Lecture #1

December 10, 2020 6:18 PM

Easy problem

Data $f(x)$ $x \in [0,1]$ given

unknown $u(x)$ $x \in [0,1]$.

$\downarrow$

relationship between u & f

$$u'' = f(x) \qquad u(0) = 0 \qquad u(1) = 0. \qquad (1)$$

DE applies at all $x \in [0,1]$.

Boundary conditions

Together BVP

Not an initial value problem.

$u(t)$ $u'' = f$ $u(0) = a$, $u'(0) = b$.

Here pick $T > 0$. $u(t)$ for $t < T$ does not depend on $f(t)$ for $t > T$.

Go back to BVP. (1).

Model the vertical displacement of a wire with mass loading $f(x)$.

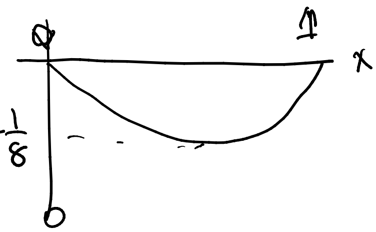
$f \equiv 1$. $u'' = 1$ $u = \frac{x^2}{2} + \underbrace{ax + b}$.

match with BC's.

$u(0) = 0$ $b = 0$.

$u(1) = 0$ $a = -\frac{1}{2}$

$u = \frac{1}{2}(x^2 - x) = \frac{1}{2}\left[(x - \frac{1}{2})^2 - \frac{1}{4}\right]$. $-\frac{1}{8}$



Ⓐ Linearity

BVP (1) is a linear problem. Linear problems have the superposition property.

If $u_1(x)$ solves $u_1'' = f_1$ $u_1(0) = 0$, $u_1(1) = 0$

and $u_2(x)$ " $u_2'' = f_2$ $u_2(0) = 0$ $u_2(1) = 0$.

If $u_1(x)$ solves $u_1'' = f_1$ $\quad u_1(0)=0,\ u_1(1)=0$

and $u_2(x)$ " $u_2'' = f_2$ $\quad u_2(0)=0,\ u_2(1)=0.$

Then $W = C_1 u_1 + C_2 u_2$

any constants.

solve $w'' = c_1 f_1 + c_2 f_2$ $\quad w(0)=0,\ w(1)=0.$

$u'' = f$ $\quad u(0) = a,\ u(1) = b.$

data, given numbers.

u is the sum of 3 solutions, $u = u_{I} + u_{II} + u_{III}$.

u_{I} solves	$u'' = f$	$u(0)=0$	$u(1)=0.$
u_{II} solves	$u''=0$	$u(0)=a$	$u(1)=0.$
u_{III} "	"	$u(0)=0$	$u(1)=b.$

$u_{II} = a(1-x)$

$u_{III} = bx.$

(B) Well-posedness

Theory for BVP (1): if f continuous then there is a unique solution $u(x)$ with 2 continuous derivatives. (strong solution).

C: space of continuous functions on $[0,1]$.

C_2: space of functions with continuous second derivatives.

$$\|u\|_C = \max_{x\in[0,1]} |u(x)|.$$

$$\|u\|_{C_2} = \|u\|_C + \|u'\|_C + \|u''\|_C.$$

More theory shows that there is a $K>0$ so that

$$\|u\|_{C_2} \leq K\|f\|_C \quad \text{for every } f.$$

solution of BVP (1).

Consider $u_1'' = f_1$ $\quad u_1(0)=0,\ u_1(1)=0$

$u_2'' = f_2$ $\quad u_2(0)=0,\ u_2(1)=0$

$u_2'' = f_2 \qquad u_2(0)=0, \quad u_2(1)=0$

$W = u_1 - u_2$ solves BVP(1) with data $f_1 - f_2$.

$$\underline{\| u_1 - u_2 \|_{C_2}} \leq K \, \underline{\| f_1 - f_2 \|}$$

Shows uniqueness ✓ If you have small changes in the data, then you have small changes in the solution. Well-posedness.

(C) Weak solutions.

Cases where data is idealized to be discontinuous.

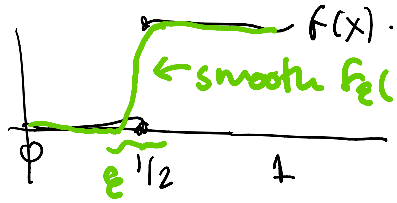
Consider $f(x) = \begin{cases} 0 & x < 1/2 \\ 1 & x > 1/2 \end{cases}$ ← corresponding $u(x)$ cannot be C_2.

What does $u'' = f$ mean in this case?

The corresponding solution does exist in this case, it is unique, well posed problem.

↓

One way to make sense is as a limit



$u_\varepsilon(x) \to u(x)$ weak solution

(I) Eigenanalysis for Matrices.

If $\mathbb{A}\underline{u} = \lambda \underline{u}$ for $\underline{u} \neq \underline{0}$

n×n matrix

n component column vector.

λ is a scalar (a number)

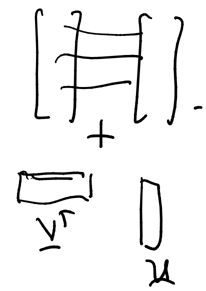
Then λ is an eigenvalue of $\mathbb{A}$ and $\underline{u}$ is the corresponding eigenvector.

Eigenvectors have arbitrary (nonzero) magnitude

(A) Symmetric Matrices (real coefficients).

Ⓐ Symmetric Matrices (real coefficients).

$\mathbb{A} = \mathbb{A}^T$



Consider $\underbrace{\underline{v} \cdot (\mathbb{A}\underline{u})}_{\text{number}} = \underline{v}^T \mathbb{A}\, \underline{u}$.

dot product $\qquad \mathbb{A} = \mathbb{A}^T$ ✓

equal to its transpose $\quad = (\underline{v}^T \mathbb{A}\, \underline{u})^T = \underline{u}^T \mathbb{A}\, \underline{v} = \underline{u} \cdot (\mathbb{A}\underline{v})$.

$\underline{v} \cdot (\mathbb{A}\,\underline{u}) = \underline{u} \cdot (\mathbb{A}\,\underline{v})$ for all $\underline{u}$ & $\underline{v}$ if $\mathbb{A}$ is symmetric.

(iff statement - consider $\underline{u}$ & $\underline{v}$ basis unit vectors).

Write in inner product notation.

$(\underline{v}, \mathbb{A}\underline{u}) = (\mathbb{A}\underline{v}, \underline{u})$. if $\mathbb{A}$ is symmetric.

Symmetric matrices have some nice properties:

- Eigen values are real ①.
- Eigenvectors from different eigenvalues are orthogonal ②
- There is an orthonormal basis $\{\underline{v}_1, \dots \underline{v}_n\}$. ③. of eigenvectors.

Proof of ②

$\mathbb{A}\,\underline{v}_1 = \lambda_1 \underline{v}_1 \qquad \mathbb{A}\,\underline{v}_2 = \lambda_2 \underline{v}_2 \qquad \lambda_1 \neq \lambda_2$.

$(\mathbb{A}\underline{v}_1, \underline{v}_2) = (\underline{v}_1, \mathbb{A}\,\underline{v}_2)$ (due to symmetry of $\mathbb{A}$).

$\lambda_1 (\underline{v}_1, \underline{v}_2) = \lambda_2 (\underline{v}_1, \underline{v}_2)$

$\underbrace{(\lambda_2 - \lambda_1)}_{\neq 0} (\underline{v}_1, \underline{v}_2) = 0$.

$\underline{v}_1$ & $\underline{v}_2$ must be orthogonal.

③ Idea of proof. Consider $V = \text{span}\{\underline{v}_1, \underline{v}_2, \dots \underline{v}_m\}$

$\underline{v_1}$ & $\underline{v_2}$ must be orthogonal.

③ Idea of proof. Consider $V = \text{span}\{\underline{v_1}, \underline{v_2}, \ldots \underline{v_m}\}$ $m<n$.
then it can be shown there is an eigenvector of $\mathbb{A}$ in $V^{\perp}$

space of vectors perpendicul to V.

Consider solving $\mathbb{A}\underline{u} = \underline{f}$

assume $\mathbb{A}$ is invertible, no $\lambda=0$ eigenvalue.

If $\underline{f} = \underline{v_j}$, then $\underline{u} = \frac{1}{\lambda_j}\underline{v_j}$ ✓.

$\underline{f} = \sum_{j=1}^{n} a_j \underline{v_j}$, then $\underline{u} = \sum_{j=1}^{n} \frac{a_j}{\lambda_j}\underline{v_j}$

normalized.

$(v_i, v_i) = \|v_i\|^2 = \underline{1}$.

If f is given, then

$\underline{v_i} \cdot \left[\underline{f} = \sum_{j=1}^{n} a_j \underline{v_j}\right]$. with $a_j = (\underline{f}, \underline{v_j})$

So the solution is $\underline{u} = \sum_{j=1}^{n} \frac{(\underline{f}, \underline{v_j})}{\lambda_j}\underline{v_j}$ scalar

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Ex $\mathbb{A} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$ symmetric ✓.

$\lambda_1 = -1$ $\quad \underline{v_1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda_2 = -3$ $\quad \underline{v_2} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

Solve $\mathbb{A}\underline{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. $\underline{f}$

$(f, \underline{v_1}) = \frac{1}{\sqrt{2}}$ $\quad (f, \underline{v_2}) = \frac{1}{\sqrt{2}}$

$\sqrt{2}$ $\qquad$ $) = \sqrt{2}$

$$\underline{u} = \frac{1}{\sqrt{2}} \frac{1}{(-1)} \begin{pmatrix} 1 \\ 1 \end{pmatrix} / \sqrt{2} + \frac{1}{\sqrt{2}} \frac{1}{(-3)} \begin{pmatrix} 1 \\ -1 \end{pmatrix} / \sqrt{2}$$

$$= \begin{pmatrix} -\frac{1}{2} \\ -1/2 \end{pmatrix} + \begin{pmatrix} -1/6 \\ 1/6 \end{pmatrix} = \begin{pmatrix} -2/3 \\ -1/3 \end{pmatrix}$$

check this solves the system ✓.

Lecture #2

January 12, 2021 11:29 AM

(B) Symmetrization

$\mathbb{A}$

not symmetric

Some are "close" to symmetric.

$\mathbb{E}$ ← SPD symmetric positive definite

symmetric, have all positive eigenvalues.

$\mathbb{E}$ define a weighted inner product.

$$(\underline{u},\underline{v})_E = (\underline{u}, \mathbb{E}\underline{v})$$

$(\underline{u},\underline{v})_E$ has all the properties of an inner product

$(\underline{u},\underline{u})_E \geq 0$ only zero if $\underline{u}=\underline{0}$

$\|\underline{u}\|_E^2$

Easy example $\mathbb{D} = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix}$

diagonal

positive entries

weighted inner product and norm.

$$\|\underline{u}\|_D^2 = \sum_{i=1}^{n} d_i u_i^2$$

$\mathbb{E}\mathbb{A}$ ← if this can be made symmetric for a given spd $\mathbb{E}$. then $\mathbb{A}$

symmetric

$(\mathbb{E}\,\mathbb{A})$ ← If this can be made symmetric for a given spd $\mathbb{E}$, then $\mathbb{A}$ is called symmetrizable.

spd ↑ symmetric.

$$\boxed{(\underline{u}, \mathbb{A}\underline{v})_E} = (\underline{u}, \mathbb{E}\mathbb{A}\underline{v}) = (\mathbb{E}\mathbb{A}\underline{u}, \underline{v}).$$

symmetric symmetric.

$$= (\mathbb{A}\underline{u}, \mathbb{E}\underline{v}) = \boxed{(\mathbb{A}\underline{u}, \underline{v})_E.}$$

showing $\mathbb{A}$ is symmetric in $(\cdot,\cdot)_E$

Eigen values must be real and eigen vectors are E orthogonal $(\underline{V}_1, \underline{V}_2)_E = 0$.

Ex $\mathbb{A} = \begin{pmatrix} -3 & 2 \\ 1 & -2 \end{pmatrix}$ not symmetric ✗

$$\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -3 & 2 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} -3/2 & 1 \\ 1 & -2 \end{pmatrix} \text{ symmetric } ✓$$

$\mathbb{E}$

$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. ↑ not symmetrizable only one eigenvector.

Eigenvalues will be real, $\underline{V}_1, \underline{V}_2$ ← E orthogonal.

$\underline{V}_1 = (V_{1,1}, V_{1,2})$

$$(\underline{V}_1, \underline{V}_2)_2 = \tfrac{1}{2} V_{11} V_{21} + V_{12} V_{22} = 0.$$

check $\lambda_1 = -1$ $\underline{V}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\underline{V}_1 \cdot \underline{V}_2 \neq 0$.

$\lambda_2 = -4$ $\underline{V}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ but $(\underline{V}_1, \underline{V}_2)_E = 0$ ✓

III Series solutions to BVP, part 1.

$u'' = f(x)$ $u(0) = 0,\ u(1) = 0.$

$u'' = f(x) \quad u(0)=0, \quad u(1)=0.$

$$\mathcal{L} u = f$$

includes the BC's.

Details to follow, but

$\mathcal{L}$ is symmetric, has real eigen values $\lambda = -n^2\pi^2$

and has an o.n. orthonormal. basis of eigen vectors

$$\phi_n(x) = \sqrt{2} \sin n\pi x. \qquad [\text{sine series}].$$

What is the inner product?

$$\boxed{(u, v) = \int_0^1 u(x) v(x)\, dx, \quad \|u\|^2 = (u,u) = \int_0^1 u^2\, dx.}$$

functions of x, $x \in [0,1]$

with $u(0)=0$, $u(1)=0$, $v(0)=0$, $v(1)=0$.

So $\{\phi_n\}$ are normalized $\|\phi_n\| = 1$ and orthogonal

can check by direct computation.

Symmetry

$$(u, \mathcal{L}v) = \int_0^1 u v''\, dx$$

satisfy BC's. $= \int_0^1 u'' v\, dx$

$$= (\mathcal{L}u, v)$$

Integrate by parts twice.

$$uv'\Big|_0^1 - u'v\Big|_0^1 = 0$$

symmetric.

From this, we could work out that the eigenvalues are real and eigenfunctions are orthogonal. Same argument as for linear algebra case

$\mathcal{L} f = \lambda f$ ...

... for linear algebra case

$$\mathcal{L}\phi_1 = \lambda_1\phi_1 \qquad \underline{\lambda_1 \neq \lambda_2}$$
$$\mathcal{L}\phi_2 = \lambda_2\phi_2$$

$$(\phi_1, \mathcal{L}\phi_2) = (\mathcal{L}\phi_1, \phi_2) \qquad \mathcal{L} \text{ symmetric}$$

$$\lambda_2 (\phi_1, \phi_2) = \lambda_1 (\phi_1, \phi_2)$$

$$\underbrace{(\lambda_2 - \lambda_1)}_{\text{not zero}} \underline{(\phi_1, \phi_2)} = 0$$

ϕ_1 & ϕ_2 are orthogonal

Analog of eigen solution to $\mathbb{A}\underline{u} = \underline{f}$, is the sine series solution to $\underline{\mathcal{L}}u = f$.

$$u = \sum_{n=1}^{\infty} \frac{2\int_0^1 f(s)\sin n\pi s\, ds}{-n^2\pi^2} \cdot \underline{\sin n\pi x}$$

coefficient.

Note: This naturally defines a weak solution. ✓.

Consider now $\underbrace{u'' + u' = f}_{\mathcal{L}u = f.}$ $\quad u(0) = 0, \; u(1) = 0.$ (✩)

$\mathcal{L}$ not symmetric. But if we multiply the equation by $\underline{e^x}$ ← "like" multiplying by a diagonal matrix with positive entries.

$$\underbrace{e^x u'' + e^x u'}_{(e^x u')'} = e^x f.$$

$$\underline{(e^x u')'} = e^x f.$$

$\mathcal{L}u = (e^x u')'$

$\underline{(u, \mathcal{L}v)} = \int_0^1 u\,(e^x v')' dx$

$= -\int u' e^x v' dx$

This is symmetric

$= -\int_0^1 u' e^x v' \, dx$

$= \int_0^1 (u'e^x)' v \, dx$

$= (\mathcal{L}u, v).$

symmetry.

This is symmetric.

So (*) has real eigenvalues, eigenfunctions are orthogonal in the weighted inner product

$$\int_0^1 e^x \phi_i(x) \phi_j(x) \, dx = 0.$$

Green's Function and Linear Algebra Warm-up.

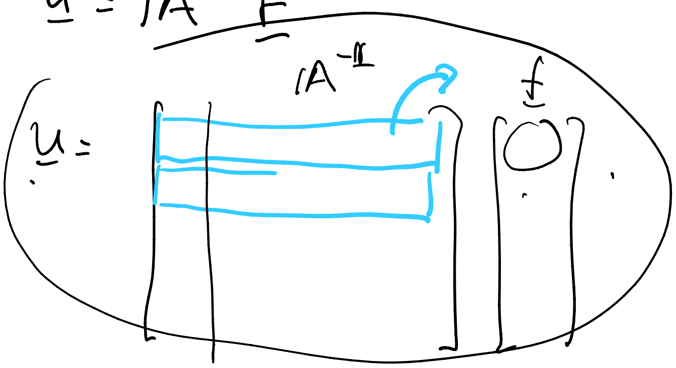
$$\underset{n\times n}{\mathbb{A}} \; \underset{n\times 1}{\underline{u}} = \underset{n\times 1}{\underline{f}}$$

Linear system.

$\mathbb{A}$ is invertible

$\underline{u} = \mathbb{A}^{-1} \underline{f}$

Note: if $\mathbb{A}$ is symmetric then so is $\mathbb{A}^{-1}$.



First column of $\mathbb{A}^{-1}$ is the solution of $\mathbb{A}\underline{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix}$.

j'th " " " is the solution of $\mathbb{A}\underline{u} = \underline{e_j}$.

So $\mathbb{A}^{-1}$ is a map from data to the solution.

Can write $\underline{u} = \sum_{j=1}^{n} \underset{\uparrow}{\underline{G_j}} f_j$.

j'th column of $\mathbb{A}^{-1}$

Green's Functions Tell you

$\boxed{\text{BVP (1)} \quad u'' = f, \quad u(0) = 0, \quad u(1) = 0.}$

$u(x) = \int^1 G(x,s) \, f(s) \, ds$

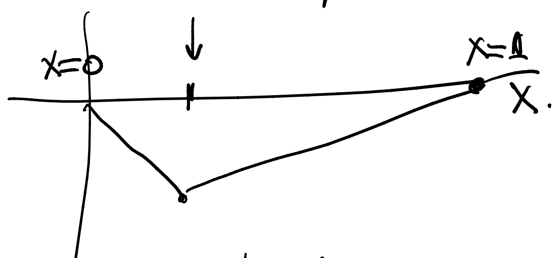
Tell you (verify later)

$$\boxed{u(x) = \int_0^1 G(x,s) \cdot f(s)\, ds}$$ ⊕

Notes: $G(x,s) = G(s,x)$

$$G(x,s) = \begin{cases} s(x-1) & \text{if } s \le x \\ x(s-1) & \text{if } s \ge x. \end{cases}$$

Another way to define a weak solution.



Let's show that this u really is a solution to the BVP.

$u(0) = 0$ $\quad$ $u(1) = 0$, $\quad$ $u'' = f$. ← check these.

$G(0,s) \equiv 0$ ✓ $\quad$ $G(1,s) \equiv 0$ ✓

$$u(x) = \int_0^{x} s(x-1) f(s)\, ds + \int_x^1 x(s-1) f(s)\, ds$$

$$u'(x) = \cancel{x(x-1)f(x)} + \int_0^x s f(s)\, ds$$

$$- \cancel{x(x-1)f(x)} + \int_x^1 (s-1) f(s)\, ds.$$

$$u''(x) = x f(x) - (x-1) f(x) = f(x) \quad ✓.$$

Comparing series solution & Green's Function Solution.

Series $\quad u(x) = \sum^{\infty} \dfrac{2 \int_0^1 \sin n\pi s\, f(s)\, ds}{-n^2\pi^2} \sin n\pi x$

Series $\quad u(x) = \sum_{n=1} \frac{2\int_0^1 \sin n\pi s f(s)\, ds}{-n^2\pi^2} \sin n\pi x$

global.
shows that large n components of f are highly damped.

Green $\quad u(x) = \int_0^1 G(x;s)\, f(s)\, ds.$

local.
Exact map of data to solution.

$u'' = f \quad u(0)=0,\ u(1)=0.$ $\quad$ BVP.

we know this has a unique solution. u.

$\boxed{u = \mathcal{L}^{-1} f}$

then $\mathcal{L} u = f.$

Lecture #3

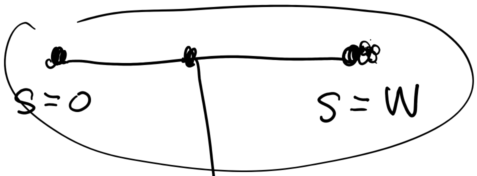
January 18, 2021 9:35 AM

Derivation of our first, simple BVP.

$u(x)$, $u'' = f$, $u(0)=0$, $u(1)=0$ (1)

Weighted, stretched wire

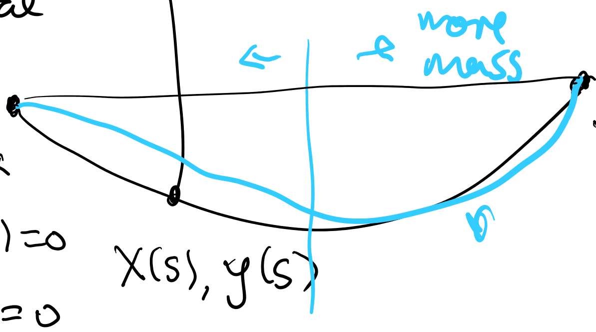
wire length W. Material coordinate in the wire s.



Attach mass, mass density $\mu(s)$ kg/m.

Stretch the wire between 2 points a distance L apart. $L > W$.

horizontal



(2)

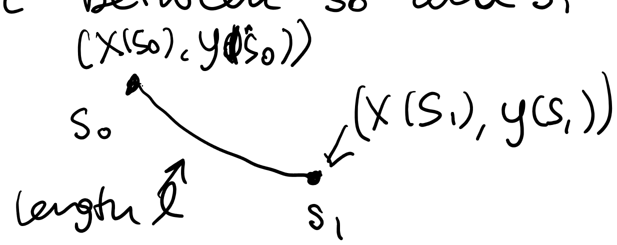
Problem is to find $x(s), y(s)$ [coupled BVP].

Note: $\mu \equiv 0$ we would have

$y(s) \equiv 0$ $\boxed{x = \frac{L}{W} s}$

and the wire would have uniform tension T_*

Consider now $\mu \neq 0$. Look at a small piece of wire between s_0 and s_1



Stretched length

$$\ell \approx \sqrt{(x(s_1) - x(s_0))^2 + (y(s_1) - y(s_0))^2}$$

Unstretched length $s_1 - s_0$. "is notation for defined as"

Unstretched length $s_1 - s_0$.

"is notation for defined as"

Ratio $\dfrac{l}{s_1 - s_0} \longrightarrow \boxed{f(s)} := \sqrt{(x')^2 + (y')^2}$

take the limit $s_1 \to s_0$.

Material assumption: The local tension in the wire is a function of f

That is $\boxed{\begin{array}{l} T(f) \\ \text{given function from experiments.} \end{array}}$

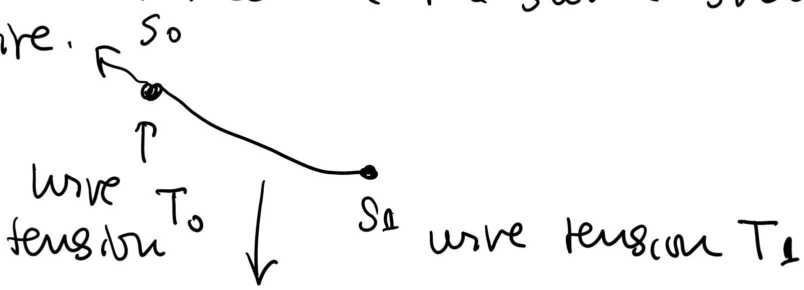
$T(1) = 0$

$T\left(\frac{L}{W}\right) = T_*$

$T'(f) > 0$ for $f > 1$.

Balance forces on the same small piece of wire.

Mass $\mu(s_1 - s_0)$



vertical gravitational force $\mu(s_1 - s_0)g$.

Tangent vector direction (x', y').

