delta_n} f(z)\,dz \right|$

length $O(2^{-n})$.

z_0 - point Δ_n have in common.

f differentiable there, can make

$f(z) - f(z_0) - f'(z_0)(z - z_0) < \varepsilon |z - z_0|$

if $z_0 = p$, need to do some more work. $\rightarrow$

$$\therefore \int_{\partial\Delta_n} f(z)\,dz = \int_{\partial\Delta_n} [f(z) - \underbrace{f(z_0) - f'(z_0)(z-z_0)}_{\text{polynomials so can throw them in.}}]\,dz$$

$$\left|\int_{\partial\Delta_n} f(z)\,dz\right| \le \varepsilon\, 4^{-n}.$$

$\Rightarrow |J| \le \varepsilon.$

(vi) Ω convex open set $p \in \Omega$.

f continuous on Ω and $f \in H(\Omega - \{p\})$.

Then $f = F'$ for some $F \in H(\Omega)$. Hence

$$\int_\gamma f(z)\,dz = 0 \quad \text{by (iv).}$$

- fix a - define $F(z) = \int_{[a,z]} f(\xi)\,d\xi$.

$F(z) - F(z_0) =$ integral over $[z_0, z]$

by (v).

$$\therefore \frac{F(z)-F(z_0)}{z-z_0} - f(z_0) = \frac{1}{z-z_0}\int_{[z_0,z]} f(\xi)-f(z_0)\,d\xi$$

continuity of $f \Rightarrow f = F'$!

need the index here. →

Show that if $f'(z_0) \neq 0$ then →
the map is invertible and that
the inverse map is holomorphic.

this shows that f is representable by power series.

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\xi)}{\xi - z}\, d\xi.$$

(use $g(\xi) = \begin{cases} \dfrac{f(\xi) - f(z)}{\xi - z} & \xi \in \Omega,\ \xi \neq z \\ f'(z) & \xi = z. \end{cases}$

(vii) Morera's theorem. If f is a continuous complex fn in an open set Ω such that $\int_{\partial\Delta} f(z)\,dz = 0$ for every closed $\Delta \subset \Omega$, then $f \in H(\Omega)$.

(31) (Complex) Open mapping Theorem. If Ω is a region and $f \in H(\Omega)$ then $f(\Omega)$ is either a region or a point.

Every nonconstant holomorphic function in a region is locally of the form ($f(z_0) = w_0$)

$$f(z) = w_0 + [\varphi(z)]^m$$

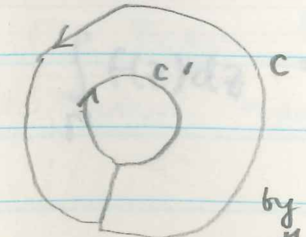
($f - w_0$ has an m'th order zero at z_0)

where $\varphi(z)$ is a bijection from a neighborhood of z_0 to a neighborhood of zero.

<u>Cor</u>: if f is 1-1 in Ω then $f'(z) \neq 0$ for every $z \in \Omega$ and the inverse of f is holomorphic. (Note the example e^z that contradicts the reverse reasoning)

(32) (Complex). Laurent series.

Let $f(z)$ be a function analytic in the ring shaped region between two concentric circles C and C', of radii R and R' ($R' < R$), and centre a, then f can be expanded in a series of positive and negative powers of $z-a$, convergent at all points of the ring-shaped region.



$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(w)\,dw}{w-z} - \frac{1}{2\pi i}\int_{C'} \frac{f(w)\,dw}{w-z}.$$

by Cauchy's theorem.

$$\frac{1}{w-z} = \sum_{n=0}^{\infty} \frac{(z-a)^n}{(w-a)^{n+1}} \rightarrow$$

converges uniformly for $|z-a| < R$.

$$\frac{1}{z-w} = \sum_{n=1}^{\infty} \frac{(w-a)^{n-1}}{(z-a)^n}$$

converges uniformly for $|z-a| > R'$.

$$\frac{1}{2\pi i} \int_C \frac{f(w)}{w-z} = \sum_{n=0}^{\infty} a_n (z-a)^n$$

where $a_n = \frac{1}{2\pi i} \int_C \frac{f(w)}{(w-a)^{n+1}} dw$.

but $\frac{1}{2\pi i} \int_{C'} \frac{f(w)}{z-w} = \sum_{n=1}^{\infty} b_n (z-a)^{-n}$.

where $b_n = \frac{1}{2\pi i} \int_{C'} f(w)(w-a)^{n-1} dw$.

when combined,

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n$$

where $a_n = \frac{1}{2\pi i} \int \frac{f(w)}{(w-a)^{n+1}} dw$.

33 (complex) Residue theorem.

Note from 32 that $\int_{C'} f(w)dw = 2\pi i\, b_1$.

denote as $Res(f;a)$.

(i) Use this to compute

$$\int_\Gamma f(z)dz = \sum_{a \in A} Res(f;a)$$

poles enclosed by Γ.

(ii) If f is analytic in C, then

$$\frac{1}{2\pi i}\int_{\partial C}\frac{f'(z)}{f(z)}\,dz = \underset{\substack{\uparrow \\ \text{number of zeros} \\ \text{in } C.}}{N}$$

(Compute using residue formula).

