Survey of Applied Mathematics Techniques

Brian Wetton [1]

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[1] www.math.ubc.ca/~wetton, wetton@math.ubc.ca

Lecture 1

Function Spaces, Finite Difference Methods & Green's Functions

1.1 Introduction

These notes are a series of nine lectures I gave at the South University of Science and Technology in Shenzhen, in the Summer of 2018. The idea was to show the basic techniques of applied mathematics: modelling, scaling and asymptotics, numerical approximation, and analysis. I planned to show just enough that the mixed audience of senior undergraduates and graduates could understand the idea of the techniques and follow up with further reading as needed. I should say that "applied mathematics" for me is about continuum problems and differential equation models and that is the focus of these notes, although there are many other types of problems coming from applications where mathematical analysis is key to understanding.

1.2 A First Problem

Consider solving the ODE boundary value problem for $u(x)$ for given data $f(x)$:

$$-u'' + u = f, \quad \text{problem A} \tag{1.1}$$

with either $x \in \mathbb{R}$ (with u bounded, problem A1) or $x \in [0, 1]$ (with u and u' 1-periodic, problem A2). If we were really applied mathematicians we would not proceed until we knew where (1.1) came from, what u and f were, and what kind of mathematical analysis would be useful to application questions. However, let us take that for granted for now. Some details of models that lead to equation (1.1) after suitable scaling will be shown in lecture #3. We will proceed with some basic mathematical questions:

1. Do solutions exist? Are they unique? Is the problem well posed? (that is, do small changes in the data f lead to small changes in the solution u?).

2. What kind of behaviour in solutions can we identify with analysis?

3. How can we compute accurate approximate solutions for a given f?

The first two questions can help to validate the modelling steps of the phenomenon that led to the equation. The third is natural, since most application questions are not answered with analytic solutions.

Notice that the first and third questions above require a notion of distance, and so the discussion begins below with the introduction of suitable normed spaces of functions.

1.3 Function spaces and weak solutions

Let C be the set of continuous, bounded functions. In the case of problem A2, the functions in C will be continuous across the periodic boundary. The space C^n is of functions whose derivatives up to order n are continuous and bounded. A natural extension is to C_0^∞, the set of infinitely differential functions that have compact support (are identically zero outside a finite interval). We can now define that a **strong** solution to (1.1) is a function $u \in C^2$ that satisfies (1.1) at each point x for a given data $f \in C$. The natural norms are

$$\|u\|_{C^n} := \max_{i=0\ldots n} \max_x |u^{(i)}(x)|$$

where $u^{(i)}$ is the i'th derivative of u. The norm $\|\cdot\|_C$ is normally denoted $\|\cdot\|_\infty$ (the maximum norm).

In applied problems of this type, data is often not continuous. The natural set for data in many applications is piecewise continuous functions, but these are not so amenable to analysis. Instead, this set is extended further to the set of measurable, square integrable functions, L_2. The easiest way to think of these functions, that include piecewise continuous functions but also a wider class, is as Cauchy sequences of C_0^∞ functions in the L_2 norm:

$$\|u\|_{L_2} := \sqrt{\int |u(x)|^2}.$$

with corresponding inner product

$$(u, v) = \int uv$$

The norm $\|\cdot\|_{L_2}$ is normally denoted $\|\cdot\|_2$. Just like the strong spaces above, we include spaces H^n of functions that have derivatives up to order n that are L_2 functions.

$$\|u\|_{H^n}^2 = \sum_{i=0}^{n} \|u^{(i)}\|_2^2.$$

For given data $f \in L_2$ (not necessarily continuous), we cannot expect u to be a strong solution (second derivative continuous). However, we can define a solution in a weaker sense. Consider now a strong solution of (1.1). Multiply by a smooth function $v(x)$ in C_0^∞ and integrate by parts to obtain

$$(u', v') + (u, v) = (f, v). \tag{1.2}$$

If $f \in L_2$ we say that $u \in H^1$ is a *weak solution* of (1.1). It is clear that if u is a strong solution (so with f continuous) then u is also a weak solution. It can also be shown that if u is a weak solution that has two continuous derivatives, it is also a strong solution. More importantly, it can be shown that if f_m is a Cauchy sequence (in L_2) of continuous functions that tends to $f \in L_2$, and $u_m \in C^2$ are the strong solutions with data f_m, then

$$\lim_{m\to\infty} \|u_m - u\|_{H^2} = 0.$$

Thus, the weak formulation is a natural extension of the original (strong) formulation of problem A.