Unit tangent vector $\hat{\tau} = \dfrac{(x', y')}{\sqrt{(x')^2 + (y')^2}} = \dfrac{1}{f}(x', y')$.

Net tension force balances gravity

$\mu_{ave} = \mu(s_*)$ somewhere in (s_0, s_1)

$$\underbrace{T(f_1)\,\hat{\tau}_1}_{\text{vector force at right end of wire.}} - T(f_0)\,\hat{\tau}_0 = \boxed{(0, \mu \cdot (s_1 - s_0) g)}$$

$\uparrow$ left side

$s_1 - s_0$ $\quad$ $s_1 - s_0$

Now divide by $s_1 - s_0$ and take the limit $s_1 \to s_0$.

$$\frac{d}{ds}\left(T(f)\,\hat{\tau}\right) = (0, \mu g)$$

(3)

with $f = \sqrt{(x')^2 + (y')^2}$ $\quad$ $\hat{\tau} = (x', y')/f$

with $f = \sqrt{(x')^2 + (y')^2}$ $\quad \hat{\tau} = (x', y')/f$. (3)

$\frac{dx}{ds}$ ← denoted x_s in the posted notes.

Together with BC's (2), this is a second order, coupled, nonlinear BVP. (2 derivatives)

Assume that y is "small" ← but still significant to the application.
W is the natural lengthscale for the problem.

$$y = \boxed{\varepsilon W} Y(s)$$

scale for size of y.

dimensional y $\quad$ dimensionless y. $\quad$ started with $\frac{Mg W}{T}$.

Consistent to assume $\boxed{x = \frac{L}{W} s + \varepsilon W X(s)}$ $\quad$ dimensionless x. $\quad$ $\varepsilon = 0$ solution.

Neglect terms of size ε^2 and smaller in what follows (linearization).

Reminder — tangent line approximation $\quad$ Taylor series.

$$g(t) = \underbrace{g(t_0) + \varepsilon g'(t_0)}_{\text{tangent line approximation}} + \underbrace{\frac{\varepsilon^2}{2} g''(t_0) + \ldots}_{\text{Higher order Taylor}}$$

$t = t_0 + \varepsilon$

$$x' = \frac{L}{W} + \varepsilon W X'$$
$$y' = \varepsilon W Y'$$
} no approximations yet.

$$f = \sqrt{(x')^2 + (y')^2} \approx |x'| = \boxed{\frac{L}{W} + \varepsilon W X'}$$

$\uparrow$ $\cancel{O(\varepsilon^2)}$

$\varepsilon W Y'$ $\quad$ $\frac{L}{W} + \cancel{\varepsilon W X'}$ $\quad$ $\frac{dX}{ds}$

$\frac{1}{1 + c\varepsilon} \approx 1 - c\varepsilon + \varepsilon^2$

$T\left(\frac{L}{W}\right)$

$T(f) \approx T(\frac{L}{W} + \varepsilon W X') = T \ldots \varepsilon W T'(\ldots) X'$

$$T(f) \approx T\left(\frac{L}{W} + \varepsilon W X'\right) \overset{\cancel{O(\varepsilon^2)}}{=} T_* + \varepsilon W T'\left(\frac{L}{W}\right) X'$$

tangent line approximation

T(f)

T' is dT/df > 0

$$\hat{\tau} = \frac{(x', y')}{f} \approx (1, \varepsilon L Y')$$

$$\boxed{\frac{y'}{f} = \frac{\varepsilon W Y'}{\left(\frac{L}{W}\right) + \cancel{X}} = \varepsilon \frac{W^2}{L} Y'.}$$

(error in posted notes)

$$T(f)\hat{\tau} \approx \left(T_* + \varepsilon W T'\left(\frac{L}{W}\right) X', \ \varepsilon \frac{W^2}{L} T_* Y'\right).$$

$$\left(T(f)\hat{\tau}\right)' \approx \boxed{\left(\varepsilon W T'\left(\frac{L}{W}\right) X'', \ \varepsilon \frac{W^2}{L} T_* Y''\right) = (0, \mu g)} \quad (4).$$

X and Y decouple in this linearized problem

First component $X'' = 0$ $\quad X(0) = 0, \ X(1) = 0$

so $X \equiv 0$

This does not mean $x = \frac{L}{W} s$, just means $x = \frac{L}{W} s +$ size ε^2.

Consider second component of (4).

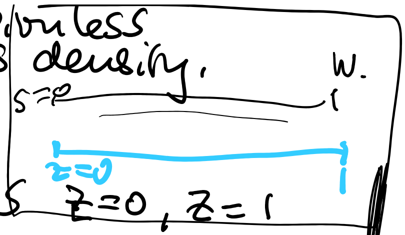
$$\varepsilon \frac{W^2}{L} T_* Y'' = \mu(s) g \quad (5).$$

let $M = \int_0^W \mu(s)\, ds$. total mass

scale μ by M/L

$$\mu(s) = \frac{M}{L} \tilde{m}(s) \leftarrow \text{dimensionless mass density.}$$

$$\mu(s) = \frac{M}{L}\,\tilde{m}(s) \leftarrow \text{dimensionless mass density.}$$



Scale s by W

$$\boxed{s = Wz} \leftarrow \text{dimensionless } z=0,\ z=1 \text{ ends of the wire.}$$

$$\varepsilon \frac{\cancel{W^2}}{\cancel{L}} T_* \frac{1}{\cancel{W^2}} \frac{d^2Y}{dz^2} = \frac{Mg}{\cancel{L}}\, m(z) \leftarrow s/W.$$

$$\varepsilon \frac{d^2Y}{dz^2} = \frac{Mg}{T_*}\, m(z)$$

check: dimensions $\frac{Mg}{T_*}$ should be dimensionless ✓.

$m(z) = \tilde{m}(zW)$.

Take $\boxed{\varepsilon = \frac{Mg}{T_*}}$

check whether this is small.

$$\boxed{\frac{d^2Y}{dz^2} = m(z) \qquad Y(0)=0 \qquad Y(1)=0}$$

This is just (1) in different letters. kg/m.

$$\boxed{m(z) = \frac{\mu(zW)\,L}{M}}$$ ✓

get Y.

$$y(s) = \frac{MgW^2}{LT_*}\, Y(s/W).$$

z

$u'' + u' - \lambda u = 0.$

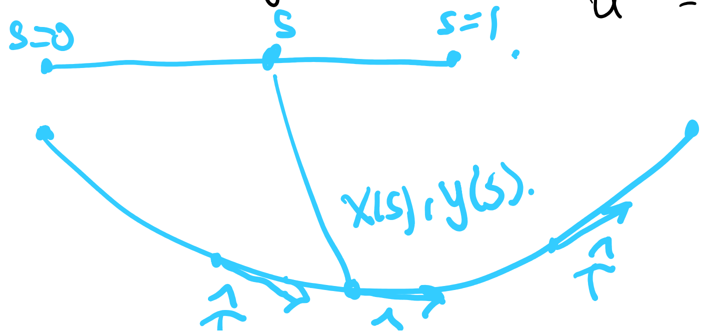
$\boxed{u(0)=0,\quad u(1)=0.}$

$u'' = \lambda u \qquad u(0)=0,\ u(1)=0.$

$\sin n\pi x.$

$u'' = -n^2\pi^2 \sin n\pi x.$

$\lambda =$ negative



Lecture #4

January 19, 2021 3:54 PM

BVP (1) $u(x)$

$$\underbrace{u'' = f(x) \quad u(0)=0,\ u(1)=0.}$$

$\mathcal{L}u = f$

includes DE and BCs.

Eigenfunctions $\phi(x)$ and eigenvalues λ

$\underline{\mathcal{L}}\phi = \lambda\phi$ and ϕ is not identically zero.

$\underline{\phi'' = \lambda\phi}$ $\quad \phi(0)=0,\ \underline{\phi(1)=0}.$

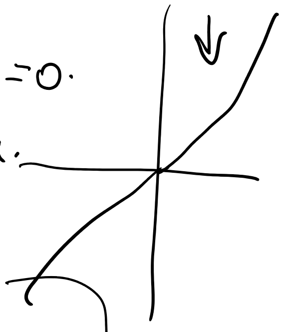
$\lambda=0 \quad \phi''=0 \quad \phi = A+Bx \quad \phi\equiv 0.$

$\lambda>0 \quad \lambda=\mu^2 \quad \phi'' - \mu^2\phi = 0 \quad (*)$

"recipe" $\phi = e^{rx}$ $\quad r^2-\mu^2=0.$

$r=\pm\mu.$

sinh x



general solution of $(*)$ is

$$\phi = Ae^{\mu x} + Be^{-\mu x}$$

equivalently $\phi = \underline{C\sinh \mu x + D\cosh \mu x}.$

$\sinh x = \dfrac{e^x - e^{-x}}{2} \qquad \cosh x = \dfrac{e^x+e^{-x}}{2}$

$\sinh 0 = 0 \qquad \cosh 0 = 1$

$\sinh'(0) = 1 \qquad \cosh'(0) = 0.$

$C=D=0$ to satisfy the BC's, $\lambda>0$ no eigenvalues.

$\lambda<0 \quad \lambda=-\mu^2$

$\lambda < 0 \quad \lambda = -\mu^2$

$$\phi'' = -\mu^2 \phi \qquad \phi'' + \mu^2 \phi = 0$$

$$r^2 + \mu^2 = 0$$

Complex roots $\quad r = \pm i\mu$.

$$\phi = A \sin \mu x + B \cancel{\cos \mu x} \quad \Leftarrow \quad u(0) = 0, \; B = 0$$

$\uparrow$

$u(1) = 0 \Rightarrow A = 0$ (not eigenvalue)

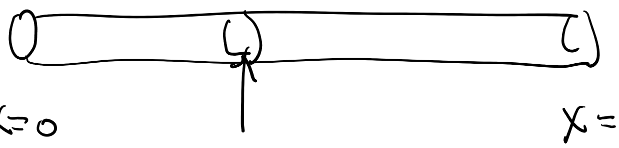
unless $\sin \mu = 0 \Rightarrow \mu = n\pi$.

$$\lambda = -\mu^2 = -n^2\pi^2$$

corresponding eigenfunction $\quad \underline{\sin n\pi x}$.

Derive the Heat Equation.

Thermal conduction in a thin rod.



$x=0 \qquad\qquad x=L$

temperature in a cross section at x

Take as approximately constant on this cross section.

$U(x,t)$.

Temperature is held constant at the ends.

$U(0,t) = 0$ and $U(L,t) = 0$.

Volumetric Heating $F(x,t)$ given data.

Volumetric Heating $F(x,t)$ given data.

Foreshadow the result. After scaling x and t and u, we will get dimensionless problem

$$u_\tau = u_{yy} + F(y,\tau) \qquad u(0)=0, \quad u(1)=0$$

$\frac{\partial u}{\partial \tau}$ / scaled time $\frac{\partial^2 u}{\partial y^2}$ PDE!

No surprise $x = Ly$

Two experimental observations.

Material sample mass m (kg) initially at temperature U_0, heat Q (J) is added and temperature increases to $U_1 > U_0$. It is found that $U_1 - U_0$ is approximately proportional to Q/m. ← with a constant c

$$\frac{Q}{m} = c\,(U_1 - U_0)$$

c does not depend on m and not strongly on U_0 in a range.

c has units $\frac{J}{kg\,°C}$

Let ρ be the mass density (kg/m³) — assume roughly constant with temperature.

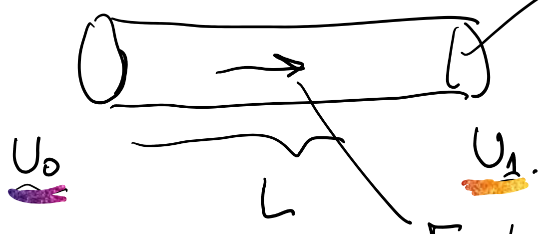
$c\rho$ ← energy required to heat 1 m³ of the material by 1°C.

(units $\frac{J}{m^3\,°C}$)

(units $\frac{J}{m^3 \, {}^\circ C}$)
Volumetric heat capacity.

Second experiment.

Constant cross-sectional area A.



F heat flow through the material at steady state.

$$F = \frac{KA}{L}(U_0 - U_1)$$

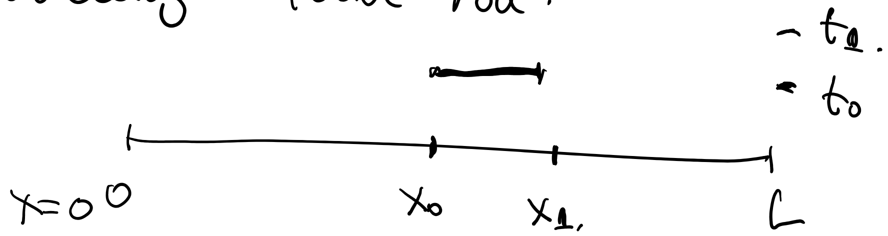
material constant K ← independent of L and A

units $\frac{J}{s \, m \, {}^\circ C}$

Heat flux per unit area. Take the limit as $L \to 0$.

$$j = \frac{F}{A} \qquad j = -K \frac{\partial U}{\partial x}.$$

Modeling. Thin rod.



At t_0, heat flow into the piece at the left.

$$j_0 A = -KA \left.\frac{\partial U}{\partial x}\right|_{x_0}.$$

heat flow into the piece at the right.

$$-j_1 A = KA \left.\frac{\partial U}{\partial x}\right|_{x_1}.$$

$$j_0 A - j_1 A + F(x, t_0)(x_1 - x_0)A \to$$

$$j_0 A - j_1 A + F(x, t_0)\underbrace{(x_1 - x_0)A}_{\text{Volume.}}$$

volumetric heating

$$= \boxed{kA\left(\frac{\partial U}{\partial x}\Big|_{x_1} - \frac{\partial U}{\partial x}\Big|_{x_0}\right) + (x_1 - x_0)A\,\underline{F}(x, t_0)}$$

Assume this heating is uniformly in $[t_0, t_1]$.

Volumetric heat capacity

$$\underbrace{(x_1 - x_0)A}_{\text{Volume.}}\, c\rho\,(U(t_1) - U(t_0)) \simeq (t_1 - t_0)\underbrace{\Big[\qquad\qquad\Big]}_{\text{amount of heat entering the piece of the rod.}} \quad (*)$$

average temperature in the piece.

increase the temperature of the piece.

Divide $(*)$ by $x_1 - x_0$ and $t_1 - t_0$ averages out

$$\frac{U_{ave}(t_1) - U_{ave}(t_0)}{t_1 - t_0} = \frac{k}{c\rho}\,\frac{\left(U_x|_{x_1} - U_x|_{x_0}\right)}{x_1 - x_0} + F_{ave}$$

in x in x and t

Take limits $t_1 \to t_0$, $x_1 \to x_0$.

$$\boxed{\frac{\partial U}{\partial t} = \frac{k}{c\rho}\frac{\partial^2 U}{\partial x^2} + \frac{1}{c\rho}F(x,t).}$$

linear!

k thermal diffusivity

units of k: m^2/s.

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2} + \frac{1}{c\rho}F(x,t).$$

Scale x, $x = 1$ u $\leftarrow$ dimensionless

Scale x, $\quad x = L y \leftarrow$ dimensionless $y \in [0,1]$.

$$\frac{\partial^2 U}{\partial x^2} = \frac{1}{L^2} \frac{\partial^2 U}{\partial y^2}$$

$$\frac{\partial U}{\partial t} = \frac{k}{L^2} \frac{\partial^2 U}{\partial y^2} + \frac{1}{c\rho} F(x,t).$$

time scale? $\quad t = T\tau \quad \leftarrow$ what should T be?

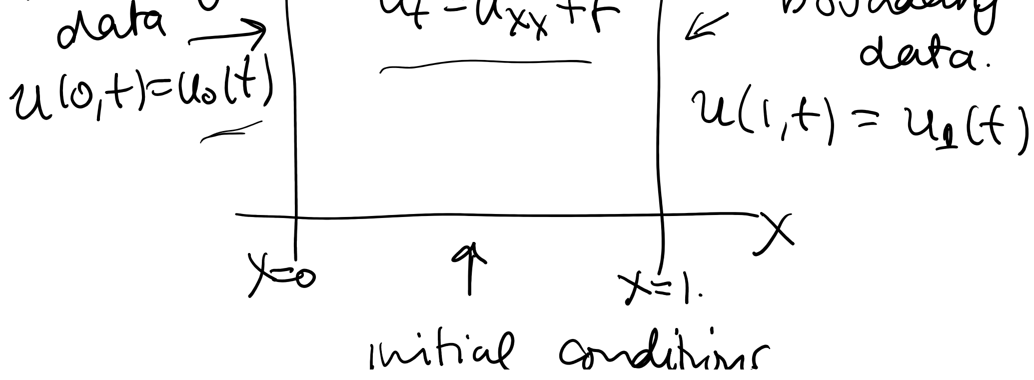
$$\frac{\partial U}{\partial t} = \frac{1}{T} \frac{\partial U}{\partial \tau}. \qquad \text{Take } T = \frac{L^2}{k}. \quad (\text{units } s\ \checkmark).$$

Then
$$\frac{\partial U}{\partial \tau} = \frac{\partial^2 U}{\partial y^2} + \underbrace{\frac{L^2}{kc\rho} F\left(Ly, \frac{L^2}{k}\tau\right)}_{f(y,\tau)}.$$

so
$$\underbrace{\frac{\partial U}{\partial \tau} = \frac{\partial^2 U}{\partial y^2}}_{} + \underline{f(y,\tau)}. \quad \leftarrow \text{form foreshadowed.}$$

U still has a scale. No specific scale for U in this linear problem - use any scale you want.

Heat Equation in a bounded domain.



$u(x,t)$.

$x=1$.

initial condition

$$u(x,0) = V(x)$$

4 types of data.

$u_0(t)$, $u_1(t)$ — Boundary data.

$V(x)$ — initial data.

$F(x,t)$ — forcing.

Theory says the problem has a unique solution, well posed to all the data types.

Reminder - we can solve these problems separately due to linearity.

I: u_I solves $u_0(t)$ ✓
$u_1(t) \equiv 0$, $V(x) \equiv 0$, $F \equiv 0$.

II u_{II} solves u_1 ✓
$u_0(t) \equiv 0$, $V(x) \equiv 0$, $F \equiv 0$.

III u_{III} solves $V(x)$ ✓
$u_0(t) \equiv 0$, $u_1(t) \equiv 0$, $F \equiv 0$.

IV u_{IV} solves F ✓
$u_0 \equiv 0$, $u_1 \equiv 0$, $V \equiv 0$.

Solution to the original problem is

$$u = u_I + u_{II} + u_{III} + u_{IV}.$$

Look at posted notes at Variable Coefficient problem.

Lecture #5

January 26, 2021 9:22 AM

Th lecture - Heat Equation

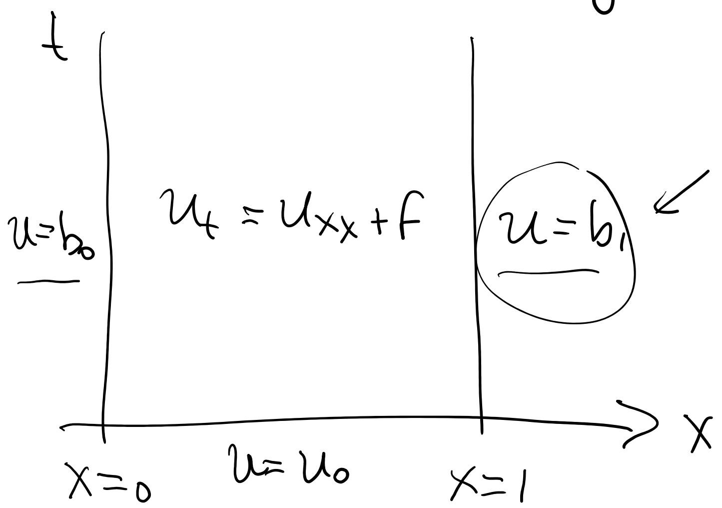
$u(x,t)$ unknown.

$$\left.\begin{aligned} u_t &= u_{xx} + \underbrace{\boxed{f(x,t)}}_{\text{data}} \\ \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} + f \end{aligned}\right\} \quad \text{PDE} \qquad t>0,\ 0<x<1.$$

$u(x,0) = \boxed{u_0(x)}$ Initial conditions

$$\left.\begin{aligned} u(0,t) &= b_0(t) \\ u(1,t) &= b_1(t) \end{aligned}\right\} \quad \text{Boundary conditions}$$

It is an initial boundary value problem.



Insulated bc.

$\frac{\partial u}{\partial x} = 0.$

$-\frac{\partial u}{\partial x}$ proportional to heat flux.

	u_0	f	b_0	b_1
u_I	✓	0	0	0
u_{II}	0	✓	0	0
u_{III}	0	0	✓	0
u_{IV}	0	0	0	✓

u_{IV} | ∪ ∪ ∪ ∨

Solution $u = u_I + u_{II} + u_{III} + u_{IV}$ only for linear problems

Look at I & II first in simple setting

↑ (review but some new perspectives)

These two solution processes are related. (DuHamel's principle).

Scalar $u(t)$

$$\dot{u} = F(u,t). \qquad \text{IVP.}$$

$\frac{du}{dt}$

Theory: If F is differentiable, there exists a unique solution in an interval $[0,T)$.

Restrict to linear problems.

given data.

$$\dot{u} = a(t)u + \boxed{f} \qquad u(0) = \boxed{u_0.} \qquad (1).$$

I $\dot{u} = a(t)u$ $\qquad u(0) = u_0.$

II $\dot{u} = a(t)u + f$ $\qquad u(0) = 0.$

Solution to (1) is

$$u = u_I + u_{II}$$

homogeneous solution

particular solution.

$\boxed{\text{solve I}}$

$\dot{u} - a(t)u = 0 \qquad u(0) = u_0$

solution

solve I

$$\dot{u} - a(t)u = 0 \qquad u(0) = u_0$$

multiply the equation by $e^{-A(t)}$

where $A(t) = \int_0^t a(s)\,ds \rightarrow A' = a.$

$A(0) = 0.$

$$\underbrace{e^{-A(t)}\dot{u} - a(t)e^{-A(t)}u}_{\frac{d}{dt}\left(e^{-A(t)}u\right)} = 0.$$

$$\frac{d}{dt}\left(e^{-A(t)}u\right) = 0$$

Integrate

$$e^{-A(t)}u(t) = C.$$

$$u(t) = e^{A(t)}C = u_0 \boxed{e^{A(t)}}.$$

Green's Function for the IC's.

match IC's $u(0) = u_0.$

problem II

$$\dot{u} - a(t)u = f \qquad u(0) = 0.$$

multiply by the same integrating factor.

$$\frac{d}{dt}\left(e^{-A(t)}u(t)\right) = e^{-A(t)}f(t).$$

Integrate from 0 to t (use s as variable of integration)

$$e^{-A(t)}u(t) = \int_0^t e^{-A(s)}f(s)\,ds.$$

$$u(t) = \int_0^t \underbrace{e^{A(t)-A(s)}}_{\text{Green's Function.}} f(s)\,ds.$$

Notes: • $u(t)$ only depends on $f(s)$ for $0 \le s \le t.$

Notes: • $u(t)$ only depends on $f(s)$ for $0 \le s \le t$.
(Causality).

• $e^{A(t)-A(s)} = e^{\int_s^t a(s)\,ds}.$

$A(t) = \int_0^t a(\tau)\,d\tau$

Green's Function for problem I with initial data at time s.

Example of DuHamel's Principle.

Ex $\dot{u} = -u + \cos t \qquad u(0) = 2.$

$u_I = 2e^{-t}$

$u_{II} = \int_0^t e^{-(t-s)} \cos s \, ds.$

$= \frac{1}{2}(\sin t + \cos t) - \frac{e^{-t}}{2}.$

$$u = \underbrace{\frac{1}{2}(\sin t + \cos t)}_{\text{persistent solution.}} + \underbrace{\frac{3}{2}e^{-t}}_{\text{Transient from the initial conditions}}.$$

vector $\underline{u}(t)$ n component column vector

$\dot{\underline{u}} = A(t)\underline{u} + \underline{f}(t) \qquad \underline{u}(0) = \boxed{\underline{u}_0}$

I. $\underline{f} \equiv 0$. Pure IVP.

Expect the solution $\underline{u}(t)$ to have the form.

$$\underline{u}(t) = \underline{\Phi}(t)\ \underline{u}_0 \qquad (3)$$

n components   n×n matrix   n components

Called the Fundamental solution matrix. (Green's Function).