(iii) Note that $\dfrac{f'(z)}{f(z)} = \dfrac{d}{dz}\log\{f(z)\}$.

so $$\int_{\partial C}\frac{f'(z)}{f(z)}\,dz = \Delta_C \log\{f(z)\}$$

$$= i\,\Delta_C \arg\{f(z)\}.$$

$$\therefore\ N = \frac{1}{2\pi}\,\Delta_C \arg\{f(z)\}.$$

(iv) (Rouché's theorem). If $f(z)$ and $g(z)$ are analytic inside and on a closed contour C and $|g(z)| < |f(z)|$ on C, then $f(z)$ and $f(z)+g(z)$ have the same number of zeros inside C.

write in terms of logarithm

$$\Delta_C \arg(f+g) = \Delta_C \arg f + \underbrace{\Delta_C \arg(1+g/f)}_{\text{zero!}}$$

✓

(34) (complex). $f(z) = \sum_{n=0}^{\infty} z^{n!}$ - function with $|z| = 1$ a natural boundary.

(35) (complex) Poisson formula.

$$P_r(\theta - t) = \mathrm{Re}\left[\frac{e^{it} + \overset{re^{i\theta}}{z}}{e^{it} - z}\right] = \frac{1 - r^2}{1 - 2r\cos(\theta - t) + r^2}$$

$$P_r(t) = \sum_{-\infty}^{\infty} r^{|n|} e^{int} \qquad (*)$$

- P harmonic for fixed t (real part of an analytic function).
- P approximates $\delta(\theta - t)$ as $r \to 1$:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(t)\,dt = 1 \qquad \text{from } (*)$$

$$0 < P_r(t) < P_r(\delta) \qquad 0 < \delta < |t| \le \pi$$

and $\lim_{r \to 1} P_r(\delta) = 0 \qquad 0 < \delta \le \pi,$

Need uniqueness for conjugate → harmonic function.

actually, prove here that →
$h(z) < \max_{\partial U} h$ for $z \in U$.

Therefore $F(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t) f(t)\,dt$

is harmonic and has boundary values f (solves the Dirichlet problem).

Define $f(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{-it}+z}\, u(e^{it})\,dt$.

If u were harmonic in the unit disk, then u is the real part of $f(z)$.

(36) (PDE) Uniqueness of solutions of the Dirichlet problem. Method 1.

Suppose h is harmonic in a bounded region U and continuous on $\bar{U}$. h cannot attain a maximum value in the interior of U.

Suppose $\max_{\partial U} h(z) = M$ and $h(z_0) > M$.

Choose $\varepsilon > 0$ so that $g(z) = h(z) + \varepsilon |z - z_0|^2$ satisfies $\max_{\partial U} g(z) < h(z_0) = g(z_0)$.

check the reasoning behind this. →

Therefore there is an interior point z_1 for which g attains a maximum. At this point $g_{xx} \le 0$ and $g_{yy} \le 0$.

But $\Delta g = 4\varepsilon > 0$ always #.

(37) (PDE) Harnack

(i) If $u_n \to u$ uniformly on compact subsets of Ω, then u is harmonic in Ω. Proof: $u(x)$ is locally a poisson integral.

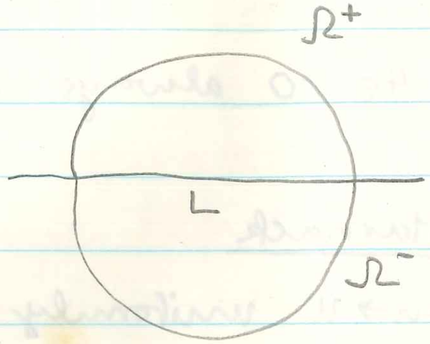
(ii) If $u_1 \le u_2 \le u_3 \le \cdots$ then either u_n converges uniformly on compact subsets of Ω or $u_n(z) \to \infty$ for every $z \in \Omega$. Proof:

$$\frac{R-r}{R+r} \le P_r(t) \le \frac{R+r}{R-r}$$

Therefore for u_n positive (add constant if necessary).

$$\frac{R-r}{R+r} u_n(a) \le u_n(a + re^{i\theta}) \le \frac{R+r}{R-r} u_n(a).$$

$\therefore$ Set where $u = \sup u_n$ is bounded is open, where ∞ is open $\Rightarrow$ result.



Note: You get similar results in 3D.

38 (Complex) Schwarz Reflection Principle.

Suppose v is harmonic in a region in the upper half plane bounded by a segment L of the real line and $v=0$ on L. Extend v to Ω^- by the formula $v(x,-y) = -v(x,y)$. Now v defined on $\Omega = \Omega^- \cup \Omega^+ \cup L$ has the mean value property $\Rightarrow$ v harmonic in Ω.

Extension: an analytic function which is real on L can be extended from Ω^+ to Ω^- by the formula $F(\bar{z}) = \overline{F(z)}$.

39 (Complex) Maps on the unit disk.

Suppose $f \in H(U)$, f is 1-1, $f(U) = U$, $\alpha \in U$, and $f(\alpha) = 0$. Then there is a constant λ, $|\lambda| = 1$ such that

$$f(z) = \lambda \frac{z-\alpha}{1-\bar{\alpha} z}$$

(40) (Complex) Monodromy theorem.
Suppose Ω is a simply connected region, (f, D) is a function element, $D \subset \Omega$, and (f, D) can be analytically continued along every curve in Ω that starts at the center of D. Then there exists $g \in H(\Omega)$ such that $g(z) = f(z)$ for all $z \in D$.

(41) (Complex) Picard's theorem.
(i) (Little) every nonconstant entire function attains each value, with one possible exception.
(ii) (Big) every entire fn which is not a polynomial attains each value infinitely many times, again with one possible exception.

(42) (PDE) Technique for PDE's - one unknown, arbitrary dimension, first order.

(i) Linear

$$a(x,y)\, u_x + b(x,y)\, u_y = c(x,y)$$

with u given on a curve $(x_0(s), y_0(s))$.