Understanding the regularity of solutions is important to the design of numerical methods for computing approximate solutions.

1.4 Uniqueness and Well-Posedness

Consider a strong solution u of (1.1). Consider its maximum value, attained at a point x. At this point, $u'' \le 0$ so

$$u_{\max} \le f(x) \le \|f\|_\infty$$

and similarly

$$-u_{\min} \le \|f\|_\infty$$

Together these show that

$$\|u\|_\infty \le \|f\|_\infty. \tag{1.3}$$

Since the equation (1.1) is linear, the difference of two solutions solves (1.1) with $f \equiv 0$ and so uniqueness follows from (1.3).

Using the Green's function form of the solution presented below, a stronger result can be shown, that for every $f \in C^n$ there is a unique solution $u \in C^{n+2}$ and

$$\|u\|_{C^{n+2}} \le K\|f\|_{C^n}$$

where K depends only on n. Again using linearity, this shows the *well posedness* of the problem, that small changes in the data lead to small changes in the solution. If an equation is not well posed, it is an indication that there is a problem in the modelling that led to it.

A similar result holds for weak solutions. If (1.2) holds for all $v \in C_0^\infty$ then by a Cauchy sequence argument it holds for all $v \in H^1$ and so we can use $v = u$ in the equation which leads to

$$\|u\|_2\|u\|_{H^1} \le \|u\|_{H^1}^2 \le \|u\|_2\|f\|_2$$

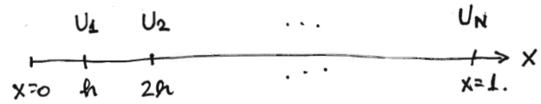

Figure 1.1: Uniform grid in spatial discretization.

and so $\|u\|_2 \leq \|f\|_2$. The argument for uniqueness follows from linearity as above.

This estimate can be extended. It can be shown that for every $f \in H^n$ there is a unique solution $u \in H^{n+2}$ and

$$\|u\|_{H^{n+2}} \leq K\|f\|_{H^n}$$

where K depends only on n.

1.5 Finite Difference Approximation of Problem A2

Assume now that $f \in C^2$ so there is a unique solution $u \in C^4$. We can approximate the solution on an evenly spaced grid of N points, $x_j = jh$, $j = 0 \dots N-1$, $h = 1/N$. The approximate values to be computed are $U_j \approx u(jh)$. This is shown in Figure 1.1.

The simplest method to use is a Finite Difference approximation in which the vector $\mathbf{U}$ solves the linear system:

$$-(U_{j-1} - 2U_j + U_{j+1})/h^2 + U_j = f(jh)$$

where we identify U_N with U_0 and U_{-1} with U_{N-1} due to the periodicity. The first term in the expression above is the centred, second order finite difference approximation of the second derivative. We will derive this approximation next lecture. The geometry and the weights of the linear combination of values used in the approximation is called the *stencil* of the discrete approximation.

We summarize the system as

$$\mathcal{A}\mathbf{U} = \mathbf{F}$$

We will leave the convergence analysis of the Finite Difference approximation to the next lecture, but it is easy to show that the $N \times N$ matrix $\mathcal{A}$ is invertible. Consider $\mathcal{A}\mathbf{U} = \mathbf{0}$. If $\mathcal{A}$ were singular, then this would have a nonzero solution $\mathbf{U}$. Let j be an index such that $|U_j| = \max_i |U_i|$ and without loss of generality take $U_j > 0$. Now

$$U_j = (U_{j-1} - 2U_j + U_{j+1})/h^2 \leq 0$$

since $U_j \geq U_{j-1}$ and $U_j \geq U_{j+1}$, leading to the desired contradiction.