This works if every column $\underline{\Phi}_j$ satisfies.

$$\dot{\underline{\Phi}}_j = A\,\underline{\Phi}_j \quad \text{with} \quad \underline{\Phi}_j(0) = \underline{e}_j. \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

standard unit basis vector.

Note: $\underline{\Phi}(0) = I$ ← identity matrix.

$$\frac{d\underline{\Phi}}{dt} = A\,\underline{\Phi}$$

n×n matrix

verify column by column.

So $\frac{d\underline{u}}{dt}$ From (3) $u(t)$

$$\frac{du}{dt} = \frac{d}{dt}\left(\underline{\Phi}\ \underline{u}_0\right) = A\,\underline{\Phi}\ \underline{u}_0 = A\,\underline{u} \quad ✓$$

II $\underline{u}_0 = \underline{0}$ $f(x,t)$ ✓

Extend the previous idea to

$$\underline{\Psi}(t,s)$$

$\Psi(t,s)$

↑ Fundamental matrix with initial data at times

$\Psi(t,0) = \Phi(t)$ ← before, ICs at time 0.

$$\frac{\partial \Psi}{\partial t} = \boxed{A(t)\,\Psi} \qquad \text{for every } s.$$

$$\Psi(t,t) = I.$$

Duhamel's principle ↓ II $\quad u_0 = 0$

f ✓.

$$u(t) = \int_0^{t} \Psi(t,s)\, f(s)\, ds. \qquad (4).$$

(Need to check ⟶).

I identity matrix

$$\frac{du}{dt} = \int_0^t \frac{\partial \Psi}{\partial t} f(s)\,ds + \underbrace{\Psi(t,t)\, f(t)}_{f(t)}$$

$$= \int_0^t A(t)\,\Psi f(s)\,ds,$$

$$= A(t) \underbrace{\int_0^t \Psi(t,s)\, f(s)\,ds}_{u(t).} + f(t)$$

$$= A\,u + f.$$

Note: This is the variation of parameters formula.

(See notes for this interesting but not essential topic).

Simplest vector case $\quad \dot{u} = A\,u \qquad u(0) = u_0.$

A is symmetric does not depend on time.

A has real eigenvalues λ_j

o.n. eigenvectors $\underline{v}_j$

$\{\underline{v}_j\}$ is a basis, so

$$\underline{u}(t) = \sum_{j=1}^{n} a_j(t)\, \underline{v}_j \qquad \text{for some } a_j(t).$$

To match the initial conditions

$$\sum_{j=1}^{n} a_j(0)\, \underline{v}_j = \underline{u}_0.$$

Same as $\mathbb{T}\, \underline{a}(0) = \underline{u}_0 \Rightarrow \underline{a}(0) = \mathbb{T}^{-1} \underline{u}_0$

$\underline{v}_j$ in columns

Note: if the columns of $\mathbb{T}$ are o.n then $\mathbb{T}^{-1} = \mathbb{T}^{T}$

put in the DE

$$\underbrace{\sum \dot{a}_j\, \underline{v}_j = \sum_{j=1}^{n} \lambda_j a_j\, \underline{v}_j}_{\text{orthogonal.}}$$

$\dot{a}_j = \lambda_j a_j \quad j = 1, \dots n. \quad a_j(t) = a_j(0)\, e^{\lambda_j t}$

$$\underline{a}(t) = \Lambda\, \underline{a}(0) = \Lambda \mathbb{T}^{T} \underline{u}_0$$

diagonal matrix entries $e^{\lambda_j t}$

$\Lambda(t)$

diagonal matrix entries $e^{\lambda_j t}$

now $\underline{u}(t) = T\,\underline{a}(t) = \underbrace{T \overset{\Lambda(t)}{\Lambda} T^T}_{} \underline{u}_0$

Fundamental solution

$\Phi(t)$

In this case, can label $e^{\underline{At}}$.

$-u'' = \mu^2 u$ $\quad$ $-u'' = \dfrac{\mu^2}{K} u$

$\underline{u=0}$ $\quad$ $\underline{u=0}$

$\underline{A, B}$ $\quad$ $\underline{C, D}$.

$u_+ = u_-$

$u'_+ = u'_-$

$\boxed{\text{Check } \underline{K=1} \to \mu = \pi.}$

$\underline{\text{Hint}}$.

Lecture #6

January 28, 2021 9:16 AM

$\underline{u}(t)$ $\qquad \dot{\underline{u}} = \mathbb{A}\underline{u}$ $\qquad \underline{u} = \underline{u}_0$

$\mathbb{A}$ symmetric. λ_j real basis o.n eigenvectors $\{\underline{v}_j\}$.

$$\boxed{\underline{u}(t) = \sum_{j=1}^{n} a_j(t)\underline{v}_j}$$

eigenvector $\mathbb{A}\underline{v}_j = \lambda_j \underline{v}_j$

$$\sum_{j=1}^{n} \dot{a}_j \underline{v}_j = \sum_{j=1}^{n} \lambda_j a_j \underline{v}_j$$

$$\underline{u}(0) = \sum_{j=1}^{n} a_j(0)\underline{v}_j$$

$$\dot{a}_j = \lambda_j a_j$$

solution $a_j(t) = a_j(0)e^{\lambda_j t}$

$$a_j(0) = \underline{u}_0 \cdot \underline{v}_j.$$

$\underline{v}_i \cdot \underline{v}_j = 0$ unless $i=j$

$\underline{v}_i \cdot \underline{v}_j = 1.$

$u(x,t)$

$u_t = u_{xx}$ $\qquad u(x) = u_0(x)$

$u(0,t) = 0, \quad u(1,t) = 0.$

$u_t = \mathcal{L}u$

includes the BC's

$\mathcal{L}$ is symmetric $\qquad (u,v) = \int_0^1 u(x)v(x)\,dx$

eigenvalues $\lambda_n = -n^2\pi^2$, o.n. eigenfunctions $\phi_n = \sqrt{2}\sin n\pi x.$

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t)\,\underline{\phi_n(x)}$$

$u_t = u_{xx}$ $\qquad \sum_{n=1}^{\infty} \dot{a}_n \phi_n(x) = \sum_{n=1}^{\infty} -n^2\pi^2 a_n \phi_n(x)$

o.n.

$\dot{a}_n = -n^2\pi^2 a_n$

$a_n(t) = a_n(0)\, e^{-n^2\pi^2 t}$

$$u(x,t) = \int_0^1 G(x,t,s)\, u_0(s)\, ds$$

$a_n(0) = \int_0^1 u_0(x) \sin n\pi x \, dx.$

put all together.

$$G(x,t,s) = 2\sum_{n=1}^{\infty} \sin n\pi s \sin n\pi x \; e^{-n^2\pi^2 t}.$$

$$u(x,t) = \boxed{\sum_{n=1}^{\infty} 2 \underbrace{\left[\int_0^1 u_0(s) \sin n\pi s \, ds\right]}_{\text{a number}} e^{-n^2\pi^2 t} \sin n\pi x}.$$

bunch of decaying terms.

Discussion:

- so $\lim_{t\to\infty} u(x,t) \equiv 0.$

- $n=1$ is the slowest decaying term.

  $e^{-\pi^2 t}$ ← time scale for decay $T = \frac{1}{\pi^2}$

  $\sin \pi x$ is the shape of the slowest decaying term.

$u_t = -u_{xx}$

Backward Heat Equation

This problem is ill-posed.
Not a physical model.

$u_0(x)$

$$u(x,t) = \sum_{n=1}^{\infty} 2\left[\int_0^1 (\sin n\pi s)\, u_0(s)\, ds\right] e^{n^2\pi^2 t} \sin n\pi x$$

$$u(x,t) = \sum_{n=1} 2\left[\int_0 (\sin n\pi s)\ldots\right] e \quad \sin n\pi x$$

some time T
This term
$e^{n^2\pi^2 T} \to \infty$
as $n \to \infty$.

Arbitrarily small changes in u_0 make arbitrarily large changes in the solution.
Ill-posed!

$$\dot{\underline{u}} = \underline{\underline{A}}\,\underline{u} + \underline{f} \qquad \underline{u}(0) = \underline{0}.$$

$$\lambda_j, \ \underline{v}_j$$

$$\underline{u}(t) = \sum a_j(t)\,\underline{v}_j \qquad \sum b_j \underline{v}_j$$

$$\sum \dot{a}_j \underline{v}_j = \sum a_j \lambda_j \underline{v}_j + \underline{f}(t) \qquad b_j = \underline{f}(t)\cdot \underline{v}_j.$$

$$\dot{a}_j(t) = \lambda_j a_j + b_j(t).$$

$$a_j(t) = \int_0^t e^{\lambda_j(t-s)} f_j(s)\,ds.$$

$$\underline{u} = \sum_{j=1}^n a_j(t)\,\underline{v}_j.$$

$$u_t = u_{xx} + f(x,t) \qquad u(x,0) \equiv 0.$$
$$u(0,t) = 0, \quad u(1,t) = 0.$$

$$f(x,t) = \sum_{n=1}^{\infty} f_n(t) \sin n\pi x$$

sine series at every t.

$$f_n(t) = \sqrt{2}\int_0^1 f(s,t)\sin n\pi s\,ds.$$

$$u(x,t) = \sum^{\infty} a_n(t) \sin n\pi x$$

$$f_n(t) = \sqrt{2} \int_0^1 f(s,t) \sin n\pi s \, ds.$$

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t) \sin n\pi x$$

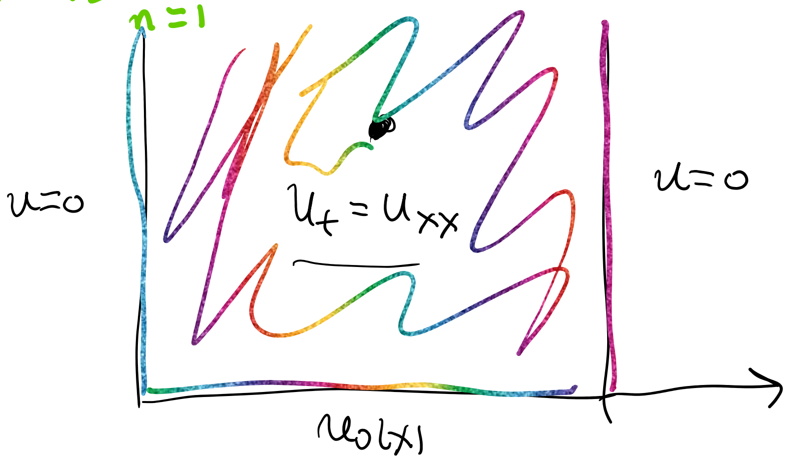
$$a_n(0) = 0.$$

$$\dot{a}_n = -n^2\pi^2 a_n + f_n.$$

$$a_n(t) = \int_0^t e^{-n^2\pi^2(t-s)} f_n(s) \, ds$$

$n=1$ eigenvalue is the most interesting here as well.

$$u(x,t) = \sqrt{2} \sum_{n=1}^{\infty} a_n(t) \sin n\pi x.$$



(*)

$$u_t - u_{xx} = 0.$$

Theorem (Maximum Principle). Let $M = \max_x \{u_0(x), 0\}$.

The solution to (*) satisfies.

$$u(x,t) \leq M \quad \text{for all } x \in [0,1],\ t \geq 0.$$

Makes physical sense.

Proof: Consider $V = u + \varepsilon x^2/2$.

$$V_t - V_{xx} = \underbrace{u_t - u_{xx}}_{0 \text{ by Heat Equation}} - \varepsilon < 0.$$

$$V_t - V_{xx} < 0.$$

V cannot have a maximum value in the interior of the domain.

interior of the domain.

So v must have a maximum at either

(i) left boundary $v = 0$

(ii) right boundary. $v = \varepsilon/2$.

(iii) initial conditions $v \leq M + \varepsilon/2$.

$$u = v - \varepsilon x^2/2 \leq v \leq M + \varepsilon/2.$$

Result without knowing the solution.

Let $\varepsilon \to 0$, $u \leq M$ ✓.

Note: A similar argument shows

$$u(x,t) \geq \min_{x \in [0,1]} \{ u_0(x), 0 \}.$$

Also shows that the problem is unique and well posed.

$u_0(x) \to u(x,t)$

$v_0(x) \to v(x,t)$.

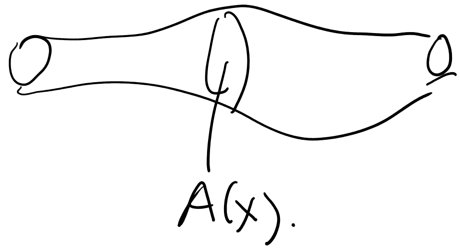
$$\boxed{\underbrace{|u(x,t) - v(x,t)|}_{\text{solves } \circledast \text{ with } u_0 - v_0} \leq |u_0(x) - v_0(x)|.}$$

Lecture #7

February 2, 2021 9:18 AM

Generalized Heat Equation.

$$C(x)\rho(x)A(x)u_t = (K(x)A(x)u_x)_x.$$



When c, ρ, K, A were independent of x, we had the scaled problem

$$u_t = u_{xx} \qquad x \in [0,1]. \leftarrow$$

vanilla Heat Equation.

$$u_t = \frac{1}{\boxed{C(x)\rho(x)A(x)}}\left(\boxed{K(x)A(x)u_x}\right)_x. \quad (\star).$$

Eigenanalysis of linear operators of the form

$$\mathcal{L}\varphi = \frac{1}{a(x)}\left(b(x)\varphi'\right)'$$

will give series solutions to $(\star)$

Think back to the derivation of the heat equation

$-u_x \leftarrow$ is proportional to the heat flux in $+x$ direction

Boundary conditions $u_x\big|_{x=0} = 0 \leftarrow$ insulated condition.

homogeneous Neumann condition.

Neumann condition.

Backtrack BC $\boxed{u\big|_{x=0} = 0} \leftarrow$ homogeneous Dirichlet condition

Dirichlet condition

$$\Big|_{x=0}$$

Dirichlet condition.

Dirichlet condition

Can't apply both! This is a choice.

$x=0$ BC third alternative. $\alpha>0$ given constant.

$$u-\alpha u_x=0\Big|_{x=0}. \qquad (\alpha u_x=u)$$

Robin condition.

if $u_x>0$ this is heat flux out of the left end.

Newton law of cooling term.

At the right end.

$$u+\alpha u_x=0\Big|_{x=L}.$$

$$\boxed{\begin{aligned} \mathcal{L}\varphi = \frac{1}{a(x)}\left(b(x)\varphi'\right)' &= \lambda\varphi \\ c_1\varphi(0)+c_2\varphi'(0)&=0 \\ d_1\varphi(1)+d_2\varphi'(1)&=0 \end{aligned}}$$

$(b(x)\varphi')'=\lambda a(x)\varphi.$

c_1, c_2 given constants, not both zero.

d_1, d_2 not both zero.

Sturm Liouville problem.

Eigenvalues & eigenfunctions, I would get a way to find series solutions for

$$\mathcal{L}u=f$$

and $\quad u_t=\mathcal{L}u \quad$ with $u(x,0)=u_0(x)$ given

and $\quad u_t=\mathcal{L}u+f \quad$ with $u(x,0)=0$

(Duhamel's principle).

(Duhamel's principle).

Theory (assumes $a(x) > 0$, $b(x) > 0$).

[1] All eigenvalues are real and negative.

[2] All eigenvalues are simple.
(there is only one eigenfunction for each eigenvalue).

[3] If $\varphi_n(x)$ and $\varphi_m(x)$ are two eigenfunctions corresponding to different eigenvalues, then

$$\int_0^1 a(x)\,\varphi_n(x)\,\varphi_m(x)\,dx = 0.$$

(a kind of orthogonality).

[4] The eigenvalues form an infinite series

$$0 > \lambda_1 > \lambda_2 > \cdots$$

and $\lambda_n \to -\infty$ as $n \to \infty$.

[5] Suppose f is continuously differentiable.
then

$$\sum_{n=1}^{\infty} f_n \varphi_n(x)$$

want this to be $f(x)$.

with $f_n = \dfrac{\int_0^1 a(x)\,\varphi_n(x) f(x)\,dx}{\int_0^1 a(x)\,[\varphi_n(x)]^2\,dx}$

Converges pointwise to $f(x)$.

$\{\varphi_n\}_{n=1}^{\infty}$ is a "basis" of functions.

Example: $\lambda \varphi = \varphi''$ $\varphi(0) = 0$, $\varphi(1) + \alpha\varphi'(1) = 0$.

$\varphi'' = \lambda\varphi$ Theory or direct check gives
$\lambda < 0$, $\lambda = -\mu^2$.

$\lambda < 0, \quad \lambda = -\mu^2.$

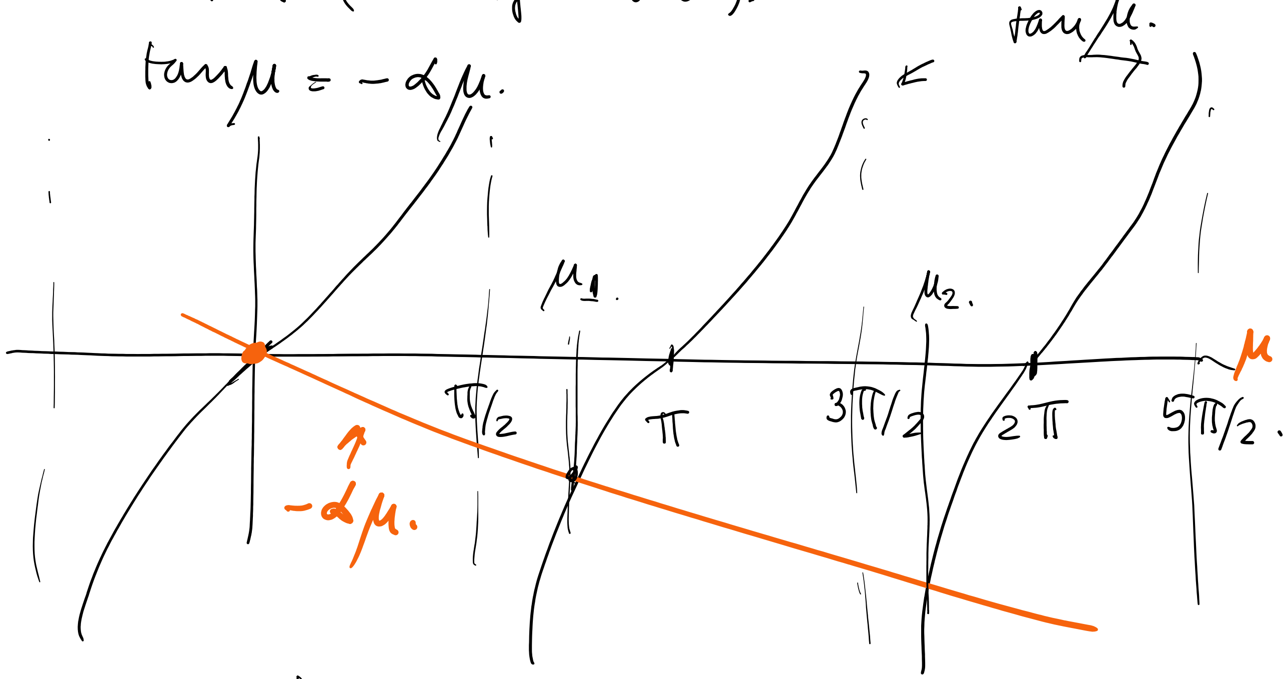
$$\varphi = A \sin \mu x + \cancel{B \cos \mu x} \qquad \varphi(0) = 0 \Rightarrow B = 0$$

Need $A \neq 0$ to get an eigenfunction, so

$$\varphi(1) + \alpha \varphi'(1) = A(\underbrace{\sin \mu + \alpha \mu \cos \mu}_{=0}) = 0$$

$\mu = 0$ is a root (no eigenvalue).

$$\tan \mu = -\alpha \mu.$$



Eigenvalues $\lambda_n = -\mu_n^2$. (exact values – no exact formula, use Wolfram Alpha)

Also, from the theory.

$$\int_0^1 \sin \mu_n x \sin \mu_m x \, dx = 0 \quad \text{if} \quad n \neq m.$$

and $\sum_{n=1}^{\infty} f_n \sin \mu_n x$ converges to $f(x)$

when $$f_n = \frac{\int_0^1 f(x) \sin \mu_n x \, dx}{\int_0^1 \sin^2 \mu_n x \, dx}$$

I call this the "sine-α" series.

Use this to solve some problems.

Use this to solve some problems.

(I) $u'' = f(x)$ $\quad u(0)=0, \quad u(1)+\alpha u'(1)=0$

$$u = \sum_{n=1}^{\infty} \frac{f_n}{-\mu_n^2} \sin \mu_n x.$$

(II) $u_t = u_{xx}$ $\quad u(x,0) = f(x)$

$$u(x,t) = \sum_{n=1}^{\infty} f_n e^{-\mu_n^2 t}.$$

$u(0,t)=0$

$u(1,t)+\alpha u_x(1,t) = 0.$

(III) $u_t = u_{xx} + f(x,t)$

$u(x,0)=0$

Series with DuHamel's principle.

Ex 2 Example with a small parameter ε.
(Linearization).

Some warm-up

2 easier examples first.

$$(x-2)(x-1) = x^2 - 3x + 2 = 0.$$

roots are $x=1, 2.$

What about $\boxed{x^2 - 3x + 2 + \varepsilon = 0.}$ $\quad \varepsilon$ is "small".

Should have a root near $x=1$.

$x(\varepsilon)$, with $x(0)=1$.

$x(\varepsilon)$, with $x(0) = 1$.

Linear approximation $x(\varepsilon) \approx 1 + \frac{dx}{d\varepsilon}(0)\,\varepsilon$.

Implicit differentiation

$$2x\frac{dx}{d\varepsilon} - 3\frac{dx}{d\varepsilon} + 1 = 0.$$

$$\frac{dx}{d\varepsilon} = \frac{-1}{2x-3}$$

evaluate at $x=0$, $x(0)=1$.

$$\left.\frac{dx}{d\varepsilon}\right|_{\varepsilon=0} = 1.$$

So root $x \approx 1 + \varepsilon$.

Could work out higher order corrections

$$x(\varepsilon) \approx 1 + \frac{dx}{d\varepsilon}(0)\,\varepsilon + \frac{1}{2}\underline{\frac{d^2x}{d\varepsilon^2}}(0)\,\varepsilon^2 + \dots$$

repeated implicit differentiation.

Alternate derivation.

$$x^2 - 3x + 2 + \varepsilon = 0.$$

plug in

Know $x(\varepsilon) \approx \boxed{1 + \beta\varepsilon}$

has the form

$$(1+\beta\varepsilon)^2 - 3(1+\beta\varepsilon) + 2 + \varepsilon = 0.$$

$$(1 + 2\beta\varepsilon + \beta^2\varepsilon^2) - 3(1+\beta\varepsilon) + 2 + \varepsilon = 0$$

Collect terms in powers of ε <Check this

Collect terms in powers of ε

$O(1)$ $\quad 1 - 3 + 2 = 0$ ✓

$O(\varepsilon)$ $\quad 2\beta - 3\beta + 1 = 0.$

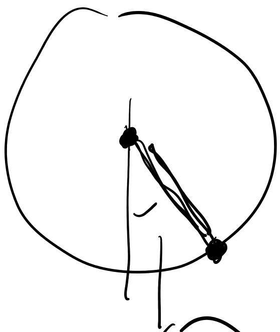
$\beta = 1$ as before.

Check that the first solution was correct.

Could work out higher order terms starting with

$$X(\varepsilon) = 1 + \beta\varepsilon + \gamma\varepsilon^2$$

and match terms of size ε^2.



$\Theta(t)$.

Scaled nonlinear pendulum problem

$$\ddot{\Theta} = -\sin\Theta.$$

$\Theta(0) = \varepsilon$

$\dot{\Theta}(0) = 0$

$|\Theta(t)| \leq \varepsilon$ for all t.

$$\Theta(t) \approx \varepsilon\,\theta(t) + \varepsilon^2 u(t) + \varepsilon^3 v(t).$$

$\theta(0) = 1$

$\dot{\theta}(0) = 0$

$$\sin x \approx x - \frac{x^3}{6}.$$

$$\varepsilon\ddot{\theta} + \varepsilon^2\ddot{u} + \varepsilon^3\ddot{v} \approx -\sin(\varepsilon\theta + \varepsilon^2 u + \varepsilon^3 v)$$

$$\approx -(\varepsilon\theta + \varepsilon^2 u + \varepsilon^3 v) - \frac{\varepsilon^3\theta^3}{6}.$$

Equate terms by powers of ε:

$O(\varepsilon)$: $\quad \ddot{\theta} = -\theta$ $\qquad \boxed{\theta = \cos t}$

$O(\varepsilon)$: $\ddot{\theta} = -\theta$ $\boxed{\theta = \cos t}$

usual linearized pendulum.

period 2π independent of ε.