Consider curves $x(t), y(t)$ along which

$$\dot{x} = a(x(t), y(t)) \qquad x(0) = x_0(s)$$

$$\dot{y} = b(x(t), y(t)) \qquad y(0) = y_0(s).$$

↑ characteristic ground curves.

Along these curves,

$$\frac{du}{dt} = c(x(t), y(t)).$$

Have u, x, y as functions of s and t. Invertible if $\partial(x,y)/\partial(s,t) \neq 0$

ie. at $s=0$ $\begin{vmatrix} a & b \\ x_0' & y_0' \end{vmatrix} \neq 0.$

need eigenvectors to be differentiable → functions of x & y.

Also need $(Ax,x) \le \lambda_{max} \|x\|^2$.

(ii) Easy extension to quasilinear equations - this is essentially a local theory.

43 (PDE) Systems in 2 space variables and N unknowns.

(*) $u_{,y} + A(x,y)\, u_{,x} + B(x,y)\, u = 0.$

A symmetric $\Rightarrow$ N real eigenvalues & eigenvectors.

Suppose A has distinct eigenvalues.

The characteristics for (*) are the curves $x_n(s), y_n(s)$ with

$\dot{x}_n(s) = \lambda_n$ ← n'th eigenvalue of A.

$\dot{y}_n(s) = 1$

Choose a point (x_0, y_0) and draw the characteristics from that point to the line $y=0$:

continued on p. 89

44 (PDE) Linear equations.

(i) (convective) $u_t = u_x$

try $u = A(t)\, e^{ikx}$

$A'(t) = A(t)\, ik$

$\Rightarrow u = A\, e^{ik(x+t)}$

(ii) (dissipative) $u_t = u_{xx}$.

try $u = A(t)\, e^{ikx}$

$\Rightarrow A(t) = A\, e^{-k^2 t}$

and $u = e^{-k^2 t}\, e^{iwx}$.

(iii) (dispersive) $u_t = u_{xxx}$.

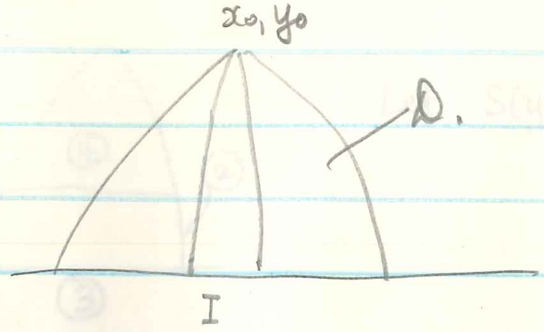
$A'(t) = -ik^3 A(t)$

$A(t) = A\, e^{-ik^3 t}$.

or $u = e^{i(kx - k^3 t)}$.

here $w = k^3$ $\frac{dw}{dk} = 3k^2$ group velocity.

continue with (43)



Thm: If $u=0$ on I then $u \equiv 0$ on D.

proof: Take inner product of (*) with u

$$\frac{1}{2}\frac{\partial}{\partial y}(u,u) + \left(u, A\frac{\partial u}{\partial x}\right) + (u, Bu) = 0$$

A symmetric $\Rightarrow$

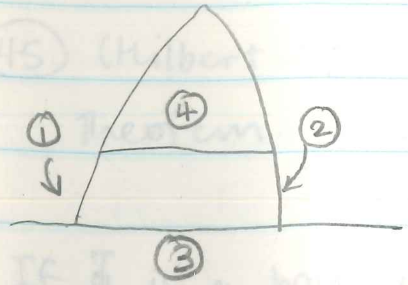
$$\frac{1}{2}\left[\frac{\partial}{\partial y}(u,u) + \frac{\partial}{\partial x}(u, Au)\right] = -(u,Bu) + \frac{1}{2}(u, A_x u)$$

can make $-B + A_x$ negative definite by a change of variables or can apply Gronwall later.

$$\therefore \int \frac{\partial}{\partial x}(u, Au) + \frac{\partial}{\partial y}(u,u) \le 0.$$

integrate by parts to get

$$\int_S n_x (u, Au) + n_y (u,u) \le 0.$$



Let $S(y) = \int |u(s,y)|^2 ds$

$$\int_{②} ds \; \underbrace{n_x}_{\substack{\geq 0 \\ \text{outward}}} \underbrace{\left[(u, Au) + \frac{n_y}{n_x}(u,u) \right]}_{\geq 0}, \qquad \frac{n_y}{n_x} = \frac{-t_x}{t_y} = -\lambda_1 .$$

≥ 0 since λ_1 smallest eigenvalue,

so $(u, Au) \geq \lambda_1 (u,u)$.

$\therefore \int_{②} ds \geq 0$; similarly $\int_{①} — ds \geq 0$

And so $S(y) \leq S(0) = 0$.

$\Rightarrow$ uniqueness of solution.

Domain of dependence

Range of influence.

(45) (Hilbert Space) Riesz Representation Theorem.

If Φ is a bounded linear functional in H, then $\exists! f \in H$ such that
$\Phi(h) = (h, f) \quad \forall h \in H$ Also $\|\phi\| = \|f\|$.

proof: let $G = \{g \in H : \Phi(g) = 0\}$ - closed linear manifold. By the projection theorem, $H = G \oplus G^{\perp}$. If $G = H$ then $f = 0$ works. Assume $f_0 \in G^{\perp}$, $f_0 \neq 0$. Then consider

$$\hat{f} = \Phi(h) f_0 - \Phi(f_0) h.$$

Now $\Phi(\hat{f}) = 0$ by linearity so $\hat{f} \in G$

so

$$(\hat{f}, f_0) = 0.$$

Solving for $\Phi(h)$, obtain

$$\Phi(h) = \left(h, \frac{\Phi(f_0) f_0}{\|f_0\|^2}\right)$$

this f works.