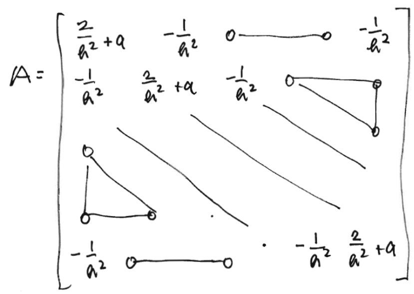

Figure 1.2: Structure of the matrix $\mathcal{A}$.

Note 1 *This argument can be used to show that any diagonally dominant matrix is invertible.*

The matrix $\mathcal{A}$ has mostly zeros as shown in Figure 1.2. As $N \to \infty$ ($h \to 0$) the fraction of non-zeros decreases. We call such matrices *sparse*.

1.5.1 Exact formula for $\mathcal{A}^{-1}$: the first Green's function

Let $\mathcal{G}$ denote $\mathcal{A}^{-1}$. It is the map from the data $\mathbf{F}$ to the approximate solution $\mathbf{U}$:

$$\mathbf{U} = \mathcal{G}\mathbf{F} \tag{1.4}$$

If we let $\mathbf{G}_i$ be the i'th column of $\mathcal{G}$, then (1.4) is equivalent to

$$\mathbf{U} = \sum_{i=1}^{N} F_i \mathbf{G}_i. \tag{1.5}$$

and so $\mathbf{G}_i$ is the response in the solution to data at location i. This can be seen more directly from the definition of the inverse, since $\mathcal{A}\mathcal{G} = \mathcal{I}$ (the identity matrix) shows $\mathcal{A}\mathbf{G}_i = \mathbf{e}_i$, where $\mathbf{e}_i$ is the standard basis vector for component i. Although not typical terminology, we can call $\mathcal{G}$ a *discrete Green's function.*

We can solve for $\mathbf{g} = \mathbf{G}_0$ explicitly, since the entries satisfy

$$g_{j+1} - (2+h^2)g_j + g_{j-1} = \begin{cases} -h^2 & \text{if } j = 0 \\ 0 & \text{otherwise} \end{cases} \tag{1.6}$$

The solutions to the homogeneous difference equation are

$$g_j = Cr^j + Dr^{-j} \tag{1.7}$$

where $r = 1 + h^2/2 + h\sqrt{1 + h^2/4}$ and $1/r$ are the roots of

$$r^2 - (2+h^2)r + 1 = 0$$

and C and D are still to be determined. To be consistent with the periodicity in the problem, the solution (1.7) must satisfy $g_N = g_0$, which gives

$$C(r^N - 1) = D(1 - r^{-N})$$

and allows (1.7) to be rewritten as

$$g_j = A\left(\frac{r^j + r^{N-j}}{r^N - 1}\right) \tag{1.8}$$

with A to be determined. Finally, to satisfy the condition at $j = 0$ in (1.6) we have

$$g_1 - (2 + h^2)g_0 + g_{N-1} = -h^2. \tag{1.9}$$

We compute

$$g_0 = A\frac{r^N + 1}{r^N - 1}$$

and

$$g_1 = g_{N-1} = A\frac{r + r^{N-1}}{r^N - 1}$$

which in (1.9) gives

$$A = \frac{h^2(r^N - 1)}{(2 + h^2)(1 + r^N) - 2(r + r^{N-1})}.$$

This value of A in (1.8) is the desired result. Note that because of translation invariance, the entries j in the i'th column of $\mathcal{G}$ are g_{j-i} where the index is taken mod N and so we can write the discrete solution as

$$U_j = \sum_{i=0}^{N-1} g_{j-i} f_i \tag{1.10}$$

Recall that $\mathcal{A}$ was sparse. We see now that $\mathcal{G} = \mathcal{A}^{-1}$ is not sparse, in fact it has no zero entries. This implies that all entries of $\mathbf{F}$ affect solution values at all locations. This is characteristic of elliptic problems.