$O(\varepsilon^2)$: $\ddot{u} = -u$ $u(0) = 0$

$\dot{u}(0) = 0$.

$u \equiv 0$.

$O(\varepsilon^3)$ $\ddot{v} + v = -\frac{\theta^3}{6} = \boxed{-\frac{\cos^3 t}{6}}$ $v(0) = 0$

$\dot{v}(0) = 0$.

For this problem, there is a resonant term that leads to a period dependence on ε.

$y(s) = \varepsilon W + \varepsilon^2 W^{(2)}$

Nonlinear $\Rightarrow$ nondimensionalize $\Downarrow$ perturbation.

Lecture #8

February 2, 2021 2:11 PM

Consider $A \equiv 1$, $K \equiv 1$, $\rho \equiv 1$, but $c(x)$ slowly varying

$x \in [0,1]$, $c(x) = 1 + \underbrace{0.01x}_{\varepsilon x}$.

$$u_t = \frac{1}{1+\varepsilon x} u_{xx} \qquad u(0,t)=0, \quad u(1,t)=0$$

$$\mathcal{L}\varphi := \frac{1}{1+\varepsilon x}\varphi'' \qquad \boxed{\varphi(0), \ \varphi(1)=0.}$$

$$\mathcal{L}\varphi = \lambda\varphi \Rightarrow \boxed{\varphi'' = \overset{(-\pi^2+\varepsilon\beta)}{\lambda}(1+\varepsilon x)\overset{(\sin\pi x+\varepsilon v)}{\varphi}}$$

We are most interested in λ_1 and $\varphi_1(x)$.

We expect $\lambda_1 \approx -\pi^2$ and $\varphi_1(x) \approx \sin\pi x$.

$$\varphi(x) \approx \sin\pi x + \varepsilon v(x)$$

$$\lambda \approx -\pi^2 + \beta\varepsilon$$

$$-\pi^2 \sin\pi x + \varepsilon v'' \approx -\pi^2 \sin\pi x + \varepsilon\beta\sin\pi x - \varepsilon\pi^2 x\sin\pi x - \varepsilon\pi^2 v$$

$O(1)$: $-\pi^2 \sin\pi x = -\pi^2 \sin\pi x$ ✓

$O(\varepsilon)$: $v'' + \pi^2 v = -\pi^2 x\sin\pi x + \beta\sin\pi x$.

$v(0) = v(1) = 0$

$$v(x) = A^{?}\sin\pi x + \cancel{B\cos x}$$
$$+ c\,x\sin\pi x + d\,x\cos\pi x$$
$$+ e\,x^2\sin\pi x + f\,x^2\cos\pi x$$

Algebra $c = -1/4$, $d = -\frac{\beta}{2\pi}$, $e = 0$, $f = \pi/4$.

Match BCs. $v(0)=0 \Rightarrow B=0$

Match BCs. $V(0)=0 \Rightarrow B=0.$

$V(1)=0 \Rightarrow d+f=0 \Rightarrow$

$$-\frac{\beta}{2\pi}+\frac{\pi}{4}=0$$

$$\beta = \pi^2/2.$$

$$\lambda_1(\varepsilon) \approx -\pi^2 + \frac{\pi^2}{2}\varepsilon.$$

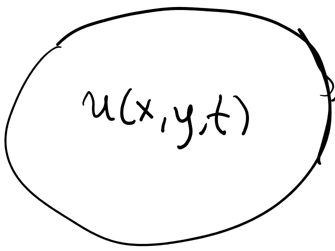
not determined. it is ok to set A=0.

$$V(x) = \boxed{A}\sin\pi x - \frac{1}{4}x\sin\pi x - \frac{\pi}{4}x\cos\pi x + \frac{\pi}{4}x^2\cos\pi x.$$

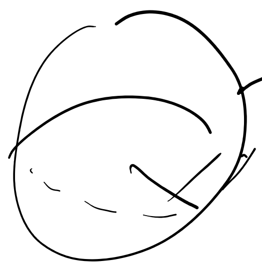
$$\varphi(x) = \sin\pi x + \varepsilon V(x).$$

On to higher dimensional problems (spatial).

2D. thin plate.



3D object



$u(x,y,z,t).$

$u(\underline{x},t).$

$$u_t = \underbrace{u_{xx}+u_{yy}}_{\text{Laplacian } \Delta u} + f(x,y,t).$$

$u(x,y,0)=u_0(x,y).$

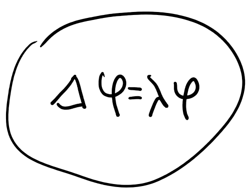
alternate notation $\nabla^2 u = \nabla\cdot(\nabla u).$

$u_t = \Delta u + f(\underline{x},t)$

$\uparrow$

$u_{xx}+u_{yy}+u_{zz}$

Series solutions



$\leftarrow$ eigenfunctions leading to series solutions.

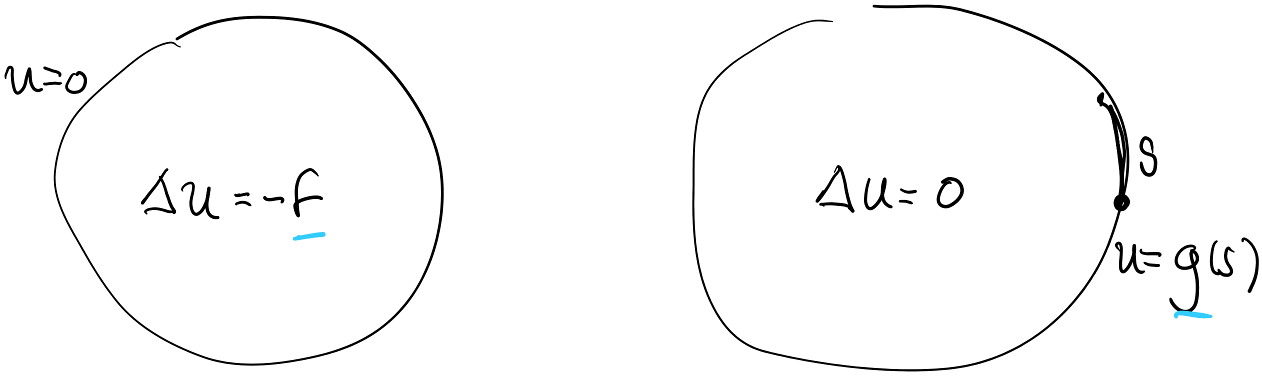
Suppose $f(x,y)$ ...

series solutions.

Suppose $f(x,y)$ ← not time dependent

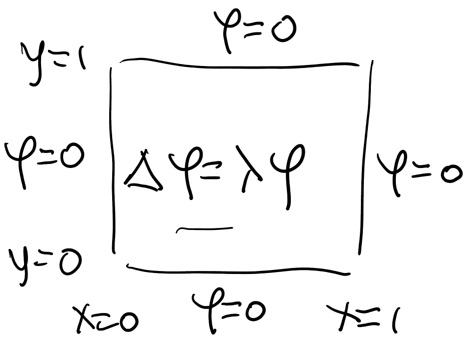
Expect as $t \to \infty$, steady state $u(x,y)$, solve

$$\Delta u = -f.$$



Poisson problem.
(will have a series solution from the eigenanalysis).

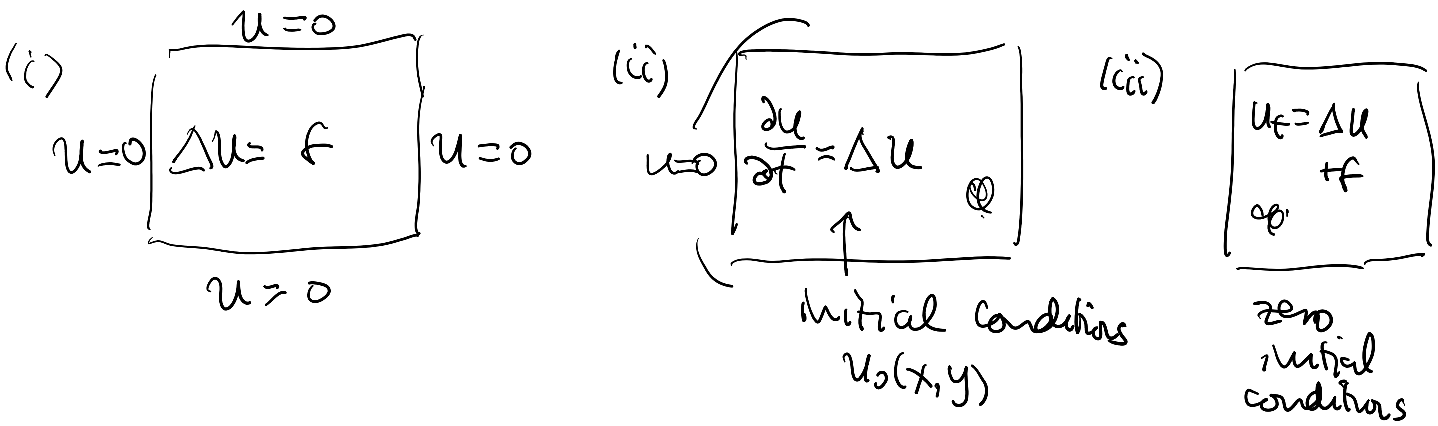
Laplace Problem.
needs other techniques.



$$\varphi_{nm} = \sin n\pi x \sin m\pi y$$

$$\lambda_{nm} = -(n^2+m^2)\pi^2$$

(Check these work).



(i) $$u = \sum_{n=1}\sum_{m=1} \frac{-f_{nm}}{(n^2+m^2)\pi^2} \sin n\pi x \sin m\pi y$$

$$f_{nm} = 4\underbrace{\int_0^1\int_0^1 f(x,y)}\sin n\pi x \sin m\pi y \, dx\,dy.$$

(ii) $$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} u_{0,n,m} e^{-(n^2+m^2)\pi^2 t} \sin n\pi x \sin m\pi y$$

$n=m=1$ term decays the slowest.

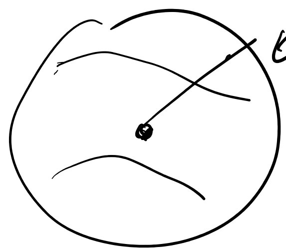
(iii) $$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \underline{u_{nm}(t)} \sin n\pi x \sin m\pi y.$$

$$f_{nm}(t) = 4\int_0^1\int_0^1 f(x,y,t)\sin n\pi x \sin m\pi y \, dx\,dy.$$

$$u_{nm}(t) = \int_0^t e^{-(n^2+m^2)\pi^2(t-s)} f_{nm}(s)\,ds.$$

Ex Heat conduction in a unit sphere, assuming radial symmetry.

$u(r,t)$.



$$\Delta u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right).$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \varphi}{\partial r}\right) = \lambda\varphi. \qquad \varphi(1)=0.$$

"Hidden" condition at $r=0$.

new variable $y = r\varphi$.

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial (y/r)}{\partial r}\right) = \lambda r y$$

$$\frac{\partial}{\partial r}\left(r^2 \left(\frac{\partial y/\partial r}{r} - \frac{y}{r^2} \right)\right) = \lambda r y.$$

$\downarrow$ simplification.

$$\frac{\partial^2 y}{\partial r^2} = \lambda y.$$

$\varphi(1) = 0 \Rightarrow y(1) = 0.$

$\varphi(r) = \dfrac{y(r)}{r}$ since $\varphi(0)$ is bounded, $y(0) = 0.$

the "hidden" condition.

$y_n(r) = \sin n\pi r \qquad \lambda_n = -n^2\pi^2.$

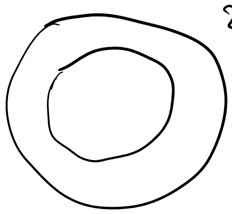
$\varphi_n(r) = \dfrac{1}{r}\sin n\pi r.$

Note: $y_n(r)$ are orthogonal "as usual",

$\varphi_n(r)$ are orthogonal in a weighted inner product.

$$\int_0^1 r^2\, \varphi_n(r)\, \varphi_m(r)\, dr = 0 \quad \text{if } n \neq m.$$

Ex



$u=0$

2D unit circle.

$\boxed{\Delta \varphi = \lambda \varphi.}$ $\qquad \varphi(r,\theta).$

$\varphi(1,\theta) = 0$

$$\Delta \varphi = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial \varphi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \varphi}{\partial \theta^2}.$$

Fourier series in θ.

Convenient to use the complex form of the Fourier series.

$$e^{in\theta} = \cos n\theta + i \sin n\theta.$$

$$e^{-in\theta} = \cos n\theta - i \sin n\theta.$$

$e^{-in\theta} = \cos n\theta - i \sin\theta.$

$$f(x) = \sum_{n=-\infty}^{\infty} f_n e^{inx}$$

with $f_n = \frac{1}{2\pi}\int_0^{2\pi} f(x) e^{-inx}\,dx$

complex, f_n and f_{-n} conjugates

Look for eigenfunctions of the form

$$\varphi(r,\theta) = \boxed{\psi(r)\, e^{in\theta}} \qquad \frac{d^2}{d\theta^2}\left(e^{in\theta}\right) = -n^2 e^{in\theta}$$

$i^2 = -1.$

$$r(r\psi')' - \lambda r^2 \psi = n^2 \psi.$$

$\frac{d\psi}{dr}$

$\psi(1) = 0$

$\psi(0)$ bounded.

No surprise, all eigenvalues will be negative

$\lambda = -\mu^2.$

$$r(r\psi')' + (\mu^2 r^2 - n^2)\psi = 0.$$

Change of variables. $x = r\mu.$ $\quad \frac{d\psi}{dr} = \mu \frac{d\psi}{dx}$

$$\boxed{x\frac{d}{dx}\left(x\frac{d\psi}{dx}\right) + (x^2 - n^2)\psi = 0}$$

$\psi\big|_{x=\mu} = 0$ $\quad \boxed{\psi(0) \text{ bounded.}}$

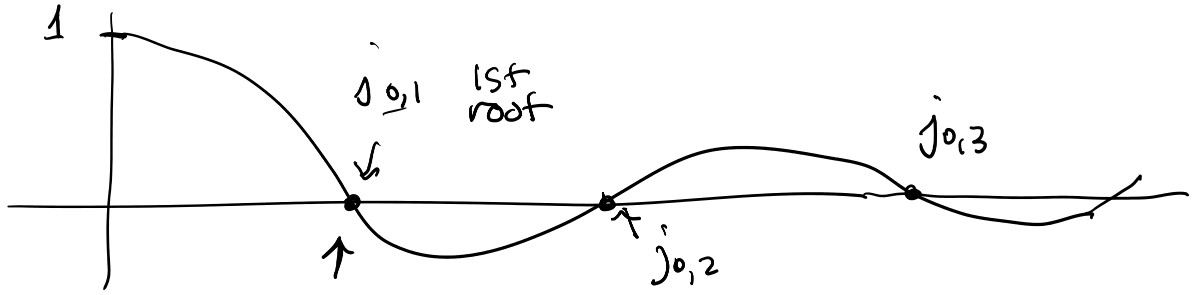
$$\psi(x) = A J_n(x)$$

Bessel function of order n.

Graph of $J_0(x)$:

Bessel function of order n.

Graph of $J_0(x)$:



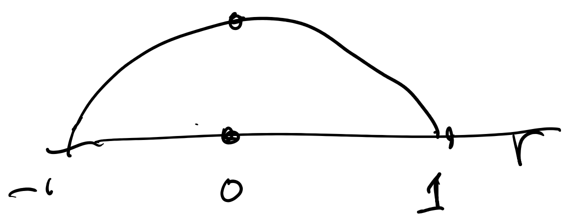
are in tables, or look up online.
You can also look up $j_{n,m}$.

$j_{0,1} \approx 2.4048$

most interested in this one - smallest magnitude.

Corresponding eigenfunction.

$\varphi(r,\theta) = J_0(r\mu)$



n=0 mode, constant in θ.

Just one eigenfunction.

$\varphi_{nm} = e^{in\theta} J_n(r j_{n,m})$

can look up in tables.

Can solve the same 3 problems with series solutions with these eigenfunctions.

Lecture #9

February 9, 2021 9:21 AM

We have looked at the Heat Equation.

$$u_t = \Delta u + f(\underline{x}, t).$$

$$u(\underline{x}, 0) = u_0(\underline{x})$$

and homogeneous BC's.

Parabolic problem.

arises as a steady state of the Heat Equation.

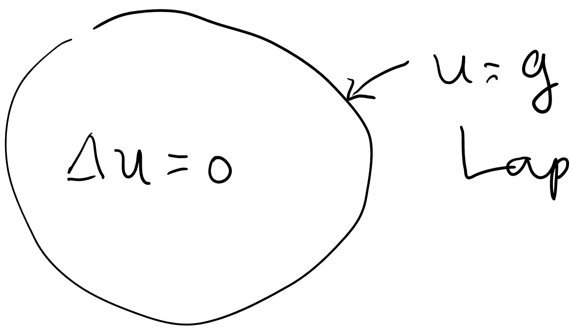
and at the Poisson problem.

$$\Delta u = f(\underline{x})$$

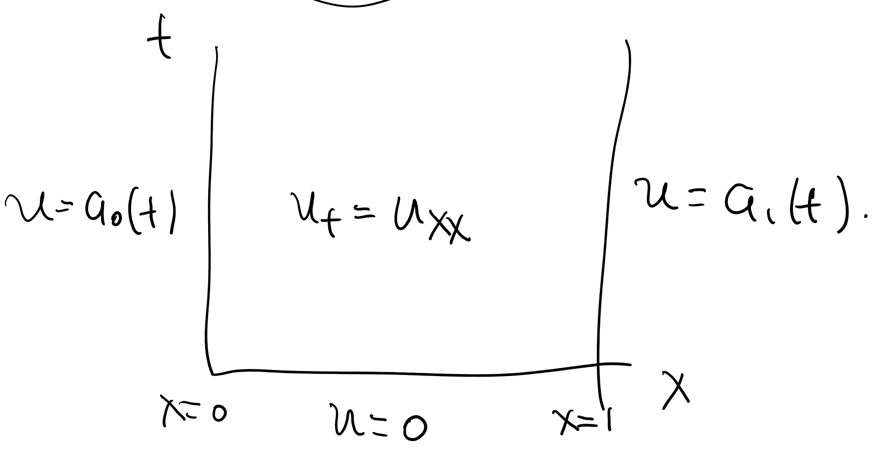
we have used series solutions.

To finish off this discussion

Elliptic problem.



Laplace Problem.

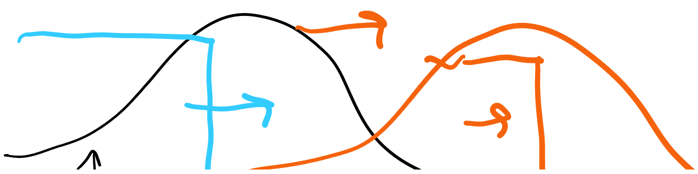


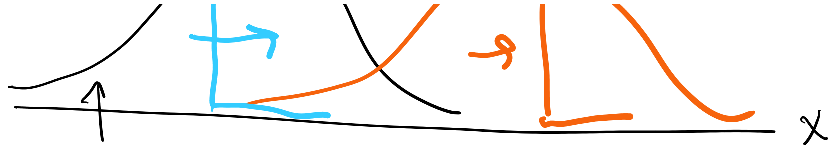
We will come back to these later in the course. (Transform methods).

Hyperbolic Problems. (wave type behaviour).

Ex One way wave equation.

wave speed 1 to the right without changing shape.





$$u(x,t) = \boxed{u_0(x-t)}$$

What PDE gives this solution?

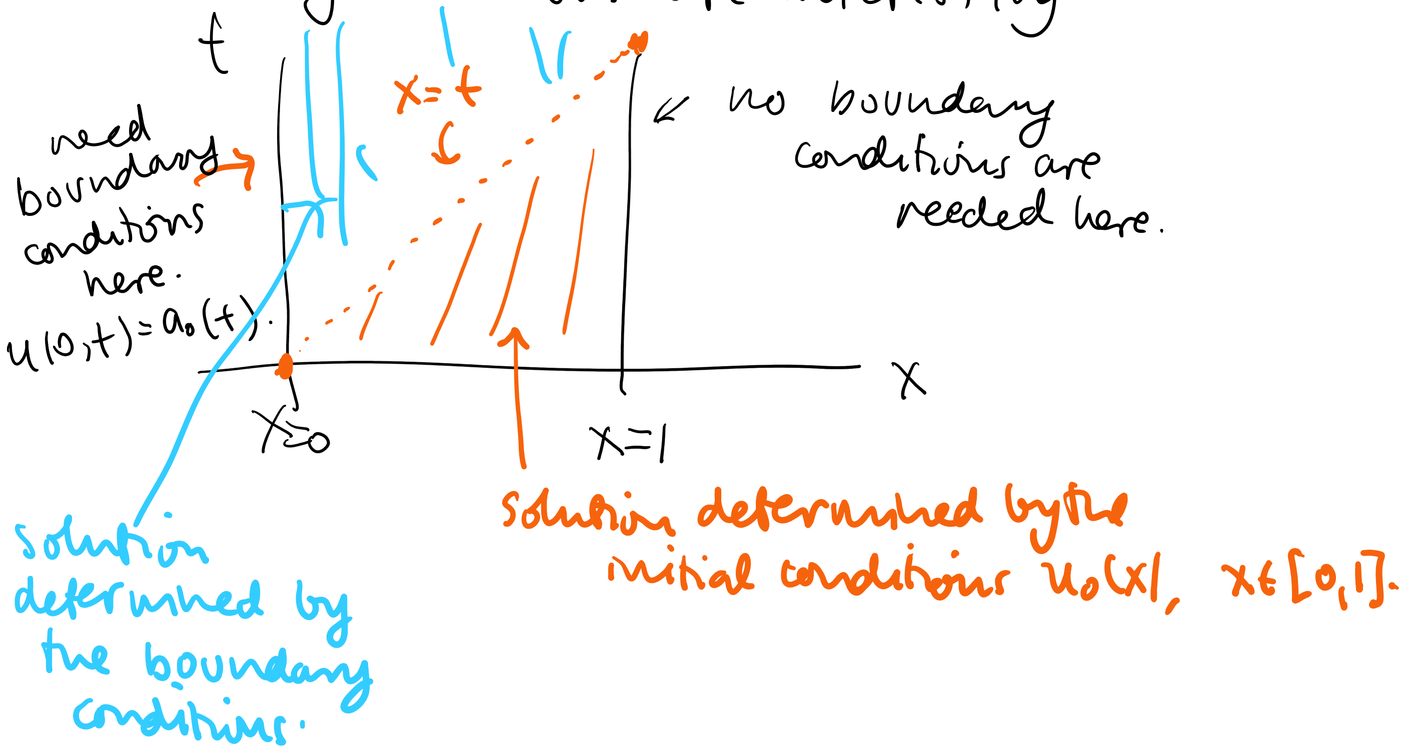
$$u_t = -u_0'(x-t)$$

$$u_x = +u_0'(x-t)$$

so $\quad u_t + u_x = 0 \quad$ one way wave equation. (W1)

~~No~~ Natural way to define weak solutions.

Boundary conditions are "interesting"



Let's forget about BC's for now.

Consider

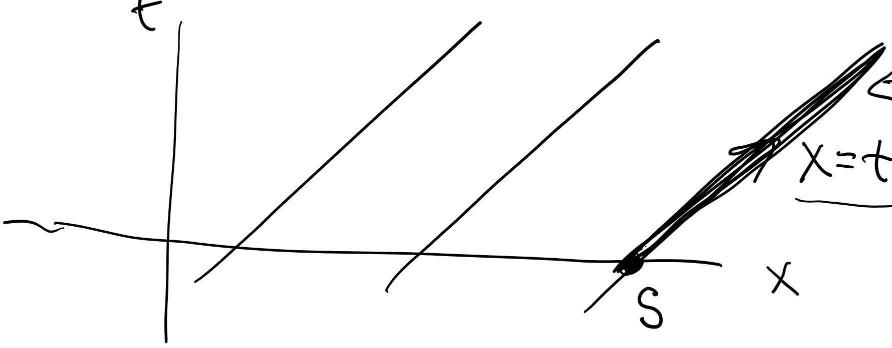
$$u_t + u_x = 0 \qquad \text{with } u(x,0) = u_0(x)$$

defined for all x. (no BC's).

with $\underline{u(x,0)} = u_0(x)$ (no BC's).