(uniqueness) if f_1 & f_2 work, then
$(h, f_1) = (h, f_2) \quad \forall h$
take $h = f_1 - f_2 \Rightarrow \|f_1 - f_2\|^2 = 0$.

(bound) $|\Phi(h)| = |(f,h)| \leq \|f\| \|h\|$ so $\|\Phi\| \leq \|f\|$
but $|\Phi(f)| = \|f\|^2$ so $\|\Phi\| = \|f\|$.

Application: showing the existence of the adjoint of a bounded operator.

46 (Hilbert Space) Lax-Milgram lemma.
Let B be a complex-valued bilinear function on $H \times H$ with
$|B(x,y)| \leq \gamma \|x\| \|y\|$
and $|B(x,x)| \geq \delta \|x\|^2$.

Then there is a ! determined linear operator S with bounded inverse S^{-1} such that $B(x,y) = (x, Sy)$ and $\|S\| \leq \gamma$; $\|S^{-1}\| \leq 1/\delta$.

(i) For fixed y, $B(x,y)$ is a bounded linear functional on H. By the Riesz Representation Theorem, there is a ! φ such that

$$B(x,y) = (x,\varphi) \quad \text{for all } x.$$

Denote by S the linear map that takes $y \to \varphi$. Since $\|\varphi\| \leq \gamma\|y\|$, S is bounded and $\|S\| < \gamma$.

(ii) Consider the following result:

$$\delta\|x\|^2 \leq |B(x,x)| = (x, Sx) \leq \|x\|\,\|Sx\|.$$

or $\|x\| \leq 1/\delta \; \|Sx\|$. ⊛

This can be used to show first that the range of S is closed and then that S is 1-1 and onto H. Therefore S^{-1} exists and by ⊛ is bounded by $1/\delta$.

(47) (Hilbert Space) Projection theorem.

If G is a closed subspace of H and $h \notin G$, then there exists a unique $g \in G$ such that

$$\delta = \inf_{x \in G} \|h - x\| = \|h - g\|$$

Furthermore, $h - g$ is in $G^{\perp}$.

(i) (uniqueness) Suppose g_1, g_2 are two such vectors, then

$$g^* = \frac{g_1 + g_2}{2} \in G.$$

so $\|h - g^*\| \geq \delta$.

But $\|h - g^*\| \leq \frac{1}{2}\|h - g_1\| + \frac{1}{2}\|h - g_2\| = \delta$.

Equality holds iff

$$h - g_1 = \lambda(h - g_2)$$

if $\lambda \neq 1$, then $(1-\lambda)h = g_1 + \lambda g_2$ so $h \in G$ #.

(ii) (existence) $\exists \{g_n\} \in G$ such that $\|h - g_n\| \to \delta$.

Need to show that $\{g_n\}$ is cauchy.

$$\|g_n - g_m\|^2 \overset{\text{parallelogram}}{=} -\|g_n - h + g_m + h\|^2 + 2\|g_n - h\|^2 + 2\|g_m - h\|^2$$

$$= 2\underbrace{\|g_n - h\|^2}_{\delta^2} + 2\underbrace{\|g_m - h\|^2}_{\delta^2} - 4\underbrace{\left\|h - \frac{g_n - g_m}{2}\right\|^2}_{\geq \delta^2}$$

(iii) (orthogonality). If $h-g$ not orthogonal to G, then there is an $x_0 \in G$ for which

$$(h-g, x_0) = a \neq 0.$$

Let $g^* = g + bx_0$.

$$\|h - g^*\|^2 = \|h - g\|^2 + |b|^2 \|x_0\|^2 - 2\mathrm{Re}(b\bar{a}).$$

choose $b = \frac{a}{\|x_0\|^2}$ and get $\|h-g^*\|^2 < \|h - g\|^2$

#.

48 (PDE) Perron's method for solving the Dirichlet problem.

$$M_\Gamma(V)(r) = \begin{cases} V(r) & r \notin \overline{\Gamma} \\ \text{for } r \in \overline{\Gamma} \text{ it is the harmonic function with boundary values } V \text{ on } \partial\Gamma. \end{cases}$$

sphere

- If $V \leq M_\Gamma(V)$ for any Γ then V is subharmonic.
- Subharmonic fn's have a maximum principle. Also if $V(r) = \max(V_1, V_2)$ where V_1, V_2 are subharmonic, then V is subharmonic.
- $M_\Gamma(V)$ is also subharmonic.
- If $\varphi(x)$ is continuous on ∂D and $V(x)$ is subharmonic and continuous in $\overline{D}$ and $V(x) \leq \varphi(x)$ on ∂D then V is called a subfunction. If F is the set of subfunctions, it is easy to see that F is nonempty (take $\inf \varphi(x)$, $x \in \partial D$).

examine this argument →
C-H p. 310.

why can we write this pair →
of inequalities? ✓
~~OK, replace by~~ $V-W$ is a subf$\underline{n}$ so
must attain max on boundary (≤ 0).

Take $V(x) = \sup_{f \in F} f(x)$ - valid since

this is bounded above by the maximum of φ. $V(x)$ is harmonic.

A point $r_0 \in \partial D$ is regular if $\exists$ a super harmonic function in D $W_{r_0}(r)$ continuous onto the boundary of D with $W_{r_0}(r) > 0$ on ∂D except for $r = r_0$, where $W_{r_0}(r_0) = 0$.