Note 2 *Even though we have an exact formula for the matrix inverse, it is faster computationally to solve* $\mathcal{A}\mathbf{U} = \mathbf{F}$ *using sparse Gaussian Elimination rather than with (1.10).*

1.6 Continuous Green's Functions

The exact solution for problem A1, whether in the weak or strong form is

$$u(x) = \int \frac{1}{2} e^{-|x-s|} f(s) ds. \tag{1.11}$$

Note that this has a very similar form to the discrete case (1.10). Let us consider the strong form now with $f \in C$. We need to show that the u from (1.11) satisfies (1.1). We need to take some care with the derivative discontinuity in the integrand, so we write

$$2u(x) = \int^x e^{s-x} f(s)ds + \int_x e^{x-s} f(s)ds$$

and then

$$2u'(x) = f(x) - \int^x e^{s-x} f(s)ds - f(x) + \int_x e^{x-s} f(s)ds$$

in which the non-integral terms $f(x)$ cancel due to continuity, leading to

$$2u'' = -f(x) + \int^x e^{s-x} f(s)ds - f(x) + \int_x e^{x-s} f(s)ds$$

and thus

$$-u'' + u = f(x)$$

for every x. Thus we have existence of strong solutions and it is straightforward to show the bound

$$\|u\|_{C^2} \leq K\|f\|_C$$

using the expressions above.

1.6.1 Derivation of the Green's function for problem A2

If you look back at the the derivation above, the conditions needed to solve (1.1) in the form

$$u(x) = \int G(x,s)f(s)ds$$

are that $G(x; s)$ (as a function of x for fixed s) solved the homogeneous boundary value problem when $x \neq s$, satisfied the boundary conditions for the problem (in this case, the far field conditions that $G(\pm\infty) = 0$), $G(x; s)$ continuous at $x = s$, and $G'(x_+; s) - G'(x_-; s) = -1$. Due to the translational invariance of the problem, the result was simplified to

$$G(x,s) = g(x-s) \tag{1.12}$$

with $g(\xi) = e^{-|\xi|}$. For problem A2, we just need to modify the approach to include the periodic conditions instead of the far field conditions. Because of the translational invariance, the solution will have the form (1.12)

$$g(\xi) = \frac{\cosh(\xi - 1/2)}{2\sinh(1/2)}$$

extended periodically from the interval [0,1].

Note 3 *When the boundary value problem is not self adjoint like in the cases we have seen here, the Green's function will solve the adjoint problem.*

1.7 Lecture #1 Problems

Problem 1 *Using the discrete green's function for the finite difference approximation, show that* $\|D_+\mathbf{U}\|_\infty \leq K\|\mathbf{F}\|$ *for a constant* K *independent of* h *and* $\mathbf{F}$ *where* D_+ *is forward differencing,*

$$D_+U_j = (U_{j+1} - U_j)/h.$$

The algebra is much easier if you use the infinite lattice results I will show you in class.

Problem 2 *Find the discrete Green's function for the fourth order wide stencil finite difference approximation to problem A:*

$$-\frac{-U_{j-2} + 16U_{j-1} - 30U_j + 16U_{j+1} - U_{j+2}}{12h^2} + U_j = F_j$$

Again, the algebra will be much easier in the infinite lattice case. Note that the matrix of this discretization is not diagonally dominant so a maximum norm stability estimate is not trivial to obtain, but can be derived from this discrete Green's function.