Called a Cauchy problem.

W1 $u_t + u_x = 0$ Solutions $u(x,t) = u_0(x-t)$.



← lines slope +1.

$\underline{x = t+s}$. ← $\underline{x(t,s)}$.

$x(t)$ givens.

Change coordinates $(x,t) \Rightarrow (s,t)$. $u(s,t) = u_0(s)$.

$\underline{u_t + c(x,t)\, u_x = 0}$ $c(\cdot,\cdot)$ given ✓.

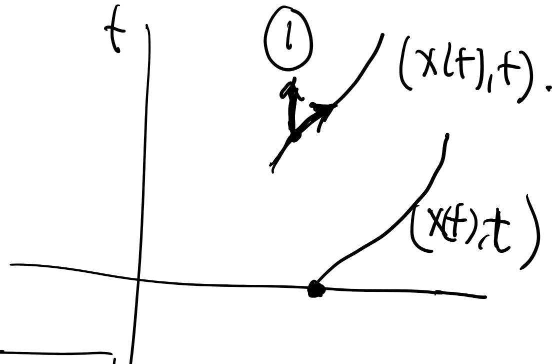
↑ Varying speed of propagation.

Curves $x(t)$ with $x(0) = s$.

Consider $\underline{u(x(t),t)}$ ↓

(2) $\dfrac{du}{dt} = \dfrac{dx}{dt}\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial t}$

↑ not $\dfrac{\partial u}{\partial t}$. (1)



Consider curves where $\boxed{\dfrac{dx}{dt} = c(x,t)}$ ← solvable ODE $x(0) = s$.

Then $\dfrac{du}{dt} = \underline{\left| c(x,t)\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial t} \right|} = 0$ ↖ characteristic lines.

by the PDE

urves.

by the PDE

$$\boxed{u(X(t),t) = u_0(s)}$$

$u(X(t,s),t) = u_0(s)$ ← form that comes from this technique an implicit form.

⇓

Conversion to $u(x,t)$.
sometimes possible to do analytically (sometimes not).

Ex $\quad u_t + \boxed{\cos t}\, u_x = 0.$

Cauchy problem IC's $\quad u(x,0) = \boxed{u_0(x)}.$

$$\frac{dx}{dt} = \cos t \qquad x(0) = s.$$

integrate $\quad x = \sin t + s. \leftarrow \quad s = x - \sin t.$

by construction, u will be constant on these curves

$$u(\sin t + s,\ t) = u_0(s).$$

$$u(x,t) = u_0(x - \sin t).$$

u is constant on these curves (characteristics).

Ex $\quad u_t + \left(\frac{1}{\ldots}\right) u_x = 0$

Ex $\quad u_t + \left(\frac{1}{1+x^2}\right) u_x = 0$

characteristics $X(t)$ satisfy $\quad | \; X(0) = s.$

$$\frac{dx}{dt} = \frac{1}{1+x^2}$$

separable equation, autonomous equation
(RHS does not depend on t).

mnemonic " $dx\,(1+x^2) = dt$ "

integrate both sides. $\quad$ constant work out at $t=0$.

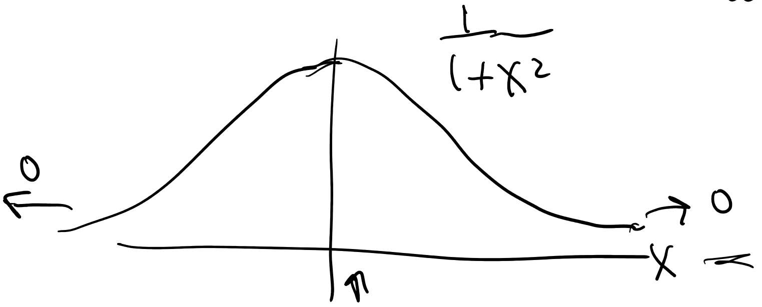
$$x + \frac{x^3}{3} = t + s + \frac{s^3}{3}.$$

determines $s(x,t)$ uniquely — would have to solve a cubic equation, with the right root.

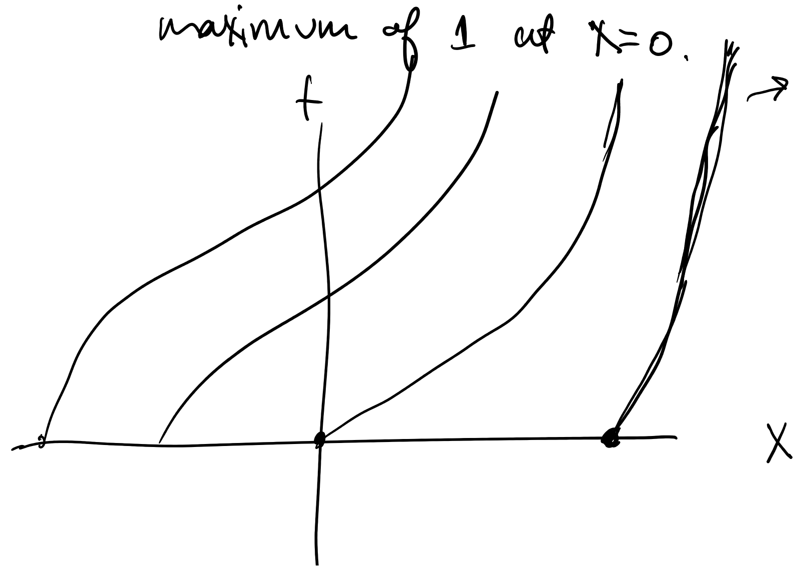
$$u(x,t) = u_0(s(x,t))$$

with that root.

at $t=0$
$s=x$
the right root is the one that continues smoothly in time from this one.



maximum of 1 at $x=0$.



$$\underline{u_t + c(x,t)\,u_x = 0.}$$ ← Has a maximum principle.

$$u(x,t) \le \sup_x u_0(x)$$

Ex $\quad u_t + u_x = -u + xt$ ← still linear in u

Cauchy problem $u_0(x)$ given.

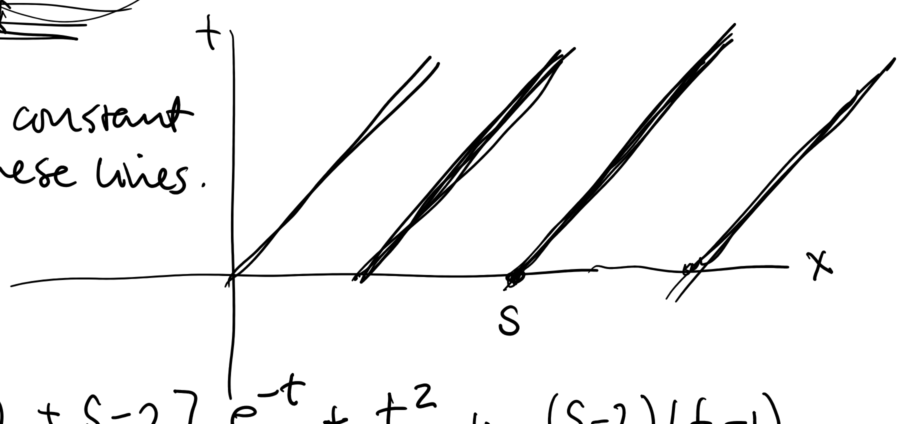
$$\frac{dx}{dt} = 1 \Rightarrow \boxed{x = t + s.} \qquad s = x - t.$$

$$\frac{du}{dt} = \frac{d}{dt}\,u(x(t),t) = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial t} = -u + xt.$$

PDE

$$\frac{du}{dt} = -u + (t+s)t.$$ along characteristics.

s is constant on these lines.



$$u(t) = [u_0(s) + s - 2]\,e^{-t} + t^2 + (s-2)(t-1).$$

$$u(x,t) = [\,u_0(x-t) + x - t - 2\,]\,e^{-t} + t^2 + (x-t-2).$$

$$\underline{\frac{du}{dt} + a(t)u = b(t)}$$

General Theory

First order equations with a time-like variable.

$$u_t + \underset{\uparrow \atop \text{speed}}{a(x,t)}\, u_x = b(x,t)\, u + f(x,t).$$

① characteristic curves. $x(t)$.

$$\frac{dx}{dt} = a(x(t),t) \qquad x(0) = s.$$

can be solved for $x(s,t)$.

② Consider $u(t) = u(x(t),t)$.

$$\boxed{\frac{du}{dt} = u_x \underbrace{\frac{dx}{dt}}_{a(x(t),t)} + u_t = u_t + a\, u_x = b(x(t),t)\, u(t) + f(x(t),t)}$$

PDE

$u(0) = u_0(s)$.

$t=0$ on characteristic $(x(t),t)$.

ODE solvable for $\underline{u(s,t)}$.

③ (if possible) solve $x(s,t)$ for $s(x,t)$ then

$$u(x,t) = u(s(x,t),t).$$

$$\underline{u_t + u_x = 0} \qquad \boxed{u(x,t) = u_0(x-t)}$$

Green's function for this problem.

Last time. $u(x,t)$ for all $x \in (-\infty, \infty)$

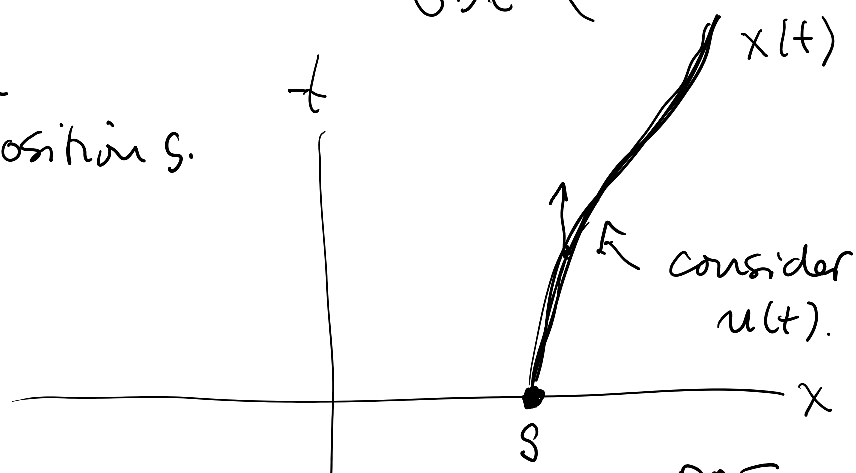
Cauchy problem $u(x,0) = u_0(x)$ given.

$$u_t + \underbrace{a(x,t)}_{\text{given local wave speed.}} u_x = \underbrace{b(x,t)}_{\text{given coefficient.}} u + \underbrace{f(x,t)}_{\text{given forcing.}}$$

$X(t)$ ← characteristic curves starting from initial position s.

$$\underline{\frac{dx}{dt} = a(x,t)} \qquad X(0) = s.$$

ODE



PDE

$$\frac{du}{dt} = \frac{d}{dt}\left(u(x(t),t)\right) = u_t + a(x(t),t)\,u_x = b(x(t),t)\,u(t) + f(x(t),t)$$

$$\dot{u} = b(x(t),t)\,u + f(x(t),t) \qquad u(0) = u_0(s)$$

known coefficients

linear problem for $u(t)$ ✓

$\underline{u(s,t)}$ implicit form of the solution.

Generalization $u(x,y)$.

$$a(x,y)\, u_x + \underline{c(x,y)}\, u_y = b(x,y)\, u + f(x,y)$$

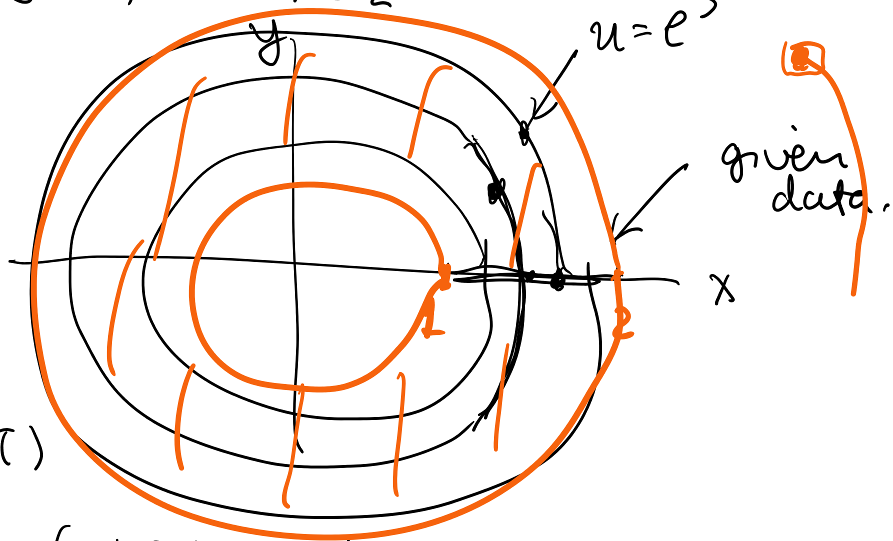
If $a(x,y) > 0$ for all x & y, divide by a and have same problem as above with x a time-like direction.

Characteristics $x(\tau), y(\tau)$, τ parameter

$\tau = 0$ curve where data is given.

<u>Ex</u> Solve $\boxed{y\, u_x - x\, u_y = 0}$

initial data $u(x,0) = e^x$ on the line segment $y = 0$, $1 \le x \le 2$.



$x(\tau)$, $y(\tau)$

$u(\tau) = u(x(\tau), y(\tau))$

$$\frac{dx}{d\tau} = y(\tau)$$

$$\frac{dy}{d\tau} = -x(\tau).$$

$$\frac{du}{d\tau} = \underline{u_x \frac{dx}{d\tau} + u_y \frac{dy}{d\tau}}$$

$\checkmark$ PDE

$= 0$

So u will be constant on these characteristic

So u will be constant on these characteristic curves.

$$\frac{dx}{d\tau} = y \qquad \frac{d^2x}{d\tau^2} = \frac{dy}{d\tau} = -x.$$

$$x = A\sin\tau + B\cos\tau. \qquad \& \; x(0) = s.$$

$$\frac{dx}{d\tau}(0) = y(0) = 0$$

$$x(\tau) = s\cos\tau$$
$$y(\tau) = s\sin\tau.$$

$$u(s,\tau) = e^{s} \qquad s = \sqrt{x^2+y^2}$$
$$= e^{\sqrt{x^2+y^2}}.$$

Convert PDE to polar coordinates, $u_\theta = 0.$

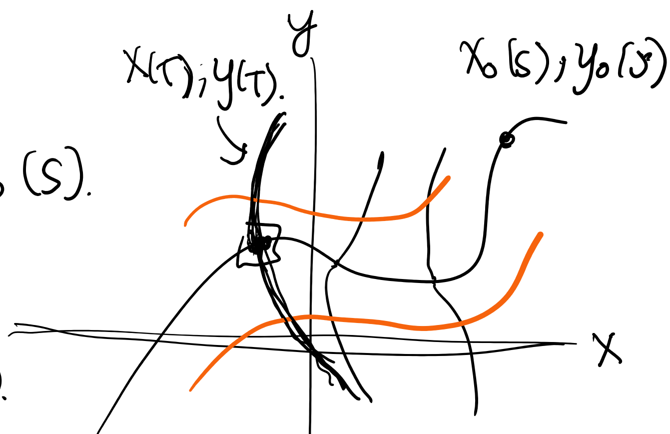
General Theory

$$a(x,y)u_x + c(x,y)u_y = b(x,y)u + f(x,y).$$

Initial data on a curve $x_0(s), y_0(s)$

$u_0(s).$

$$u(x_0(s), y_0(s)) = u_0(s).$$



For each s, there is a characteristic $x(\tau), y(\tau)$.

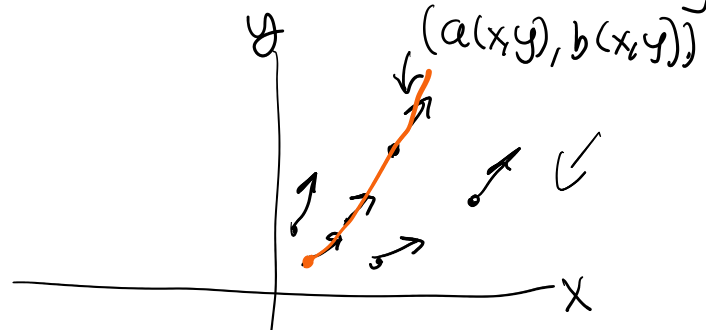
$$\frac{dx}{d\tau} = a(x,y) \qquad \frac{dy}{d\tau} = c(x,y)$$

$$x(0) = x_0(s) \qquad y(0) = y_0(s)$$

Autonomous system in the plane.

$y \qquad .(a(x,y), b(x,y))$

$y_0(s)$]] plane.

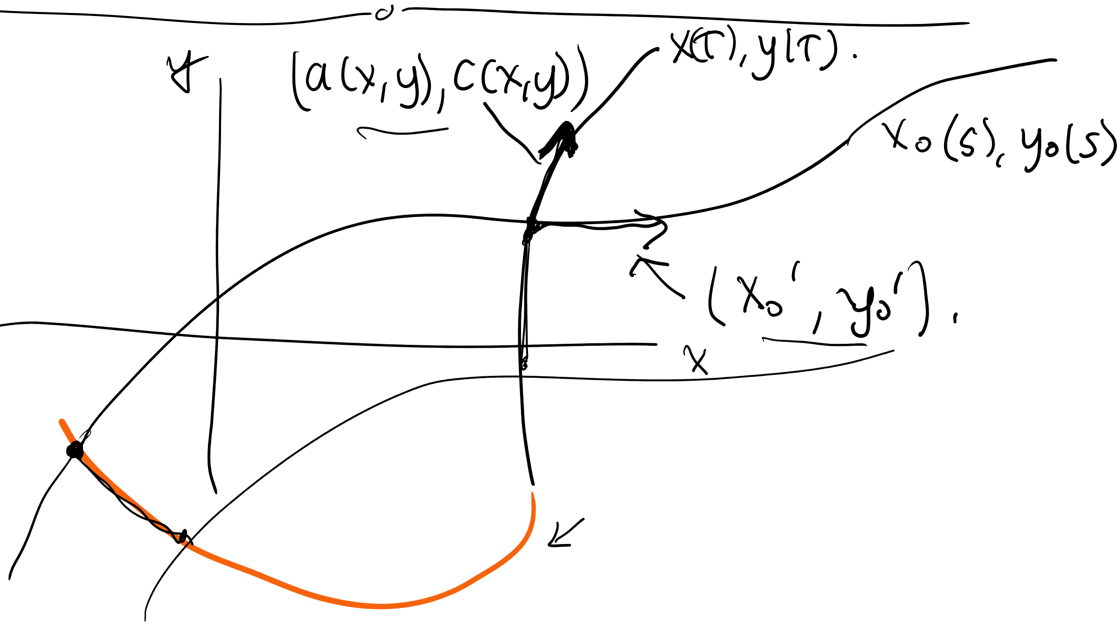


$\boxed{X(\tau), y(\tau)}$ solved ✓

on these curves $u(t) = u(X(\tau), y(\tau))$

$$\frac{du}{d\tau} = u_x \frac{dx}{d\tau} + u_y \frac{dy}{d\tau} \underset{\text{PDE}}{=} \boxed{b(x(\tau), y(\tau))}\, u(\tau) + \underline{f(x(\tau), y(\tau))}$$

↑ linear scalar problem $u(t)$ ↑ $u(0) = u_0(s)$.



Those two directions cannot be parallel.

$$\det \begin{vmatrix} a(x_0(s), y_0(s)) & c(x_0(s), y_0(s)) \\ \frac{dx_0}{ds} & \frac{dy_0}{ds} \end{vmatrix} \neq 0.$$

$a y_0' - c x_0' \neq 0$ for all s

$a y_0' - c x_0' \neq 0$ for any s.

If true, solution is defined at least locally (in a neighborhood of the data curve (x_0, y_0)).

Nonlinear problems.

go back to a time-like direction.

$$\underline{u_t} + g(u,x,t)\, u_x = \underline{h(u,x,t)}$$

quasi-linear problems.

nonlinear in u but not u_x.

Ex Inviscid Burger's Equation

$$u_t + \left(\tfrac{1}{2}u^2\right)_x = 0$$

written in conservation form.

$$u_t + u u_x = 0$$

Toy model of 1D gas dynamics

3 component model

density ρ

velocity u

Thermodynamic Variable S (entropy),

or temperature
or internal energy.

More physical model - includes viscosity

$$u_t + \boxed{\left(\tfrac{1}{2}u^2\right)_x} = \nu u_{xx}$$

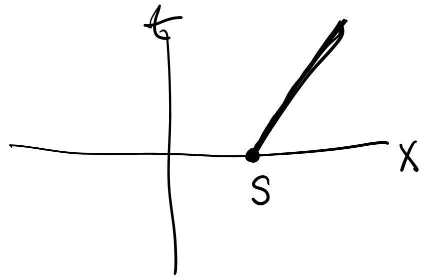
very small.

...y small.

Cauchy problem for $\underline{u_t + u u_x = 0}$.

$X(t)$ characteristics.

$u(X(t), t)$ along characteristics as before.



$$\frac{du}{dt} = \underbrace{u_x \frac{dx}{dt}}_{\frac{dx}{dt} = u} + u_t = 0$$

PDE

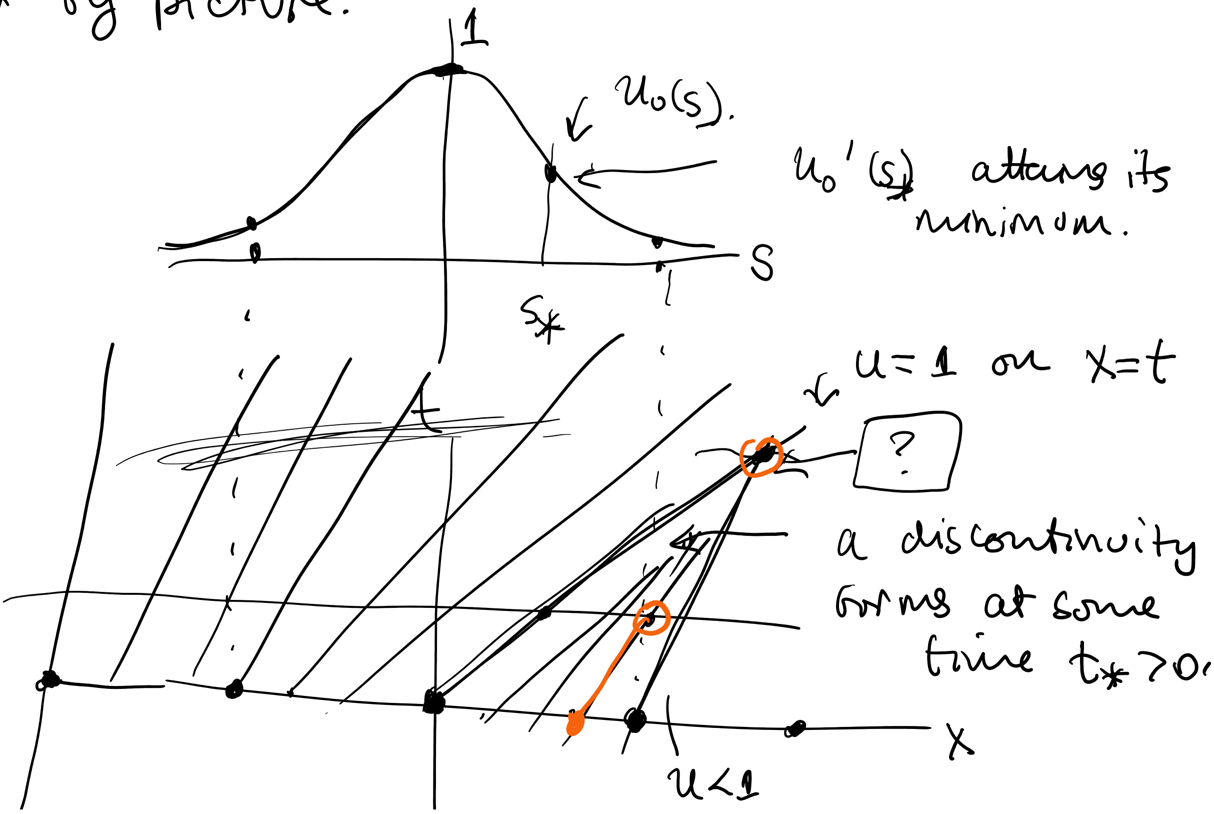
$\frac{dx}{dt} = u$ ✓

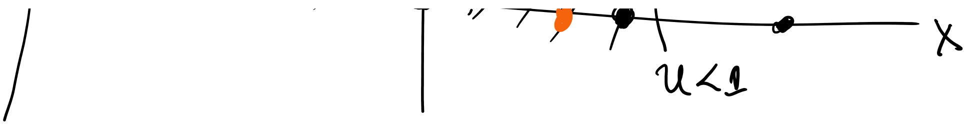
so $u(t) = u_0(s)$.