Then for any $\varepsilon > 0$ and a constant k sufficiently large,

$$V(r) = \varphi(r_0) - \varepsilon - k\, W_{r_0}(r)$$

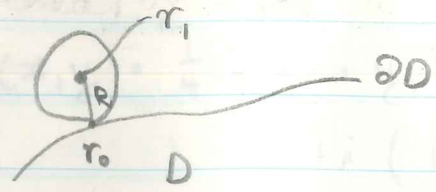
$$W(r) = \varphi(r_0) + \varepsilon + k\, W_{r_0}(r)$$

are subfunctions and superfunctions, respectively.

$\therefore \ \phi(r_0) - \varepsilon - k\, W_{r_0}(r) \le u(r) \le W(r)$.

This shows that $u(r) \to \phi(r_0)$ as $r \to r_0$.

If the boundary looks like



use the Barrier function $W_{r_0}(r) = \frac{1}{R} - \frac{1}{|r - r_1|}$

(49) (Hilbert Space) Miscellaneous

(i) Cauchy - Schwarz.

$$|(f,g)| \le \|f\| \, \|g\|. \quad (*)$$

To prove this, consider (λ real)

$$\|f + \lambda (f,g) g\|^2 = \|f\|^2 + 2\lambda |(f,g)|^2 + \lambda^2 |(f,g)|^2 \|g\|^2 \geq 0.$$

Discriminant must be zero $\Rightarrow$ $(*)$.

(ii) Generalized C-S : if A is a positive operator, then

$$|(Af,g)|^2 \le (Af,f)\,(Ag,g).$$

If a Hilbert space satisfies the parallelogram law, can define an inner product with the polarization identity.

(iii) Polarization identity

$$(x,y) = \frac{1}{4}\{\|x+y\|^2 - \|x-y\|^2 + i(\|x+iy\|^2 - \|x-iy\|^2)\}.$$

(iv) Generalized polarization identity:

$$(Ax,y) = \frac{1}{4}\{(A(x+y),(x+y)) - (A(x-y),(x-y)) + i[(A(x+iy), x+iy) - (A(x-iy), x-iy)]\}.$$

(v). Parallelogram Law

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

50 (PDE) Characteristics.

(i) A surface is non-characteristic if you can solve for the highest normal derivative using the equation on that surface.

to replace x_n say.

introduce ϕ as a coordinate, then

$$\frac{\partial u}{\partial x_i} = \frac{\partial u}{\partial x_i} + \frac{\partial u}{\partial \phi}\frac{\partial \phi}{\partial x_i}$$

so coefficient of $\frac{\partial u}{\partial \phi}$ in the equation

will be $\sum \frac{\partial \phi}{\partial x_i} a_{-\,1\,-}$

ith position.

can generalize to $a_{ij\alpha}(x,u)$ quasilinear systems.

i.e. if $L = \sum_{|\alpha|\le m} a_\alpha(x)\,\partial^\alpha$, the hypersurface $\{x_n = 0\}$ is noncharacteristic at the origin if $a_\alpha(0) \neq 0$ where $\alpha = (0,0,\cdots,m)$.

For a non-planar surface, $\phi = 0$.

(*) $\sum_{|\alpha|=m} a_\alpha(0)\,\xi^\alpha \neq 0$ where $\xi = \nabla\phi(0)$.

Note A function is called elliptic at a point if (*) is satisfied for every vector $\xi \neq 0$. An operator can only be elliptic if m is even.

Ex $L = a_{ij}(x)\,\partial_{ij}u + b_i(x)\,u_i + cu = f$

$$\sum_{|\alpha|=2} a_\alpha \xi^\alpha = a_{ij}(x)\,\xi_i\xi_j$$

So the equation is elliptic iff a_{ij} is definate.

(ii) For a system, $a_{ij\alpha}\frac{\partial u_j}{\partial x^\alpha} = F_i(x,u)$, need the matrix

$$a_{ij\alpha}\,\xi^\alpha$$

to be nonsingular.

(iii) Relationship between that and travelling waves. Suppose equations only have derivatives of order m and that coefficients are constant

$$a_{ij\alpha} \frac{\partial u}{\partial x^\alpha} = 0 \qquad |\alpha| = m.$$

$\alpha = 0$ is identified with time.

look for solutions of the form $\underline{U}\, e^{i(\omega t + \underline{k}\cdot x)}$.

$$\Rightarrow \left(a_{ij(1-)}\, \omega + a_{ij\alpha} k^\alpha \right) \underline{u} = 0.$$

ie the matrix $a_{ij(1-)}\,\omega + a_{ij\alpha} k^\alpha$ is singular.

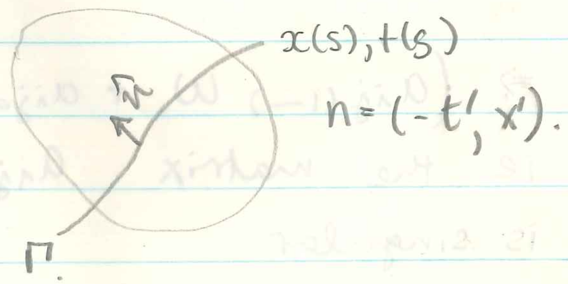
In the case $a_{ij(1-)} = I$, this reduces to the statement that $a_{ij\alpha} k^\alpha$ has an eigenvalue $1/\omega$ and $\bar{u}$ is a corresponding eigenvector.

If for every choice of $k^\alpha \neq 0$, the matrix $a_{ij\alpha} k^\alpha$ has n distinct eigenvectors, then the equations are

$$\frac{\partial u_i}{\partial t} + A_{ij}\frac{\partial u_j}{\partial x} = 0 \quad \rightarrow$$

recall $u = f(x+t) + g(x-t)$ $\rightarrow$

see that discontinuities propagate along characteristics.



with t a timelike direction.

called hyperbolic. In the case of one space dimension, this reduces to the statement that A_{ij} has n real eigen values.