Problem 3 *Find the Green's function representation of problem A on* $x \in [0, 1]$ *but with homogeneous Dirichlet boundary conditions* $u(0) = u(1) = 0$*. Note that in this case,* $G(x, s)$ *does not have the representation of* $g(x - s)$*.*

Problem 4 *Find the Green's function representation of the fourth order problem*

$$u'''' + u = f$$

on the infinite interval.

Problem 5 *Use the Green's function to extend the result above to show the estimate*

$$\|u\|_{C^{n+2}} \leq K\|f\|_{C^n}$$

for $n > 2$ *with* K *depending only on* n*.*

Appendix: Some Notes on the Fourier Transform

1.8 Fourier Series on a finite interval

Consider a function $f(x)$ defined and periodic on the interval $[0, 2\pi]$. We can expand the function in fourier series as follows:

$$f(x) = \frac{1}{2}a_0 + \sum_1^\infty [a_n \cos nx + b_n \sin nx]. \tag{1.13}$$

The coefficients a_n and b_n that make this "work" are

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx dx \tag{1.14}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx dx. \tag{1.15}$$

That these are the coefficients that we must take is seen from the fact that the functions $\{1, \sin x, \cos x, \sin 2x, \cos 2x \ldots\}$ are orthogonal in the following sense:

$$\int_0^{2\pi} \cos nx \cos mx dx = \begin{cases} 0 \text{ if } m \neq n \\ \pi \text{ if } m = n \neq 0 \\ 2\pi \text{ if } m = n = 0 \end{cases}$$

$$\int_0^{2\pi} \sin nx \sin mx dx = \begin{cases} 0 \text{ if } m \neq n \\ \pi \text{ if } m = n \end{cases}$$

$$\int_0^{2\pi} \cos nx \sin mx dx = 0 \quad \text{for all } m, n$$

These results can be easily verified. We can now multiply (1.13) by $\sin mx$ or $\cos mx$ and integrate from 0 to 2π and use the results above to see that the coefficients must be given by (1.14, 1.15).

1.8.1 Validity

So far we know that if the expansion (1.13) is going to work, the coefficients must be given by (1.14, 1.15). Under what conditions do we actually get convergence of the right hand side of (1.13) and then when do we have equality in (1.13)? I will quote two results without proof:

1.8.2 Piecewise continuous functions

Suppose that $f(x)$ is a piecewise continuously differentiable function. This means that $f(x)$ is continuous and has a continuous derivative at all but a finite number of points, where it may suffer a jump (but remains bounded). For instance, the function

$$f(x) = \begin{cases} 1, & 0 \leq x < \pi \\ -1, & \pi \leq x < 2\pi \end{cases} \tag{1.16}$$

is a piecewise continuously differentiable function. Note that this function has jumps at π and at 0 (or 2π) since when we say differentiable we mean it in the sense of 2π periodicity.

We now state our theorem:

Theorem 1 *If f is a piecewise continuously differentiable function then the series on the right of (1.13) converges at all x to*

$$\frac{1}{2}[f(x+0) + f(x-0)]$$

where $f(x+0)$ is the limit of $f(y)$ as y approaches x from the right and $f(x-0)$ is the limit of $f(y)$ as y approaches x from the left.

At all points where f is continuous (away from the jumps), $f(x+0) = f(x-0)$ and the series converges to $f(x)$.

The series associated with the function in eq. (1.16) converges to

$$\begin{cases} 1, & 0 < x < \pi \\ 0, & x = 0, \pi, 2\pi \\ -1, & \pi < x < 2\pi \end{cases}$$

1.8.3 Square integrable functions

Now consider f to be in the space of functions, $L_2[0, 2\pi]$. This space contains all functions defined on $[0, 2\pi]$ that are *measurable* (see [2] for example for this technical definition but let me say any functions we will deal with are measurable) and such that

$$\int_0^{2\pi} |f(x)|^2 dx$$

is finite. Notice that this allows some unbounded functions as well as the piecewise smooth functions above.