$\frac{dx}{dt} = u_0(s)$ $\qquad$ $x = u_0(s)t + s$.

$u(\underline{u_0(s)t} + s, t) = u_0(s)$ ← implicit form of the solution.

Ex by picture.





Q: At what time t_* does the shock form.

For $t < t_*$ the solution is well defined,
↑ continuous (smooth).

At time t_*, two s values lead to the same x value. Therefore, the map $s \to x$ at t_* is no longer invertible. Look for values of t and s where $\frac{dx}{ds} = 0$.

$$t^* = \min_t \left\{ t : \frac{\partial x}{\partial s}(t,s) = 0 \text{ for some } s \right\}.$$

Burger's Equation $\quad x = u_0(s)t + s.$

$$\frac{\partial x}{\partial s} = \underline{1 + u_0'(s)t}.$$

from picture above.

Let $M = \min_s u_0'(s) = \underline{u_0'(s_*)}$

If $M < 0$ a shock will form and

$$t_* = -\frac{1}{M}.$$

$$u_0(x) = e^{-x^2}$$

$$u_0' = -2x\, e^{-x^2}$$ minimize.

$$u_0'' = (4x^2 - 2)\, e^{-x^2}$$

$u_0'' = 0$ when $x = \pm \frac{1}{\sqrt{2}}$.

(minimum $x = + 1/\sqrt{2}$).

(minimum $x = +1/\sqrt{2}$).

$$M = u_0'\left(\frac{1}{\sqrt{2}}\right) = -\sqrt{2}\, e^{-1/2}.$$

So a shock will form at $t_* = -\frac{1}{M} = \frac{1}{\sqrt{2}} e^{1/2}$

Next time: look at how to continue the solution after t_*.

will have a discontinuity → weak solutions.

introduce viscosity solutions
use conservation.

$$\boxed{u_t + (f(u))_x = 0.}$$

Lecture #11

February 23, 2021 9:17 AM

$u(x,t)$

$u_t + \left(\tfrac{1}{2}u^2\right)_x = 0$ smooth solutions

$u_t + u\,u_x = 0.$

Cauchy problem all x

initial conditions $u_0(x)$ given.

solution on characteristics. $x(t)$.

$u(t) = u(x(t),t).$

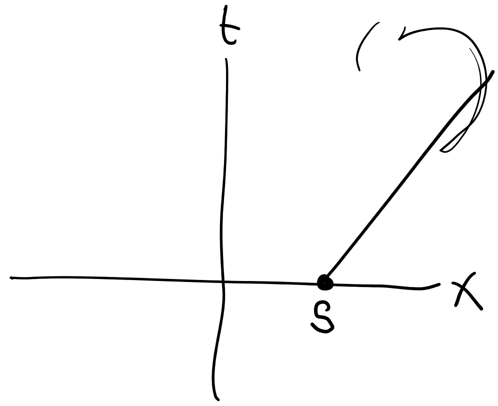
$$\frac{du}{dt} = \frac{dx}{dt}\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t}$$

[true for any curve $x(t)$].

$$\frac{dx}{dt} = u(t)$$

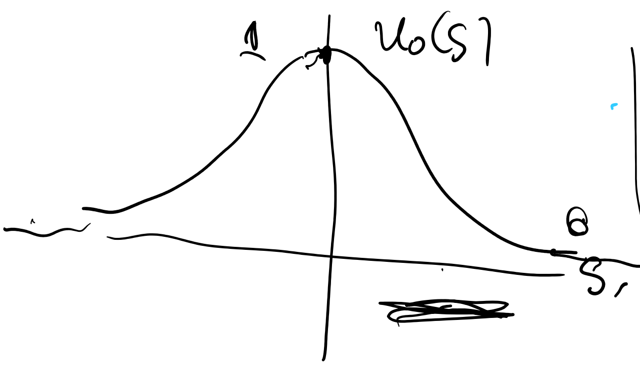
$$\frac{du}{dt} = 0 \quad u \text{ is constant}$$

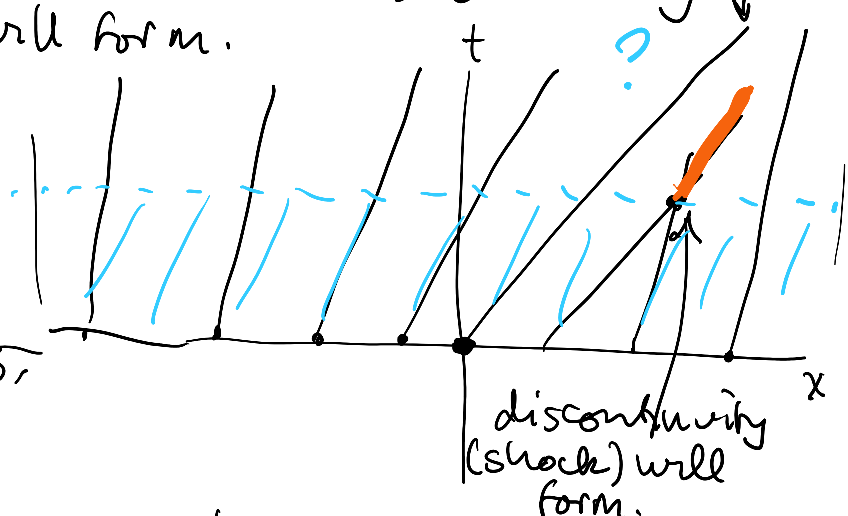
by the PDE.



$$\boxed{\begin{aligned} x(t) &= u_0(S)\,t + S \\ u(u_0(S)\,t + S, t) &= u_0(S). \end{aligned}}$$

last time: indentified when a discontinuity will form.





$\boxed{u_t + (f(u))_x = 0}$ $\quad f(u) = \frac{1}{2}u^2$

(shock) weak form.

$$\boxed{u_t + (\underbrace{f(u)}_{})_x = 0} \qquad f(u) = \frac{1}{2}u^2$$

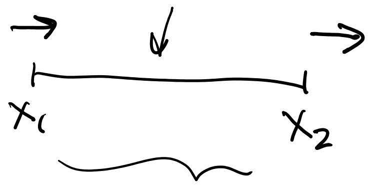
for Burger's Equation

conservation form.

interperet as a flux of u.

(positive to the right, negative to the left).

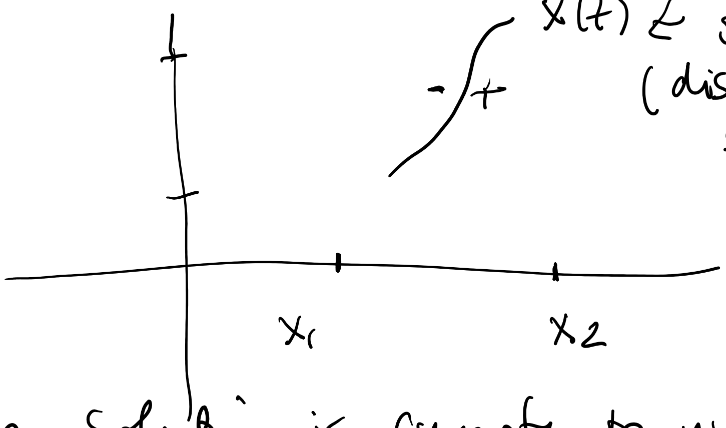
$$M(t) = \int_{x_1}^{x_2} u(x,t)\,dx.$$



amount of u in an interval $[x_1, x_2]$

$$\boxed{\frac{dM}{dt} = \int_{x_1}^{x_2} u_t(x,t)\,dx} = -\int_{x_1}^{x_2} (f(u))_x\,dx$$

$$= -f(u(x_2,t)) + f(u(x_1,t)). \quad (*)$$



$X(t) \leftarrow$ shock curve.

(discontinuous solution across the curve).

Assume solution is smooth to right & left of the shock.

$$M(t) = \int_{x_1}^{x_2} u(x,t)\,dx = \int_{x_1}^{X(t)} u(x,t)\,dx + \int_{X(t)}^{x_2} u(x,t)\,dx$$

$$\frac{dM}{dt} = \int^{X(t)} u_t\,dx + \frac{dX}{dt} u(X(t)_-, t)$$

$$\frac{d}{dt} = \int_{x_1} u_t \, dx + \underbrace{\frac{dx}{dt}}_{\text{shock speed } s} u(x(t)_-, t)$$

$$+ \int_{x(t)}^{x_2} u_t \, dx - \underbrace{\frac{dx}{dt}}_{s} u(x(t)_+, t)$$

$$= -\int_{x_1}^{x(t)} (f(u))_x dx + \underline{s\, u_-} - \int_{x(t)}^{x_2} (f(u))_x dx$$

$$\underline{- s\, u_+}$$

$$= f(u(x_1, t)) - f(u(x(t)_-, t)) + s u_-$$

$$- f(u(x_2, t)) + f(u(x(t)_+, t)) - s u_+.$$

Conservation implies these four terms must sum to zero.

$$s(u_- - u_+) + f(u_+) - f(u_-) = 0.$$

Notation: $[u] = u_+ - u_-$ $\quad [f] = f(u_+) - f(u_-)$

jump in u.

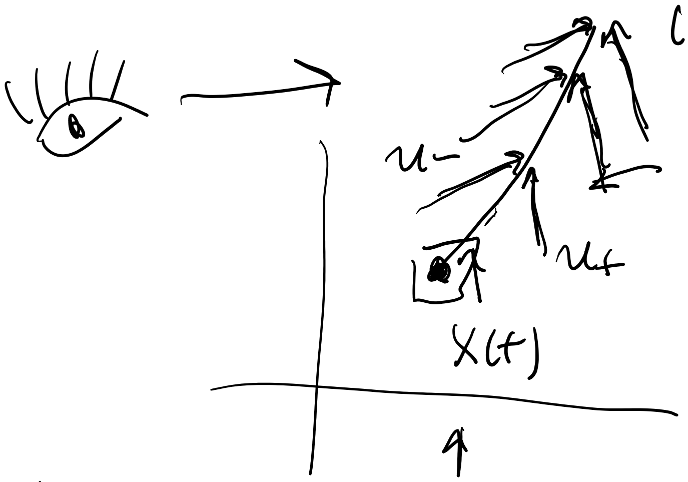
$$s = \frac{[f]}{[u]}.$$ Rankine - Hugoniot condition (RH).

Consider Burger's Equation. $f(u) = \frac{1}{2} u^2$.

$$s = \frac{\frac{1}{2}(u_+^2 - u_-^2)}{\quad} = \boxed{\frac{1}{2}(u_+ + u_-)}$$

$$s = \frac{\frac{1}{2}(u_+^2 - u_-^2)}{u_+ - u_-} = \boxed{\frac{1}{2}(u_+ + u_-)}$$

(average of the characteristic speeds).



$s \leftarrow \frac{dx}{dt} = \boxed{s = \frac{[f]}{[u]}}$ ★

Look at this picture,

$$\underbrace{f'(u_-)}_{\substack{\text{Left side} \\ \text{characteristic} \\ \text{Speed}}} > s > \underbrace{f'(u_+)}_{\substack{\text{right} \\ \text{side.}}}$$

Lax entropy condition.

For Burger's Equation $f'(u) = u$.

s for BEq.

$$u_- > \frac{1}{2}(u_+ + u_-) > u_+$$

Defn A Riemann problem is a cauchy problem for $\boxed{u_t + (f(u))_x = 0}$ with initial data.

$$u_0(x) = \begin{cases} U_L & x < 0 \\ U_R & x > 0. \end{cases}$$

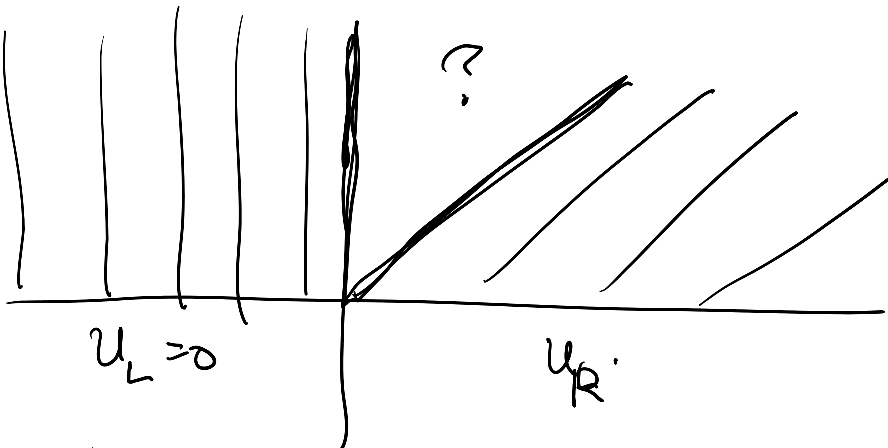
Ex Solve the RP for BEq with $U_L = 1$, $U_R = 0$

($u_L > u_R$ shock will form)

$$s = \frac{1}{2}(U_L + U_R) = 1/2$$

$$s = \frac{1}{2}(U_L + U_R) = 1/2$$

Ex 2 $U_R = 1$, $U_L = 0$ $U_R > U_L$ so a shock should not form.



Don't put a shock in? the lax entropy condition will not be satisfied.

Characteristic speed $f'(u) = u$ for B Eq.

Characteristic speed $f'(u) = u$

$u\left(\frac{x}{t}\right) = \frac{x}{t}.$

$$u(x,t) = \begin{cases} 0 & x < 0 \\ \frac{x}{t} & 0 < x/t < 1. \\ 1 & x > t. \end{cases}$$

New conservation law.

Traffic flow model.

single lane, one direction, no stop lights.

Let $\rho(x,t)$ be density of cars.

$V(x,t)$ velocity.

$0 \le \rho \le \rho_{max}$ (bumper to bumper).

$0 \le V \le V_{max}$ (speed limit).

ρV flux of cars.

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho V)}{\partial x} = 0 \qquad \checkmark$$

Make an assumption $V(\rho) \leftarrow$ model.

Example $V(\rho) = V_{max}(1 - \rho/\rho_{max})$.

$$f(\rho) = \rho V = \underline{\rho V_{max}(1 - \rho/\rho_{max})}.$$

scale and nondimensionalize.

$u_t + [f(u)]_x = 0 \qquad f(u) = u(1-u)$

$$u_t + (f(u))_x = 0 \qquad f(u) = u(1-u).$$
$$= u - u^2.$$

RH $\quad S = \dfrac{[f]}{[u]} = \dfrac{u_+ - u_+^2 - u_- + u_-^2}{u_+ - u_-} = 1 - (u_+ + u_-).$

LE $\quad f'(u_-) > S > f'(u_+) \qquad f'(u) = 1 - 2u$

$$1 - 2u_- > 1 - (u_+ + u_-) > 1 - 2u_+$$
$$u_- < \tfrac{1}{2}(u_+ + u_-) < u_+$$

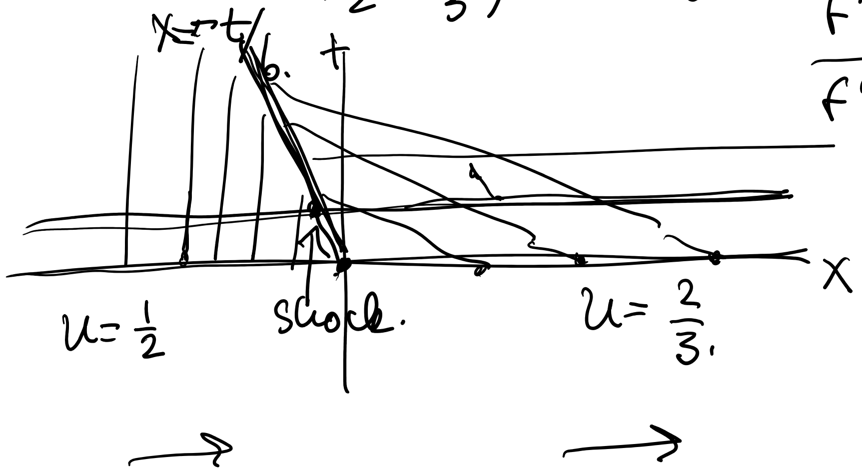
(Makes sense — traffic makes a shock only when the density ahead is less than the density behind),

<u>Ex</u> (i) $\underline{U_L = \frac{1}{2}}$, $\underline{U_R = \frac{2}{3}}$ shock will form.

Riemann problem.

$$S = 1 - \left(\tfrac{1}{2} + \tfrac{2}{3}\right) = -1/6.$$

$f'(u) = 1 - 2u$
$f'(U_L) = 0$
$f'(U_R) = -1/3.$



$$U_t + (f(u))_x = 0$$

Lecture #12

February 25, 2021 9:19 AM

Consider the RP for $u_t + (f(u))_x = 0$.

(i) $f(u)$ is convex
$(f'' > 0$ for all $u)$.

Ex BEq
$f(u) = \frac{1}{2}u^2$.

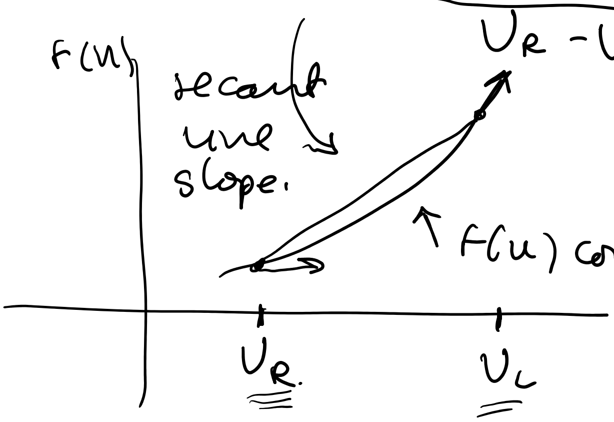
(ii) $f(u)$ is concave
$(f'' < 0$ for all $u)$

Ex traffic flow
$f(u) = u - u^2$.

f' is increasing.

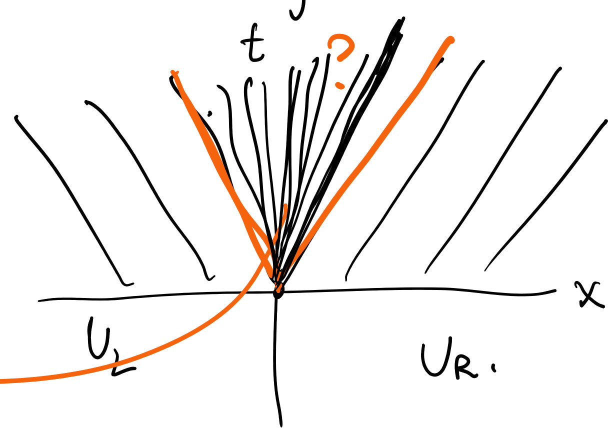
$\underline{U_L > U_R}$ shock wave.

shock speed $S = \dfrac{f(U_R) - f(U_L)}{U_R - U_L}$.



$f'(U_R) < S < f'(U_L)$ Lax Entropy condition

Stick with (i) f convex
$U_L < U_R$.
$f'(U_L) < f'(U_R)$



rarefaction wave.

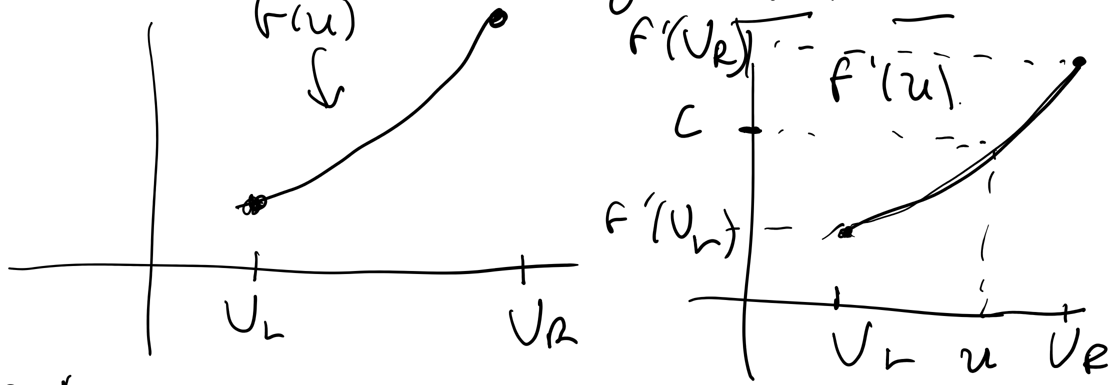
Solution constant on lines $x = \underline{c}t$.

$\underline{f'(U_L)} < c < \underline{f'(U_R)}$

Value u on has to satisfy $f'(u) = \underline{c}$.

f(u)

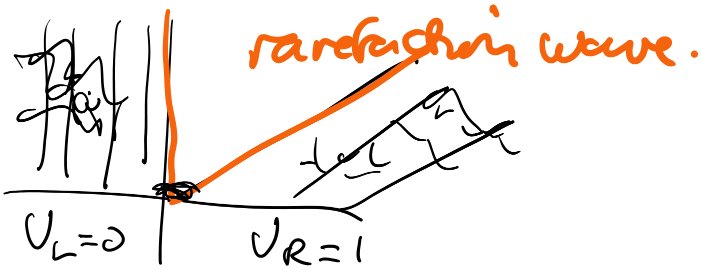
Value u on ' has to satisfy $f'(u) = c$.



$u = [f']^{-1}(c)$.

This leads to a continuous solution for $t>0$.

Ex BEq $U_L = 0$, $U_R = 1$ rarefaction.



$f'(u) = u \Rightarrow [f']^{-1}(c) = c$.

$$u(x,t) = \begin{cases} 0 & x<0 \\ x/t. & 0<x<t. \\ 1 & x>t \end{cases}$$

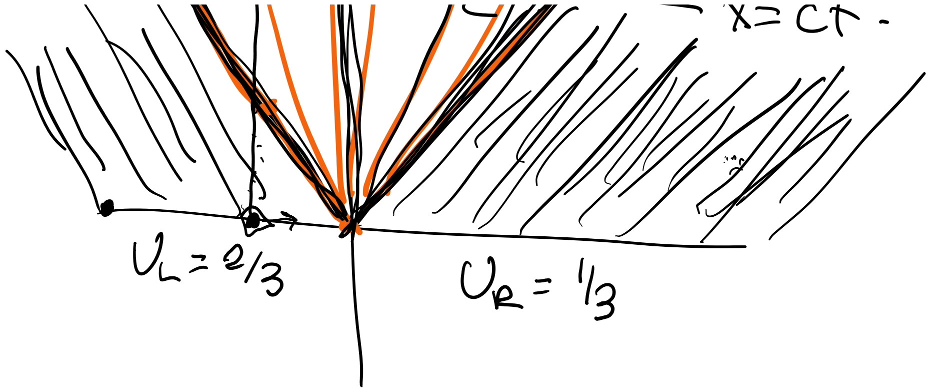
Ex Traffic flow $U_L = \frac{2}{3}$, $U_R = \frac{1}{3}$.

$f(u) = u - u^2$

$f'(u) = 1 - 2u$.

$f'(U_L) = -1/3$, $f'(U_R) = \frac{1}{3}$.

rarefaction wave.



$$f'(u) = 1 - 2u = c \quad \Rightarrow \quad u = (1-c)/2 .$$

$$u(x,t) = \begin{cases} 2/3 & x < -t/3 . \\ (1 - \frac{x}{t})/2 & -\frac{1}{3} < \frac{x}{t} < \frac{1}{3} . \\ 1/3 & x > t/3 \end{cases}$$

Fully nonlinear first order scalar equations.

Ex Eikonal Equation. $u(x,y)$.

$$u_x^2 + u_y^2 = \frac{1}{c^2(x,y)} \qquad [u_x]^2 \qquad (E)$$

Consider some examples $c \equiv 1$.

Data $u(x,0) = 0$.