(iv) Discontinuities in weak solutions are allowed only across characteristics.

$u_{xx} - u_{tt} = 0.$

Weak solution $\Rightarrow$ $\int (w_{xx} - w_{tt})\, u = 0$ for every $w \in C_0^\infty$.

Suppose u is C^2 except along a curve Γ. Integrate by parts on each side $\Rightarrow$

$$\left[\int n_x (w_x u - u_x w) - n_t (w_t u - w u_t)\right] = 0$$

$$\int (n_x w_x - n_t w_t)\,[u] - w\,(n_x [u,x] - n_t [u,t]).$$

take $w \equiv 0$ $\Rightarrow$ $\int (t' w_x + x' w_t)\,[u] = 0.$

what about higher space dimensions? →

W arbitrary (except for $W \equiv 0$ on Γ)

so if $[u] \neq 0$, need

$(\dot t, \dot x) = k(\dot x, \dot t)$. (only direction so that ∇W is zero).

$\Rightarrow \dot t = \pm \dot x$ or $x \pm t$ = constant.

Now allow $W \neq 0$ on Γ:

$$\int_\Gamma \frac{dW}{ds}[u] - \frac{d[u]}{ds} W = 0$$

integrate by parts and combine $\Rightarrow$

$$\int_\Gamma \frac{d[u]}{ds} W = 0 \Rightarrow \frac{d[u]}{ds} = c$$

so $[u]$ is constant along C.

(51) (Hilbert Space) The Fredholm Alternative.

$f - Kf = g$ (1)

$f - K^*f = 0$. (2).

The solutions of (2) and $f - Kf = 0$ (3) form subspaces of the same finite (zero or positive) dimension D. If $D = 0$, the

Finite dimension $H = R(T) \oplus N(T^*)$ →

inverse $(I-K)^{-1}$ exists, so that (1) is solvable for every element g and $\|f\| \le C\|g\|$. If $D > 0$, (1) is solvable iff g is orthogonal to every solution of (2) and the solution is not unique (can add an arbitrary solution of (3)).

Any completely continuous operator K defined on a Hilbert space can be expressed as the limit of degenerate operators (proof not too hard in separable case).

Let $K = \underbrace{K_1}_{\text{degenerate}} + K_2 \ni \|K_2\| < 1$.

Then $(I-K_2)^{-1}$ exists (Neumann series), and (1) can be rewritten

$$(I - (I-K_2)^{-1}K_1)f = (I-K_2)^{-1}g.$$

$I - (I-K_2)^{-1}K_1$ is degenerate, so it has the same nullity as its adjoint

$$I - K_1^*(I-K_2^*)^{-1}$$
$$= (I-K^*)(I-K_2^*)^{-1}$$

and so has the same nullity as $(I - K^*)$....

(52) (Hilbert Space) If K is compact and self-adjoint, then the set $\{\phi_k\}$ of eigenfunctions of K are orthogonal and the eigenvalues μ_k tend to zero. Also,

$$Kf = \sum_{k=1}^{\infty} \mu_k (f, \phi_k) \phi_k$$

and the elements $\{\phi_k\}$ span H iff the equation $Kf = 0$ possesses only the trivial solution $f = 0$.

(i) Let $N_T = \sup_{\|f\|=1} |(Tf, f)|$. $N_T = \|T\|$.

(ii) There exists a sequence $\{f_n\}$ so that $(Kf_n, f_n) \to \mu_1$ where $\mu_1 = \pm \|K\| \neq 0$. Show that $Kf_n \to \mu f_n$ extract a convergent subsequence, get $Kf = \mu f$ ✓. Repeat on orthogonal subspace, ... $\mu_n \to 0$ by the compactness of K.

(53) (Real) A function is an indefinite integral iff it is absolutely continuous. Every absolutely continuous function is the indefinite integral of its derivative.

<u>Key lemma</u>: If f is absolutely continuous on $[a,b]$ and $f'(x)=0$ a.e. then f is constant.

(54) (Real) $\alpha^{\lambda}\beta^{1-\lambda} \leq \lambda\alpha + (1-\lambda)\beta$. Equality holds iff $\alpha=\beta \Rightarrow$ Hölder inequality (if $f \in L^p$ and $g \in L^q$ then $f \cdot g \in L^1$ and $\int |fg| \leq \|f\|_p \|g\|_q$) $\Rightarrow$ Minkowski ($\|f+g\|_p \leq \|f\|_p + \|g\|_p$).

X complete iff every absolutely summable series is summable. Bounded convergence theorem $\Rightarrow$ L^p spaces are complete (Riesz-Fischer).

need proof of this. / OK. →

$$p(\lambda) = c \prod (\lambda - \alpha_i) \prod (\beta_j - \lambda) \prod_k [(\lambda-\gamma_k)^2 + \delta_k^2]$$

$c \geq 0 \quad \alpha_i \leq m \;,\; \beta_j \geq M.$

complex conjugate roots.

(55) Spectral theorems for bounded operators:

(i) Every bounded monotone sequence of self-adjoint operators converges strongly to a self adjoint operator A.

+ve

$$\|(A_m - A_n)f\|^4 = ((A_m - A_n)f, (A_m - A_n)f)^2$$
$$\leq (A_{mn}f, f)\,(A_{mn}^2 f, A_{mn} f).$$
$$\leq |(A_m f, f) - (A_n f, f)|\, K \|f\|^2$$

Generalized C-K.

monotone, bounded so converges.

(ii) $P^2 = P$ and P self-adjoint $\Rightarrow$ P projection onto $\{h \in H : Ph = h\}$.