Theorem 2 *If $f \in L_2$ then the fourier series of f converges to f in the following sense:*

$$\lim_{N\to\infty} \int_0^{2\pi} |f(x) - \frac{a_0}{2} - \sum_{n=1}^{N} [a_n \cos nx + b_n \sin nx]|^2 dx = 0.$$

This certainly does not guarantee pointwise convergence.

1.8.4 Complex form

For notational convenience (but this is very important as you'll see) it's much nicer to abandon the real form of the fourier series for the equivalent complex one. Remember the complex exponential

$$e^{(x+iy)} = e^x(\cos x + i \sin y),$$

one of the first complex analytic functions you learned about. This complex exponential obeys all of the normal rules for exponentials, *i.e.*

$$e^{z_1} \times e^{z_2} = e^{z_1+z_2}$$

for any complex numbers z_1 and z_2. Switching the roll of x above we get

$$e^{ix} = \cos x + i \sin x.$$

Now define the following complex fourier coefficients

$$c_n = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-inx} dx \tag{1.17}$$

for $n = 0, \pm 1, \pm 2, \ldots$. Clearly,

$$c_n = \begin{cases} \frac{a_0}{2}, & n = 0 \\ \\ \frac{1}{2}(a_n - ib_n), & n > 0 \\ \\ \frac{1}{2}(a_n + ib_n), & n < 0 \end{cases}$$

It is then possible to rewrite eq. (1.13) as

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}.$$

We think of c_n as being the fourier coefficients of $f(x)$ and so can naturally label them as $\hat{f}_n$ or $\mathcal{F}(f)_n$.

1.8.5 Normalization and alternate forms

There are several other ways to define the transform that you may see in other texts. The various forms have merely to do with the normalization and conjugation of the transform coefficients. Some other valid transform and inverse transform pairs are

$$\hat{f}_n = \int_0^{2\pi} f(x)e^{-inx}dx \qquad f(x) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\hat{f}_n e^{inx}$$

$$\hat{f}_n = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{inx}dx \qquad f(x) = \sum_{n=-\infty}^{\infty}\hat{f}_n e^{-inx}$$

$$\hat{f}_n = \frac{1}{\sqrt{2\pi}}\int_0^{2\pi} f(x)e^{-inx}dx \qquad f(x) = \frac{1}{\sqrt{2\pi}}\sum_{n=-\infty}^{\infty}\hat{f}_n e^{inx}$$

This last form is particularly nice and is the one I use when I can. Its symmetric weighting leads to the nice property

$$\int_0^{2\pi} |f(x)|^2 dx = \sum_{n=-\infty}^{\infty} |\hat{f}_n|^2.$$

Note 4 *It is clear that Fourier series can be applied to functions in any finite interval by scaling and translation.*

1.9 Fourier Transform on the Real Line

When $f(x)$ is defined for all x the situation with the fourier transform and its inverse is a bit different. Here, the transform takes on a continuous number of values

$$\hat{f}(\alpha) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\alpha x}dx$$

for all real α and the inverse is given by

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(\alpha)e^{i\alpha x}d\alpha \qquad (1.18)$$

As with fourier series, there are alternate forms with normalization and conjugation. Again, this is a convenient form since here

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |\hat{f}(x)|^2 dx.$$

Also, the same kind of validation arguments apply in this case (note that some kind of decay in f has to be assumed as $|x| \to \infty$ for the transform of a piecewise smooth function to even exist, though).

1.10 Some Properties of the Fourier Transform

1.10.1 Derivatives

If f has a derivative with a “valid” fourier transform, then the fourier transform of the derivative is given by

$$\mathcal{F}(f')(\alpha) = i\alpha\mathcal{F}(f)(\alpha).$$

This result can be found formally just by differentiating eq. (1.18). Here we have used $\mathcal{F}(f)$ instead of $\hat{f}$ to avoid confusion with the primes for derivatives. This is a marvelous result. In words, it says that the fourier transform reduces differentiation to multiplication. It is no wonder that fourier transform and series are used heavily in the study of differential equations.