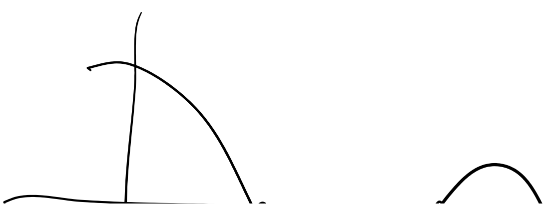
two solutions $\to u = +y$ $\qquad u_y = \pm 1 \quad u_x = 0$

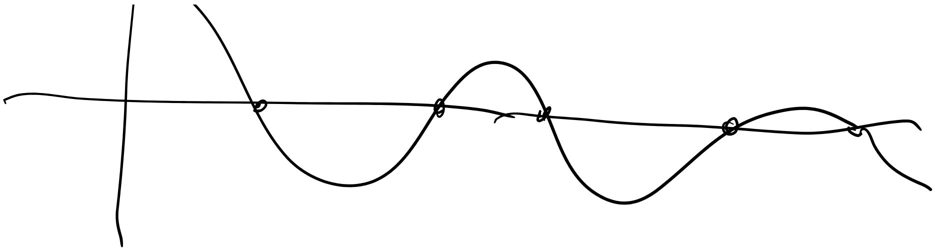
$\searrow u = -y$

Ex Data $u = 0$ on the unit circle.

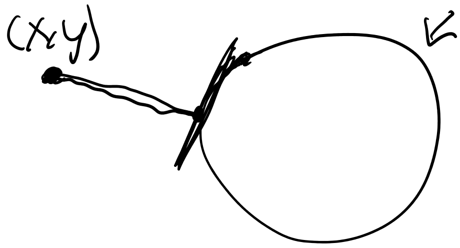
two solutions $u = \pm\left(\sqrt{x^2+y^2} - 1\right)$ $\leftarrow$ polar coordinates $\pm(r-1)$.

$$u_x = \pm \frac{x}{\sqrt{x^2+y^2}} \qquad u_y = \frac{y}{\sqrt{x^2+y^2}}$$





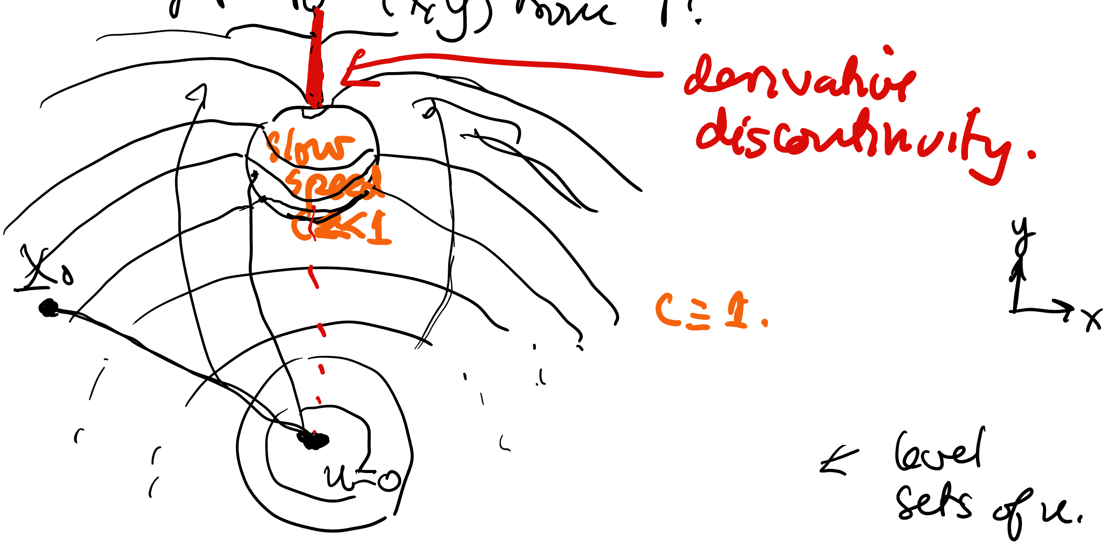
Derivation of E.



Data curve Γ $\underline{u=0}$.

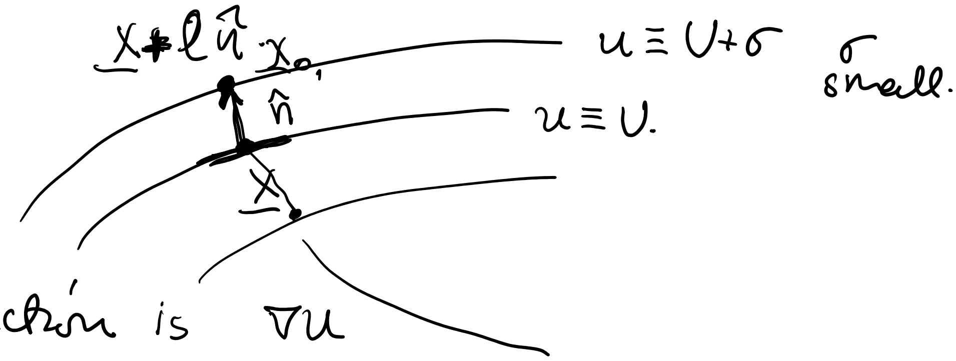
- ~~to~~ local speed $c(x,y)$.

The solution $u(x,y)$ to E is the minimal time it takes to get to (x,y) from Γ.



selection principle - shortest time level set.

(E) derive from a level set argument.



normal direction is ∇u

unit normal $\hat{n} = \dfrac{\nabla u}{|\nabla u|}$.

$c(x,y)$.

$$\sigma = u(\underline{x} + l\hat{n}) - u(\underline{x}) \approx \frac{l}{c(\underline{x})} \leftarrow c(x,y).$$

If $\underline{v}$ is small

$$u(\underline{x} + \underline{v}) - u(\underline{x}) \approx \nabla u(\underline{x}) \cdot \underline{v}.$$

Math 200 linear approx.

$$u(\underline{x} + \underbrace{l\hat{n}}_{\underline{v}}) - u(\underline{x}) = \nabla u(\underline{x}) \cdot (l\hat{n}) \approx \frac{l}{c(\underline{x})}.$$

$$\hat{n} = \frac{\nabla u}{|\nabla u|}.$$

$$\frac{\overbrace{\nabla u \cdot \nabla u}^{|\nabla u|^2}}{|\nabla u|} \cancel{l} = \frac{\cancel{l}}{c(\underline{x})}.$$

$$|\nabla u| = \frac{1}{c(\underline{x})}$$

Square

$$|\nabla u|^2 = \frac{1}{c^2(\underline{x})}$$

$$u_x^2 + u_y^2 = \frac{1}{c^2(\underline{x})}$$

Once you have found $u(x,y)$, you can find shortest time paths.

$\underline{x}(t)$ $\qquad$ $\underline{x}(0) = \underline{x_0} \leftarrow$ destination point.

$$\dot{\underline{x}} = -\nabla u(\underline{x}).$$

Nice computational method to solve for $u(x,y)$
Fast Marching Method.

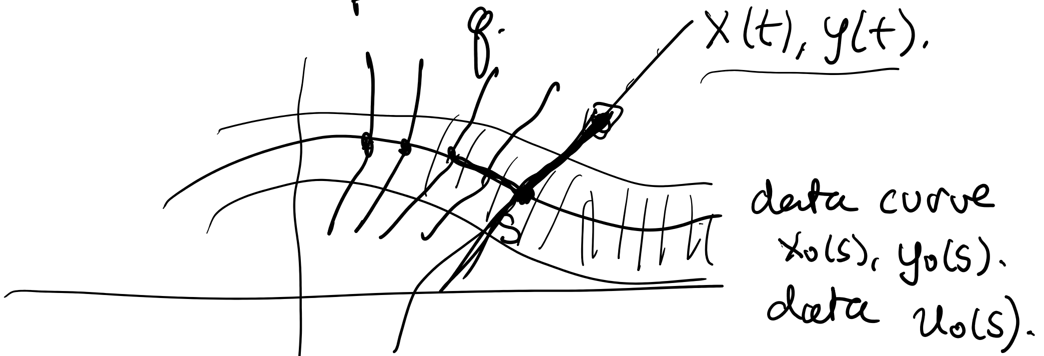
General ~~nonlin~~ nonlinear problem.

$$F(x, y, u, u_x, u_y) = 0.$$

Ex. $u_x^2 + u_y^2 - 1 = 0.$

$$\underline{F(x, y, u, \overset{p}{u_x}, \overset{q}{u_y}) = 0.}$$

Ex: $\underline{u_x^2 + u_y^2 - 1 = 0.}$



Consider
$$u(t) = u(x(t), y(t))$$
$$p(t) = u_x(x(t), y(t))$$
$$q(t) = u_y(x(t), y(t))$$

We want to replace nonlinear PDE with an ODE system for $x(t), y(t), u(t), p(t), q(t)$.

$$\dot x = F_p \quad (1) \qquad \dot y = F_q \quad (2)$$

$$\dot u = u_x \dot x + u_y \dot y = \underline{p\dot x + q\dot y}$$

$$\frac{d}{dt} u(x(t), y(t)) \qquad \dot u = pF_p + qF_q \quad (3)$$

$$\dot p = \frac{d}{dt} u_x(x(t), y(t))$$
$$= \underline{u_{xx}\dot x + u_{xy}\dot y}$$
$$\dot q = \underline{u_{xy}\dot x + u_{yy}\dot y}$$

$F(x, y, u, u_x, u_y) = 0$? Take x derivative.

Looks like I should take $\dot x = F_p$, $\dot y = F_q$

$$F_x + F_u u_x + F_p u_{xx} + F_q u_{xy} = 0$$
$$[F_y + F_u u_y + F_p u_{xy} + F_q u_{yy} = 0.$$

$$\dot p = -(F_x + F_u p) \qquad (4)$$

$y = F_y$

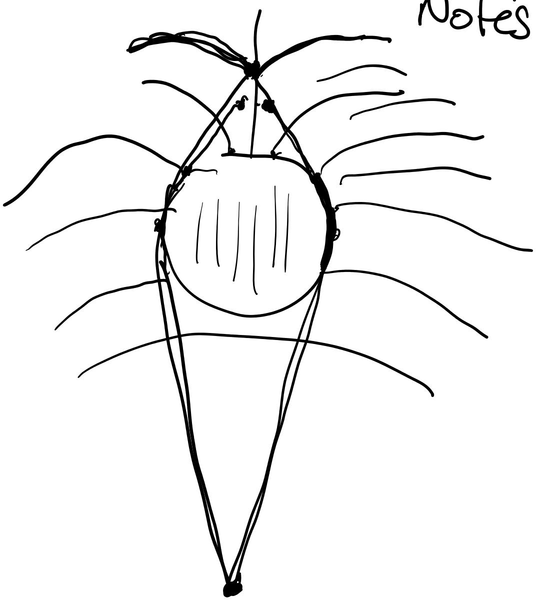
$$\dot{p} = -(F_x + F_u\, p) \qquad (4)$$

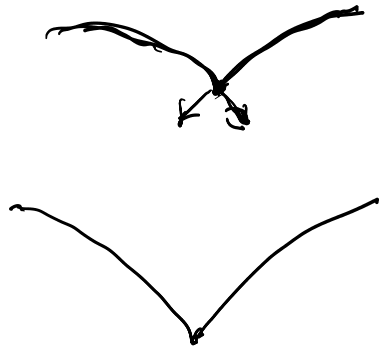
similarly $F_y=0$ $\quad \dot{q} = -(F_y + F_u\, q)$. (5)

(1)-(5) are the ODE system for $(x(t), y(t), u, p, q)$

$x_0(s), y_0(s), u_0(s)$ $\quad \boxed{p_0(s),\ q_0(s).}$

Notes. (nonuniqueness comes from these).





$$\int_0^{2\pi} e^{in\theta} e^{-im\theta} = \begin{cases} 2\pi & n=m. \\ 0 & \text{otherwise.} \end{cases}$$

$$\boxed{f(\theta) = \sum_{n=-\infty}^{\infty} c_n e^{in\theta}.}$$

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} f(x)\, e^{-in\theta}\, d\theta.$$

conjugate.

$c_n = c_{-n}^*$

n & $-n$.

$$c_n e^{in\theta} + c_n^* e^{-in\theta}$$

$a+ib + a-ib$. real result.

Lecture #13

March 2, 2021 9:19 AM

Go back to linear, constant coefficient equations.

$$\underline{u_t + a u_x = 0}$$

Cauchy Problem.

$$u(x,0) = u_0(x).$$

We wrote characteristics $x(t) = s + \underline{a}t$.

$$\frac{du}{dt} = \frac{d}{dt}\left(u(x(t),t)\right) = \dot{x}\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t}$$

$$= a\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = 0$$

← u is constant on these characteristics

$$u(\underbrace{s+at}_{x}, t) = u_0(s)$$

x , $s = x - at$.

$$u(x,t) = u_0(x-at)$$

← Character of a Green's Function.

↑ wave moving with speed a.

Let's go on to vector first order equations.

↑ linear, constant coefficient ones.

$\underline{u}(x,t)$ n components.

$$\underline{u}_t + \mathbb{A}\,\underline{u}_x = 0$$

← $\mathbb{A}$ constant in time.

↑ $n \times n$ matrix

$\underline{u}(x,0) = \underline{u}_0(x)$ Cauchy problem.

$\mathbb{A}$ symmetric, so real eigenvalues λ_j

orthonormal $\underline{v}_j$

$$\mathbb{A}\underline{v}_j = \lambda_j \underline{v}_j \qquad \|\underline{v}_j\| = 1.$$

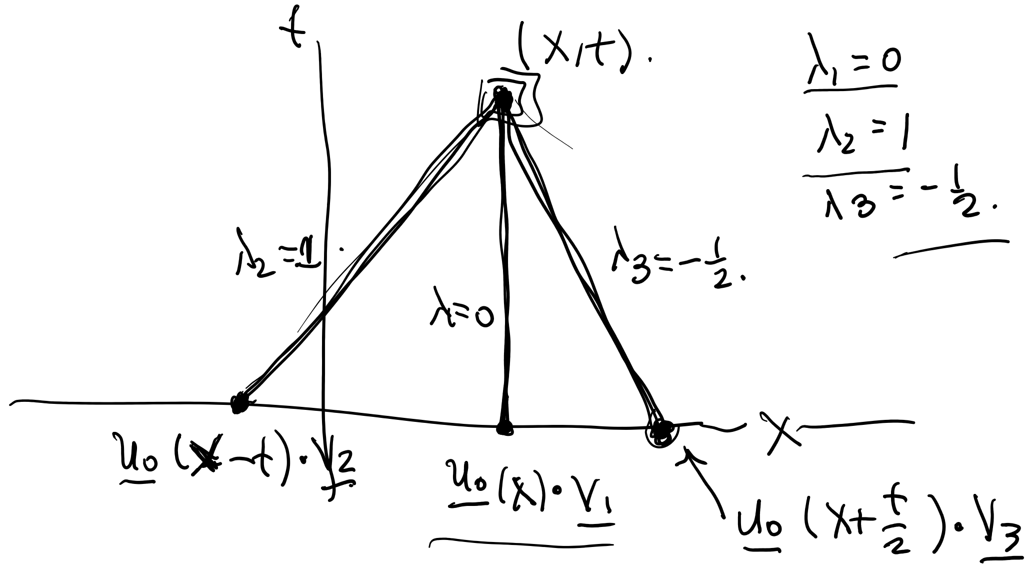
We can write $u(x,t) = \sum^{n}$ ...

We can write $\underline{u}(x,t) = \sum_{j=1}^{n} c_j(x,t)\,\underline{v}_j$ (A)

$$c_j(x,0) = \underline{u}_0(x) \cdot \underline{v}_j$$

$$\frac{\partial c_j}{\partial t} + \lambda_j \frac{\partial c_j}{\partial x} = 0$$ ← this is just our first problem today for all j.

$$c_j(x,t) = c_j(x - \lambda_j t, 0) = \underline{u}_0(x - \lambda_j t) \cdot \underline{v}_j.$$



$\lambda_1 = 0$

$\lambda_2 = 1$

$\lambda_3 = -\frac{1}{2}$.

$$\underline{u}_t + \mathbb{A}\,\underline{u}_x = \underline{f}(x,t) \qquad \underline{u}(x,0) = \underline{0}.$$

Consider now $n=2$, $\mathbb{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ case.

$$\underline{u}_t + \mathbb{A}\,\underline{u}_x = \underline{0}$$ * $\lambda_1 = 1$ $\lambda_2 = -1$

$\underline{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\underline{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

$\underline{u} = \begin{bmatrix} u \\ v \end{bmatrix}$

$$u_t + v_x = 0 \overset{\partial/\partial t}{\Longrightarrow} u_{tt} + v_{xt} = 0.$$

$$v_t + u_x = 0 \overset{\partial/\partial x}{\Longrightarrow} v_{xt} + u_{xx} = 0$$

$\frac{\partial}{\partial x}$ eliminate v.

$$\boxed{u_{tt} = u_{xx}} \leftarrow \text{"standard" wave equation.}$$

Cauchy problem $\quad u(x,0) = u_0(x)$
$\quad u_t(x,0) = u_1(x).$

There is a general solution to the WE.

$$u(x,t) = f(\underline{x-t}) + g(\underline{x+t}). \leftarrow \text{D'Alembert's Formula.}$$

two ways — one is from the vector form.

other is to do a coordinate transformation.

$$\xi = x+t, \quad \eta = x-t.$$

$$\text{WE} \rightarrow \underline{u_{\xi\eta} = 0}.$$

WE with speed c

$$u_{tt} = c^2 u_{xx} \quad \text{or} \quad \frac{1}{c^2} u_{tt} = u_{xx}$$

general solution $u(x,t) = f(x-ct) + g(x+ct)$

(just scale time to make $c=1$).

Solve Cauchy Problem with u_0 given
$u_1 \equiv 0$.

$$u(x,t) = f(x-t) + g(x+t).$$

$$u(x,0) = \underline{f(x) + g(x) = u_0(x)}.$$

$$u_t(x,0) = -f'(x) + g'(x) = 0.$$

$$\Rightarrow \quad -f(x) + g(x) = C \;\therefore$$

$$g(x) = \frac{1}{2}(u_0(x) + C)$$
$$f(x) = \frac{1}{2}(u_0(x) - C).$$

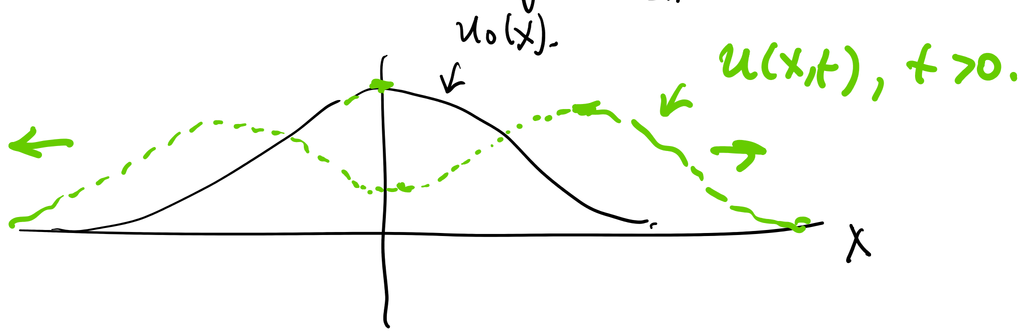
C cancels ✓.

$$\boxed{u(x,t) = \frac{1}{2}\left(u_0(x-t) + u_0(x+t)\right)}$$

map from u_0 to $u(x,t)$

Notation: $u(t) = \Phi_0\, u_0$

In words, $u_0(x)$ breaks up into two equal pieces, one moves to the right with speed 1, one to the left with speed 1.



Now solve $u_0 \equiv 0$

u_1 given $\leftarrow \frac{\partial u}{\partial t}(x,0)$.

$$u(x,0) = f(x) + g(x) = 0.$$

$f(x-t) + g(x+t)$.

$$-f'(x) + b'(x) = u_1(x)$$

integrate.

$$-f(x) + b(x) = U_1(x)$$

$\leftarrow$ antiderivative of $u_1(x)$.

$$u(x,t) = \frac{1}{2}\left(U_1(x+t) - U_1(x-t)\right)$$
$$= \frac{1}{2}\int_{x-t}^{x+t} u_1(x)\,dx.$$

x-t

Notation $u(t) = \Phi_1(t)\, u_0$.

Now consider $u_{tt} = u_{xx} + f(x,t)$. $u_0 \equiv 0$, $u_1 \equiv 0$.

Aside: let's derive DuHamel's principle, for problems second order in time.

Aside: let's derive Du Hamel's principle in formal way for first order problems.

$$u_t = \mathcal{L} u + f(t) \qquad u(0) \equiv 0 \qquad (1).$$

Look first at the pure IVP.

$$u_t = \mathcal{L} u \qquad u(0) = u_0.$$

Write the solution in the form

$$u(t) = \Phi(t)\, u_0.$$

$\Phi(0) = \text{Identity}$.
$\Phi'(t) u_0 = \mathcal{L}\Phi(t) u_0$
any t
any u_0

The problem (1) is formally solved by

$$u(t) = \int_0^t \Phi(t-s)\, f(s)\, ds.$$

check. $u(0) = 0$ ✓.

$$u_t = \underbrace{\underbrace{\Phi(0)}_{\text{Identity}} f(t)}_{f(t)} + \int_0^t \Phi'(t-s)\, f(s)\, ds.$$

$$+ \mathcal{L} \underbrace{\int_0^t \Phi(t-s) f(s)\, ds}_{u}.$$

$u_t = f(t) + \mathcal{L} u$ ✓.

---

go up one aside. $u(x,0)$ $u_t(x,0)$.

go up one aside.

$$u_{tt} = \mathcal{L}u + F(t) \qquad \begin{array}{c} u(x,0) \\ \downarrow \\ u_0 = 0, \end{array} \begin{array}{c} u_t(x,0). \\ \downarrow \\ u_1 = 0 \end{array} \quad (2)$$

need $\Phi_1(t) \leftarrow$ solves $u_0 = 0$, u_1 given.

$$\underline{\Phi_1(0)} = 0, \quad \underline{\Phi_1'(0)} = \text{Identity}.$$

$u = \Phi_1(t)\, u_1$ solves

$$u_{tt} = \mathcal{L}u \qquad u(x,0) = 0, \ u_t(x,0) = u_1.$$

$$\underline{\Phi_1''}\, u_1 = \mathcal{L}\,\underline{\Phi_1}\, u_1 \qquad \text{any time, any } u_1.$$

$$u = \int_0^t \Phi_1(t-s)\, F(s)\, ds \quad \text{solves} \quad (2)$$

Generic statement of DuHamel's principle for second order problems in time.

forcing problem with zero IC's.

Lecture #14

March 4, 2021 9:26 AM

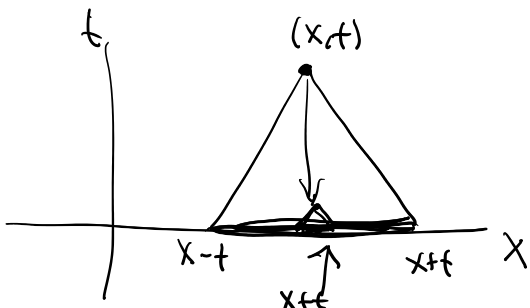
Wave equation $u_{tt} = u_{xx}$

Cauchy Problem all x $\quad u(x,0) = u_0(x)$

$u_t(x,0) = u_1(x)$.

① u_0 ✓ $\quad u_1 \equiv 0$

② $\boxed{u_0 \equiv 0}$ $\quad u_1$ ✓



$$u(x,t) = \frac{1}{2}\int_{x-t}^{x+t} u_1(\theta)\, d\theta.$$

$$u = \frac{1}{2}\int_{x-t}^{x+t} u_1(\theta)\, d\theta$$

$$u(t) = \Phi_1(t)\, u_1.$$

Example is $u_{tt} = u_{xx} + F(x,t)$

$$u_{tt} = \mathcal{L}u + F(t) \qquad u_0 \equiv 0,\ u_1 \equiv 0.$$

Formally, $\boxed{u = \int_0^t \Phi_1(t-s)\, F(s)\, ds}$ solves (A)

$\Phi_1(0) = 0$

$\Phi_1'(0) = \text{Identity}.$

$\Phi_1 u_1$ solves with $F \equiv 0$ for any u_1.

$\Phi_1'' u_1 = \mathcal{L}\Phi_1 u_1$ for any t, any u_1.

$u(0) = 0.$

$$u_t = \cancel{\Phi_1(0) F(t)} + \int_0^t \Phi_1'(t-s) F(s)\, ds.$$

zero.