(iii) polynomial functions of A ✓ If $mI \leq A \leq MI$ and if $p(\lambda) \geq 0$ for $m \leq \lambda \leq M$ then $p(A) \geq 0$. Extend definition of $f(A)$ to non-polynomial functions to monotone limits of polynomials.

so E_μ is a projection →

why is $f(A)g(A) = f \cdot g(A)$? →

works for polynomials.

(iv) we want $e_\mu(\lambda) = \begin{cases} 1 & \lambda \le \mu \\ 0 & \lambda > \mu \end{cases}$

(right continuous).

$\{E_\mu\}$ is a resolution of the identity: self adjoint operators:

$$E_\alpha = 0 \qquad E_\beta = I$$

$$E_{t+0} = E_t$$

$$E_\mu E_\nu = E_{\min(\mu,\nu)}.$$

(v) $\mu(e_\nu(\lambda) - e_\mu(\lambda)) \le \lambda(e_\nu(\lambda) - e_\mu(\lambda))$

$\le \nu(e_\nu(\lambda) - e_\mu(\lambda)).$

$\Rightarrow \mu(E_\nu - E_\mu) \le A(E_\nu - E_\mu) \le \lambda(E_\nu - E_\mu).$

then $\sum_{k=1}^{n} \mu_{k-1}(E_{\mu_k} - E_{\mu_{k-1}}) \le A \le \sum_{k=1}^{n} \mu_k(E_{\mu_k} - E_{\mu_{k-1}})$

↑ telescoping series.

Are they really the uniform limit? yes!

$$\varphi(\lambda) = \sum_{k=1}^{n} u(\nu_k) \left[e_{\mu_k}(\lambda) - e_{\mu_{k-1}}(\lambda) \right].$$

$$\pm \left[u(\lambda) - \varphi(\lambda) \right] \leq \omega$$

$$\Rightarrow \quad \pm \left[u(A) - \varphi(A) \right] \leq \omega I$$

$$\Rightarrow \quad \| u(A) - \varphi(A) \| \leq \omega.$$

So if λ_k is any number in $(\mu_k ; \mu_k)$ and $\max_k (\mu_k - \mu_{k-1}) < \varepsilon$ then

$$\| Af - \sum_{k=1}^{n} \lambda_k (E_{\mu_k} - E_{\mu_{k-1}}) f \| < \varepsilon \|f\|.$$

$\therefore$ A is the uniform limit of the sums, which are denoted by

$$\int_{m-0}^{M} \lambda \, dE_\lambda.$$

(vi) $$\left[\sum_{n=1}^{n} \lambda_k (E_{\mu_k} - E_{\mu_{k-1}}) \right]^r = \sum_{k=1}^{n} \lambda_k^r (E_{\mu_k} - E_{\mu_{k-1}})$$

$$\Rightarrow \quad A^r = \int_{m-1}^{M} \lambda^r \, dE_\lambda.$$

extend now to polynomials and continuous functions.

(vii) for U unitary,

$$Uf = \int_0^{2\pi} e^{it} \, dE_t \, f.$$

independent variables $(t, \underline{x})$. →

(56) (PDE) C-K Theorem.

(i) (Linear) Can consider only a first order system WLOG (see (ii)).

$$u_t = \sum_{\ell=1}^{n} A^\ell(x,t) \frac{\partial u}{\partial x_\ell} + B(x,t)\, u + f$$

$$u(0,x) = v(x) \quad \text{given.}$$

With A^ℓ, B, f, v analytic in x in a neighborhood of $(0,0)$.

Note: can make $v \equiv 0$ by changing f.

Write as $u(t,\cdot) = \int_0^t (T(\sigma)u + f(\sigma))\, d\sigma$. (*)

Introduce complexity $z_j = x_j + i y_j$ and assume that the power series converge for $|z_j| \le R = 1$ WLOG.

Use Cauchy's inequality: if w is a bounded holomorphic function of z in $|z| < R$, $|w(z)| \le K$, then

$$\left| \frac{dw}{dz}(z) \right| \le \frac{K}{R - |z|}.$$

there are some details missing →
about Holomorphic functions of more
than one variable.

Introduce the spaces used:

$$K_s = \{ \tilde{z} : |z_j| \le s \quad j=1, \dots n \} \quad 0 < s \le 1.$$

H_s is the space of functions $u = (u^1, \dots u^N)$
which are continuous on K_s and
holomorphic in the interior of K_s.

Norm $\|u\|_s = \max\limits_{z \in K_s} |u(z)|$.

$H(K(s))$ is complete in this norm (ie.
analytic functions that converge
uniformly converge to an analytic
function).

$T(t)$ maps $H(K(s))$ into $H(K(s'))$ and
is bounded as follows:

$$|T(t)u|_{s'} \le \frac{c}{s-s'} |u|_s$$

depends on A^2 and B.

Start with $u_0(z,t) \equiv 0$ and proceed by
iteration on ⊛.

uniqueness of analytic solution: →
just compute all derivatives at zero.

Prove by induction that

$$|u_k(t) - u_{k-1}(t)|_s \leq M \left(\frac{cet}{1-s}\right)^k$$

so $u_k(t;\cdot)$ will converge in $H(K(S))$ provided

$$\frac{ce|t|}{1-s} < 1 \quad \text{ie} \quad s < 1 - K|t|.$$

(ii) Theorem for arbitrary m

$$\left(\frac{\partial}{\partial t}\right)^m u = \sum_{j=0}^{m-1} \partial_t^{\,j} A_j u + f \; ; \; \left(\frac{\partial}{\partial t}^{j}\right) u = 0 \quad j = 1, \dots m-1.$$

differential operator of order $m-1$

Holmgren uniqueness theorem for linear problems.