1.10.2 Convolution

The convolution of two functions on the real line is given by

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-s)g(s)ds$$

The fourier transform of a convolution is extremely simple:

$$\mathcal{F}(f * g)(\alpha) = \hat{f}(\alpha) \times \hat{g}(\alpha)$$

It reduces to pointwise multiplication of the transforms. The opposite is also true: the transform of a product is the convolution of their transforms.

1.11 Discrete Fourier Transform

We now consider the corresponding situation on a finite lattice ($\mathbf{R}^n$ or $\mathbf{C}^n$). For a N vector U with components U_j (index j from 0 to $N-1$ here) the DFT is a complex N vector $\hat{U}$ with components given by

$$\hat{U}_\alpha = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} U_j e^{-2\pi(ij\alpha)/N} \tag{1.19}$$

for $\alpha = 0, 1, \ldots N-1$. The inverse transform is given by

$$U_j = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} \hat{U}_\alpha e^{2\pi(ij\alpha)/N} \tag{1.20}$$

The transform (1.19) is just a fancy way of describing the change of basis of the vector U to the ortho-normal coordinates $\{E_\alpha\}$ where $(E_\alpha)_j := e^{2\pi(ij\alpha)/N}/\sqrt{N}$.

As always, these formulas are arbitrary up to conjugation and scaling. With the scaling above we have

$$\|U\| = \|\hat{U}\|$$

where $\|\cdot\|$ is the usual Euclidean norm (which can be denoted $\|\cdot\|_2$ to distinguish it from other norms on $\mathbf{C}^n$),

$$\|U\| = \sqrt{\sum_{j=0}^{N-1} |U_j|^2}$$

This is just a consequence of the fact that the DFT is an orthonormal change of basis (such changes are Euclidean norm preserving).

1.11.1 Some Linear Algebra

The operation in eq. (1.19) is equivalent to multiplication by a matrix which we will call $\mathcal{F}$. The operation in eq. (1.20) must then be described by multiplication by $\mathcal{F}^{-1}$. By inspection we see that $\mathcal{F}^{-1} = \mathcal{F}^*$ where $*$ denotes the Hermitian (or conjugate) transpose. This result follows from the fact that the DFT is just an orthonormal change of coordinates.

Note 5 *Multiplication by an $N \times N$ matrix is in general an operation that takes $O(N^2)$ computational work. However, the DFT in eq. (1.19) can be done faster than this and stably (in $O(N \log N)$ work) using the Fast Fourier Transform (FFT) method. This algorithm has made it possible to compute approximations using fourier transform ideas.*

1.12 Transform on an Infinite Lattice

We can come full circle back to section 1.8 by considering the transform of vectors on an infinite lattice, *i.e.* U_j with $j = 0, \pm 1, \pm 2, \ldots$ (with suitable decay as $|j| \to \infty$). In this case, the transform $\hat{u}(\alpha)$ takes on values for continuous values of α and $\hat{u}$ is 2π-periodic. We have

$$U_j = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \hat{u}(\alpha) e^{ij\alpha} d\alpha$$

with

$$\hat{u}(\alpha) = \frac{1}{\sqrt{2\pi}} \sum_{j=-\infty}^{\infty} U_j e^{-ij\alpha}.$$

This is just the complex Fourier Series from section 1.8 written backwards where integration is taken over the interval $[-\pi, \pi]$ instead of $[0, 2\pi]$ (equivalent since the integrand is 2π-periodic). This change allows correct interpretation of $\hat{u}(\alpha)$ values.

Note 6 *The DFT plays the same role in the analysis of finite difference equations as the FT plays in the analysis of differential equations. In this role, the use of the DFT is known as von Neumann analysis.*

Note 7 *I owe much of my understanding and many of the details in these notes to [1].*

Bibliography

[1] Marvin Shinbrot, “The Solution of Partial Differential Equations,” unpublished lecture notes.

[2] Royden, “Real Analysis”.