(iii) Can also have nonlinear dependence

(classical) $f_t = G(x,t,f,f_x)$ $f(x,0) = 0$.

has ! local analytic solution.

(abstract) $f_t = \mathcal{G}(x,t,f)$, analytic in x and continuous in t. Also need

$$|\mathcal{G}(x,t,u) - \mathcal{G}(x,t,v)|_{\rho'} < \frac{c}{\rho-\rho'} |u-v|_\rho$$

for $|u|_\rho$ and $|v|_\rho$ small enough.

so $\mathcal{G}$ has a dependence on →
something like the derivative of f.

(57) (PDE) Sobolev imbeddings.

(i) $\|u\|_s^2 := \int_{\mathbb{R}^n} (1+|\xi|^2)^s \, |\hat{u}(\xi)|^2 d\xi.$

if $s > \frac{n}{2}$, then $C_0 \subset H^s$.

$|u(x)| \leq \int |\hat{u}(\xi)| d\xi$

$$\leq \int |\hat{u}(\xi)| (1+|\xi|^2)^{s/2} \cdot \frac{1}{(1+|\xi|^2)^{s/2}} d\xi.$$

$$\leq \|u\|_{H^s} \cdot \sqrt{\int \frac{d\xi}{(1+|\xi|^2)^s}} \qquad \text{C-S.}$$

$\int \frac{r^{n-1}}{r^{2s}} dr$ need $n-1-2s < -1$

$\Rightarrow n < 2s.$

Can also get a cone condition argument.

(ii) If $u \in W^{1,p}$ and $p > n$, then $u \in C^0$ also.

Can get mean value property from here - or from poisson's integral formula. →

58 (PDE) Potential theory.

(i) Variational methods to solve

$$-\Delta w = g \quad , \quad w\big|_{\partial\Omega} = 0.$$

Minimize $E(v) = \int_{\Omega} |\nabla v|^2 \, dx + \int_{\partial\Omega} vg$

with $v\big|_{\partial\Omega} = 0$.

(ii) $-\Delta u = f \quad \Rightarrow \quad u(x) = \int \frac{f(y)}{4\pi|x-y|} \, dy.$

(iii) Green's identity:

$$\int_{\Omega} (u\Delta v - v\Delta u) = \int_{\partial\Omega} \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right).$$

use $u = \frac{1}{4\pi|x-y|} = G \Rightarrow$

double layer. single layer

$$v(y) = -\int G \Delta v + \int_{\partial\Omega} \left(u \frac{\partial G}{\partial n} - G \frac{\partial u}{\partial n} \right).$$

↑

arbitrary function, written as the sum of double layer, single layer, and point distribution.

(iv) Single layer

$$V(x) = \int \sigma(y)\ G(x-y)\ dS(y).$$

has the following jump conditions:

V continuous, but

$$\frac{\partial V^+}{\partial n_x} = \sigma(x) + \frac{1}{2}\int \sigma(y) \frac{\partial}{\partial n_x} G\ dS(y)$$

$$\frac{\partial V^-}{\partial n_x} = -\sigma(x) + \frac{1}{2}\int \sigma(y) \underbrace{\frac{\partial}{\partial n_x} G}_{K(x,y)}\ dS(y)$$

To solve the interior Neumann problem, want to find σ so that

given $\frac{\partial u}{\partial n} = g = (-I + K)\sigma$.

Apply Fredholm alternative – g must satisfy $\int g = 0$ and u determined only up to a constant.

(v) For the scattering problem,

$$(\Delta + k^2) u = 0$$

there is a Green's function

$$G = e^{ikr}/4\pi r.$$

Duhamel $\Rightarrow$ solving $u_t = L(t)u$ for any initial data is just as good as solving nonhomogeneous problem. $\rightarrow$

Incoming plane wave $e^{i\underline{k}\cdot\underline{x}}$. Scattered wave looks like

$$\underbrace{f(\hat{k},\hat{x})}_{\text{amplitude, phase of scattered wave.}} \; \underbrace{\frac{e^{ik|x|}}{|x|.}}_{\text{sherical wave originating at 0.}} \quad \text{for } \underline{x} \text{ large.}$$

(59) (PDE) Wave equation

(i) $u_{tt} - u_{xx} - u_{yy} = 0$ Has unique solutions in D if D is bounded by a space like surface $t_\nu < -\sqrt{x_\nu^2 + y_\nu^2}$ (ν is the inner normal) since the energy in D $\iint_{T(t)} \frac{1}{2}(u_t^2 + u_x^2 + u_y^2)$ decreases in time.

(ii) $u(x,t) = t\int d\Omega_{\hat{n}}\, f(\underline{x} + t\hat{n})$

solves $u(\underline{x},0) = 0$ $\quad \frac{\partial u}{\partial t} = f(\underline{x})\,4\pi.$

when $u(x,0) = g(x)$ $\quad u_t = 0$

solve for v with $v(x,0) = 0$ and $v_t(\cdot,0) = g$

then take $u = \frac{\partial v}{\partial t}$.

method of descent to lower dimensions.

(iii) in 1D, $u(0,x) = f(x)$
$u_t(0,x) = g(x) \Rightarrow$

$$u(t,x) = \frac{1}{2}\left[f(x+t) - f(x-t) \right] + \frac{1}{2}\int_{-t}^{t} g(x+\xi)\, d\xi$$

(iv) in 3D, the inhomogeneous problem
$u_{tt} = \Delta u + F(t,\underline{x})$ is solved by

$$u(t,\underline{x}) = \frac{1}{4\pi} \int_{\rho \le t} \frac{F(t-\rho, \underline{y})}{\rho}\, d\underline{y}.$$

$|\underline{x} - \underline{y}|$.