$u_t(0) = 0$ ✓

$$u_{tt} = \Phi_1'(0) F(t) + \int_0^t \Phi_1''(t-s) F(s)\, ds.$$

$$u_{tt} = \underbrace{\Phi_1'(0)f(t)}_{f(t)} + \underbrace{\int_0 \Phi_1''(t-s)f(s)\,ds}_{L\int_0^t \Phi_1(t-s)f(s)\,ds}$$

$$\underbrace{L\int_0^t \Phi_1(t-s)f(s)\,ds}_{Lu}$$

$$u_{tt} = Lu + f(t) \checkmark$$

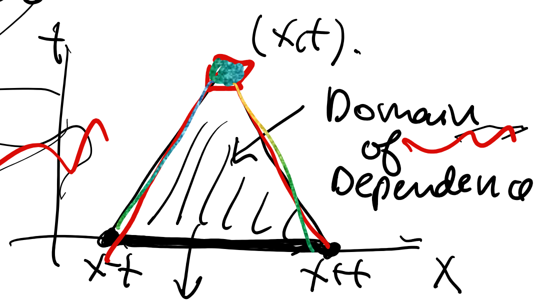
$$u_{tt} = u_{xx} + f(x,t)$$

$$\Phi_1(t)u_1 = \frac{1}{2}\int_{x-t}^{x+t} u_1(\theta)\,d\theta.$$

By Formal process above.

$$u(x,t) = \int_0^t \Phi_1(t-s)f(\cdot, s)\,ds.$$

$$= \frac{1}{2}\int_0^t \int_{x-(t-s)}^{x+(t-s)} f(\theta, s)\,d\theta\,ds.$$



$u(x,t)$ is the integral of f on this Δ.

?

Check this explicitly.

$u(0) = 0$ ✓, $u_t(0) = 0$ ✓

$$\boxed{u_{tt} = u_{xx} + f(x,t)}$$ all x, t.

$$u_t = \underbrace{\frac{1}{2}\int_x^x f(\theta,s)\,d\theta}_{\text{zero}} + \frac{1}{2}\int_0^t \Big(f(x+t-s, s) + f(x-t+s, s)\Big)\,ds.$$

$$u_{tt} = \overset{f(x,t)}{f(x,t)} + \frac{1}{2}\int_0^t \left[f'(x+t-s,s) - f'(x-t+s,s)\right] ds.$$

$\frac{\partial f}{\partial x}$.

$$u_x = \frac{1}{2}\int^t \ldots$$

$$u_x = \frac{1}{2}\int_0^t \left(f(x+t-s,s) - f(x-t+s)\right)ds. \quad \frac{\partial f}{\partial x}.$$

$$u_{xx} = \frac{1}{2}\int_0^t \left(\frac{\partial f}{\partial x}(x+t-s,s) - \frac{\partial f}{\partial x}(x-t+s,s)\right)ds.$$

$$u_{tt} = u_{xx} + f \quad \checkmark.$$

One last DuHamel thing.

It is possible to get Φ_1 from Φ_0

both solve

$$u_{tt} = L u$$

Φ_1 has $u(0) = 0$ ✓
$u_t(0)$ given

Φ_0 has
$u(0)$ given
$u_t(0) = 0$

Recall $\Phi_0(0) = \text{Identity}$

$\Phi_0'(0) = 0$

$$\Phi_0'' u = L \Phi_0 u$$

any time, any u.

short hand.

$$\Phi_0'' = L\Phi_0.$$

$$\Phi_1(t) = \int_0^t \Phi_0(t-s)\,ds.$$

$\Phi_1(0) = 0$ ✓

$$\Phi_1'(t) = \underbrace{\Phi_0(0)}_{\text{Identity}} + \int_0^t \Phi_0'(t-s)\,ds. \qquad \Phi_1'(0) = \text{✓} \; \text{Identity}.$$

$$\Phi_1''(t) = \int_0^t \Phi_0''(t-s)\,ds$$

$$= L\int_0^t \Phi_0(t-s)\,ds = L\Phi_1 \quad \checkmark$$

$$= \mathcal{L} \int_0^t \Phi_0(t-s)\,ds = \mathcal{L}\,\Phi_1 \quad ✓$$

Concrete check.

$$\Phi_1(t) = \int_0^t \Phi_0(t-s)\,ds \quad \text{for WE.}$$

$$[\Phi_0(t-s)v](x) = \tfrac{1}{2}\big(v(x-(t-s)) + v(x+(t-s)) \big)$$

$$\int_0^t [\Phi_0(t-s)v](x)\,ds$$

$$= \frac{1}{2}\int_0^t \big(v(x-(t-s)) + v(x+t-s) \big)\,ds.$$

$v(\cdot)$: $x-t$ to x $\qquad$ $v(\cdot)$: $x+t$ to x.

$$= \frac{1}{2}\int_{x-t}^{x+t} v(s)\,ds \quad ✓ \leftarrow \Phi_1(t)\,v \;✓$$

Boundary conditions for the wave equation.

$x>0$ $\qquad$ x

$u=0$ for all t.

$$u_{tt} = u_{xx}$$

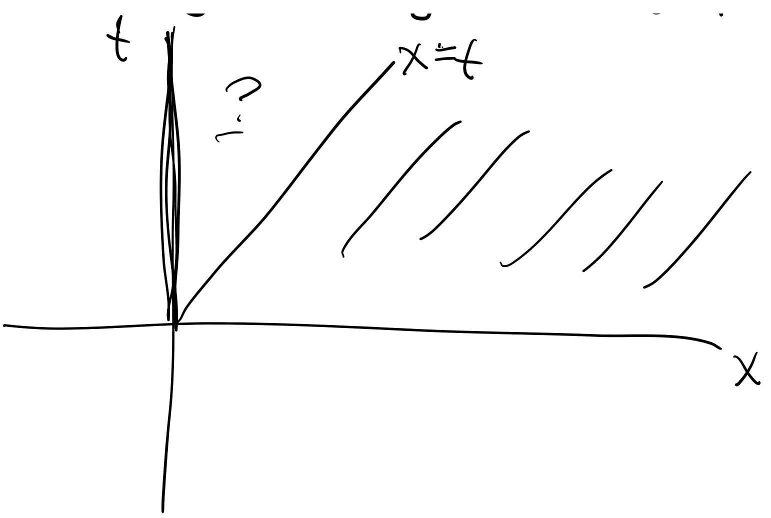
$$u(x,0) = u_0, \quad u_t(x,0) = 0.$$

$$u(x,t) = f(x-t) + b(x+t). \quad \text{for some } f(\cdot),\ b(\cdot)$$

Know
$$\left.\begin{aligned} f(s) &= \tfrac{1}{2}u_0(s) \\ b(s) &= \tfrac{1}{2}u_0(s) \end{aligned}\right\} \text{ for } s>0.$$

positive arguments are enough for $b(\cdot)$.
but we need negative arguments for f.

t $\quad$? $\quad$ $x=t$



look at BC.

$$u(0,t) = \underline{f(-t) + b(t)} = 0 \qquad t \geq 0.$$

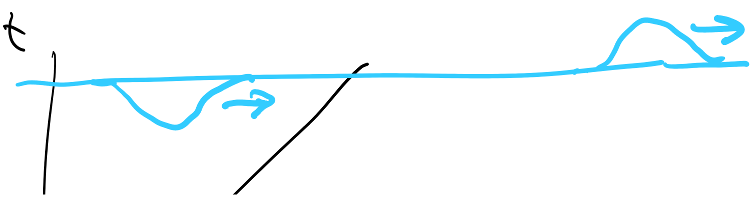
$$f(s) = -b(-s) = -\tfrac{1}{2} u_0(-s). \qquad s \leq 0.$$

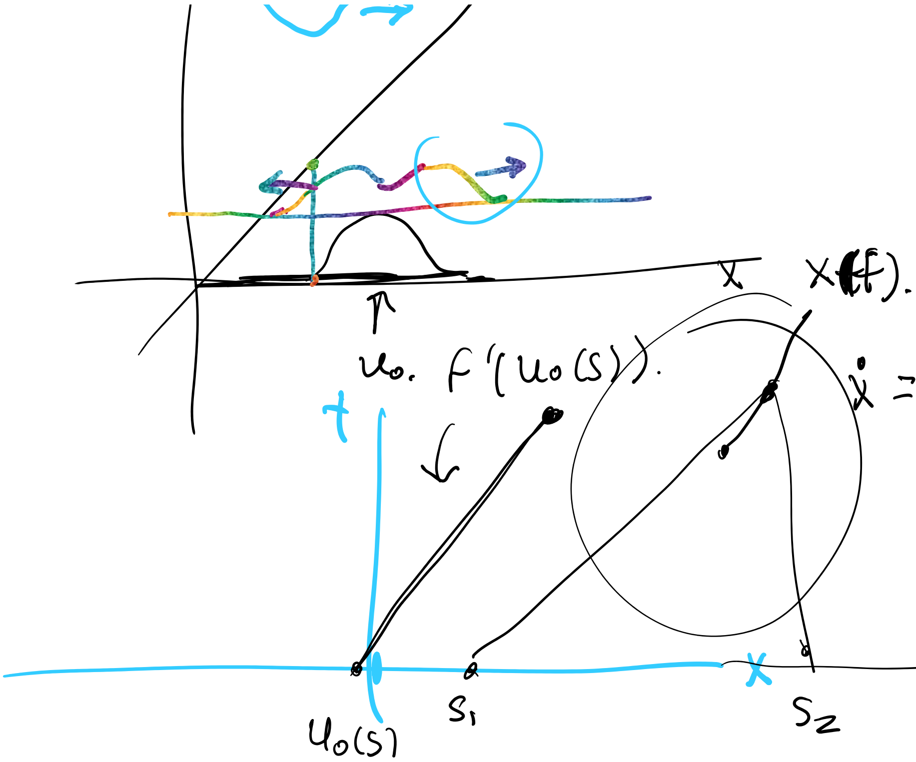
Solution.

$$u(x,t) = \begin{cases} \tfrac{1}{2} u_0(x-t) + \tfrac{1}{2} u_0(x+t) & x > t. \\ -\tfrac{1}{2} u_0(-(x-t)) + \tfrac{1}{2} u_0(x+t) & x < t \end{cases}$$

continuous at $x=t$.

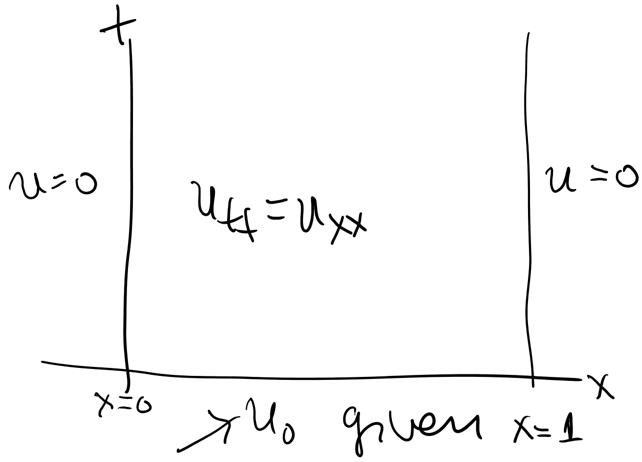
Next time - bounded domains with BC's.





$$\dot{x} = \frac{f(u_0(s_2)) - f(u_0(s_1))}{u_0(s_2) - u_0(s_1)}$$

$$u(s + f'(u_0(s))t,\ t) = u_0(s).$$



$u(x,0) = u_0(x)$

$u_t(x,0) = u_1(x) = 0.$

Consider the PDE in the form

$$u_{tt} = \mathcal{L} u$$

$\mathcal{L} u = u_{xx}$ and boundary conditions

$u(0)=0 , \ u(1)=0$

Naturaley look for solutions in terms of the eigenfunctions of $\mathcal{L}$.

$$u(x,t) = \sum_{n=1}^{\infty} c_n(t) \sin n\pi x. \quad \text{standing wave series.}$$

put into wave equation.

$$\ddot{c}_n = -n^2\pi^2 c_n.$$

$$c_n = a_n \sin(n\pi t) + b_n \cos(n\pi t).$$

Match the Initial conditions

$a_n = 0$ $\qquad b_n = 2\int_0^1 u_0(x) \sin n\pi x \, dx$

$u_1(x) \equiv 0$

$$u_1(x) \equiv 0$$

$$u(x,t) = \sum_{n=0}^{\infty} b_n \boxed{\cos(n\pi t)\sin(n\pi x.)}$$

trig identity $\Rightarrow$ D'Alembert's formula.

(This will define a weak solution if $u_0(x)$ is discontinuous).

$$u(x,t) = 2\sum_{n=0}^{\infty}\left\{\cos(n\pi t)\sin(n\pi x)\int_0^1 u_0(s)\sin n\pi s\, ds.\right\}$$

recognize this as a Green's function for the problem.

This is a form $u(t) = \Phi_0(t)\, u_0$.

$u_{tt} = \mathcal{L}u$ with BC's for $\mathcal{L}$.

Apply our S-L work.

Variable mass per unit length $\mu(x)$.

$$\mu(x)\, u_{tt} = u_{xx} \qquad u(0,t) = u(1,t) = 0$$

$$\varphi'' = \lambda\mu(x)\varphi. \qquad \varphi(0) = 0,\ \varphi(1) = 0.$$

All λ's are negative, $\varphi_j(x)$ are the shapes of the standing waves, angular frequency $\sqrt{-\lambda}$.

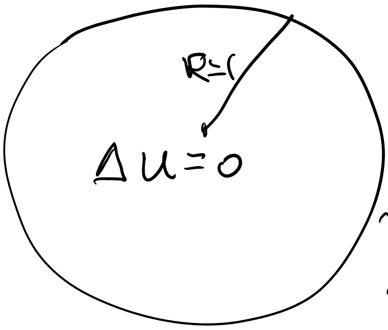
If μ is discontinuous, then φ and φ' are continuous at the discontinuity.

Chat: $u_{tt} = u_x$?

---

wave: $u_{tt} = u_x$?

Poisson's Formula in the Disk.



$u = g(\theta)$.

$u(r,\theta)$ $u(1,\theta)$.

$$\Delta u = u_{xx} + u_{yy} = \frac{1}{r}(r u_r)_r + \frac{1}{r^2} u_{\theta\theta}.$$

Look for a solution in the form

$$u(r,\theta) = \sum_{n=-\infty}^{\infty} u_n(r)\, e^{in\theta}$$

$$\frac{1}{r}(r u_n')' - \frac{n^2}{r^2} u_n = 0$$

$$(r u_n')' - \frac{n^2}{r} u_n = 0$$

n'th fourier term of $g(\theta)$.

$$\boxed{u_n(1) = g_n}$$

bounded. $u_n(0)$

$$\frac{1}{2\pi}\int_0^{2\pi} g(\theta)\, e^{-in\theta}\, d\theta.$$

$x = \ln r$, $r = e^x$

$$\frac{d^2 u_n}{dx^2} - n^2 u_n = 0$$

$n = 0$ $u_0 = A + Bx$.

$n \neq 0$.

$$u_n(x) = A e^{|n|x} + B e^{-|n|x}$$

$$u_n(r) = A r^{|n|} + B r^{-|n|} \quad B = 0$$

$u_0 = A + B \ln r$

$B = 0$, $A = g_n$.

$$u(r,\theta) = \sum_{n=-\infty}^{\infty} g_n\, r^{|n|} e^{in\theta}.$$

$$u(r,\theta) = \sum_{n=-\infty}^{\infty} J_n \quad 1 \quad t$$

High frequency components of the BC g do no penetrate far into the disk.

Algebra (notes)

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \underbrace{\frac{(1-r^2)}{1-2r\cos(\theta-\theta')+r^2}}_{G(r,\theta;\theta')} g(\theta')\,d\theta'$$

Green's Function Representation of the Solution.

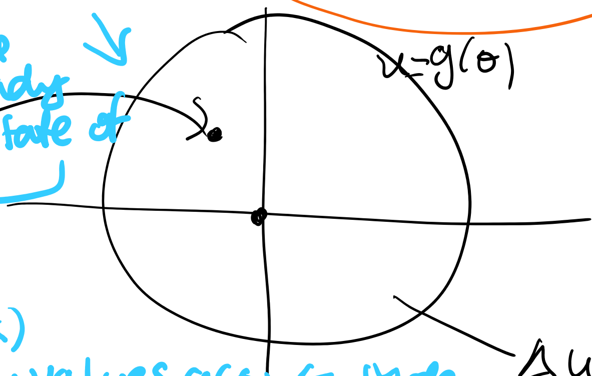
Laplace problem is the steady state of

$$\frac{\partial u}{\partial t} = \Delta u$$

$$u(x,y,0) = u_0(x)$$

Same Boundary values $g(\theta)$ ← independent of time.

$u(r,\theta)$



$u(r,\theta)$ depends on $g(\theta')$ at all θ'.

Note: $G > 0$ $\quad \frac{1}{2\pi}\int_0^{2\pi} G(r,\theta;\theta')\,d\theta' = 1$ for all r,θ.

$u(r,\theta)$ is ~~an~~ a weighted average of $g(\theta')$ values.

$u(0,\cdot)$ ← an exact average of $g(\theta')$ values.

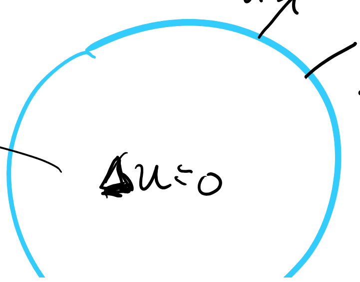
(come back to this, this prove the maximum principle for the laplace problem).

Neumann problem.

u = constant

$\Delta u = 0$

$\frac{\partial u}{\partial n} = 0$



$\frac{\partial u}{\partial n} = g(\theta)$

↑

heat flux in to the disk.

$\dfrac{\partial u}{\partial n} = 0$

heat flux in to the disk.

Also come as a steady state of the Heat Equation.

This problem is only solvable if

$$\int_0^{2\pi} g(\theta)\, d\theta = 0.$$

Then the solution is only determined up to a constant.

There is a formula for this problem as well.

$$\mathbb{A} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \qquad \lambda_1 = 0 \quad \underline{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\lambda_2 = 1 \quad \underline{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

$\mathbb{A}\underline{x} = \underline{b}. \rightarrow \underline{b} \cdot \underline{v}_1 =$ first component of b $= 0$.

solution has an arbitrary multiple of $\underline{v}_1$.

$$\iint_{\Omega} \Delta u = \int_{\Gamma} \frac{\partial u}{\partial n}$$

zero. $\partial\Omega$

Wave Equation

$$u_{tt} = u_{xx} \quad \text{all } x, \qquad u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x).$$

Cauchy Problem.

plug in

$$u(x,t) = \int_{-\infty}^{\infty} \hat{u}(\alpha,t)\, e^{i\alpha x}\, d\alpha$$

$$\frac{d^2 \hat{u}}{dt^2} = -\alpha^2 \hat{u}.$$

$$Ce^{i\alpha t} + De^{-i\alpha t}$$

D'Alembert's formula emerges.

$$\hat{u}(\alpha) = \begin{cases} A(\alpha)\cos\alpha t + B(\alpha)\sin\alpha t & \alpha \neq 0 \\ A_0 + B_0 t & \alpha = 0. \end{cases}$$

Laplace Problem as a Cauchy Problem.

$$u_{xx} + u_{yy} = 0$$

$$u_{tt} + u_{xx} = 0 \qquad u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x).$$

$$\frac{d^2 u}{dt^2} - \alpha^2 \hat{u} = 0$$

$$\hat{u}(\alpha,t) = \boxed{A(\alpha)\, e^{|\alpha| t}} + B(\alpha)\, e^{-|\alpha| t} \qquad \alpha \neq 0.$$

unbounded exponential growth.
This problem is ill-posed.

Student posed equation $\boxed{u_{tt} = u_x}$

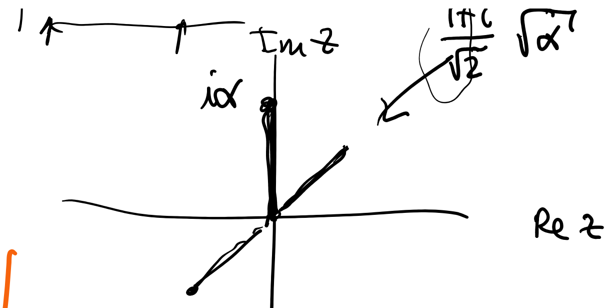
$$r(\alpha) = \frac{1 \pm i}{\sqrt{2}}\sqrt{\alpha}$$

Im z

$i\alpha$

$$\frac{d^2 \hat{u}}{\ldots}$$

$$\frac{d^2\hat{u}}{dt^2} = i\alpha\,\hat{u}$$



$$\hat{u}(\alpha,t) = A(\alpha)\, \boxed{e^{\frac{t\sqrt{\alpha}}{2}}\, e^{it\sqrt{\alpha/2}}} \quad -r(\alpha) = -\frac{(1+i)}{\sqrt{2}}\sqrt{\alpha}.$$

$+ \ldots$

unbounded exponential growth - ill posed.

Telegrapher's Equation (easy coefficients).

$$u_{tt} - u_{xx} + \underbrace{2u_t + u}_{} = 0.$$

$$\frac{d^2\hat{u}}{dt^2} + \alpha^2\hat{u} + 2\frac{d\hat{u}}{dt} + \hat{u} = 0$$

$$\underbrace{\frac{d^2\hat{u}}{dt^2} + 2\frac{d\hat{u}}{dt} + (\alpha^2+1)\,\hat{u} = 0.}_{\hat{u} = e^{rt}}$$

$$r^2 + 2r + (\alpha^2+1) = 0.$$

$$r = -1 \pm \sqrt{1-(\alpha^2+1)}$$

$$= -1 \pm i\alpha.$$

$$\hat{u}(\alpha,t) = e^{-t}\left(A(\alpha)\cos\alpha t + B(\alpha)\sin\alpha t\right) \leftarrow \alpha \neq 0.$$

damping independent of α.

wave equation behaviour

Laplace problem as a Cauchy problem was ill posed.

Let's consider it as a boundary value problem in half space.

bounded as $y \to \infty$ (this acts as a ... condition)

... ... space.

bounded as $y \to \infty$ (this acts as a boundary condition).

$$\Delta u = 0$$

$\uparrow y$

$\to x$.

$$u(x,0) = u_0(x).$$

$$\hat{u}(\alpha) = \begin{cases} A(\alpha) e^{|\alpha| y} + \underline{B(\alpha)} \underline{e^{-|\alpha| y}} & \alpha \neq 0 \\ A_0 y + \underline{B_0} & \alpha = 0. \end{cases}$$

these are zero by boundedness condition

$$\Rightarrow B(\alpha) = \hat{u}_0$$

$$\underline{\hat{u}(\alpha, y)} = \underbrace{\hat{u}_0(\alpha)\, e^{-|\alpha| y}}.$$

well posed ✓.

$u(x,y)$ is a convolution at every y. with u_0 and $g \leftarrow \hat{g} = e^{-|\alpha| y}$.

look in tables, Wolfram alpha, or contour integration.

$$g(x;y) = \frac{2y}{x^2+y^2}.$$

Green's function representation.

$$\boxed{u(x,y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{u_0(s)\, y}{(x-s)^2 + y^2}\, ds. \quad (\star)}$$

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y}{(s^2+y^2)}\, ds = 1 \qquad \text{for every } y > 0.$$

integrand > 0.

integrand > 0,
leads to a maximum principle.

$u = b_0(t)$ $\quad$ $u_t = u_{xx} + f(x,t)$ $\quad$ $u = b_1(t)$.

$u(x,0) = u_0(x)$

$u = b_0(t)$ $\quad$ $u_t = u_{xx}$ $\quad$ $u \equiv 0$ $\qquad$ (★)

$u \equiv 0$

One path forward: turn boundary data into interior forcing.

Let $v(x,t) = u(x,t) - b_0(t)\,\boxed{(1-x)}$.

use any $f(x)$ with $f(0)=1$ and $f(1)=0$.

$$v_t - v_{xx} = u_t - u_{xx} - \dot{b}_0\,(1-x).$$

Solve for v, get u

$v=0$ $\quad$ $v_t = v_{xx} - \dot{b}_0\,(1-x)$ $\quad$ $v=0$

$v_0 = -\dot{b}_0(0)\,(1-x)$

As an alternative, we can solve (★) with the Laplace transform.

Laplace transform.

Look at the Laplace Transform in review.

of a function $f(t)$ defined for $t \geq 0$.

$$\tilde{f}(s) = \int_0^\infty e^{-st} f(t)\,dt \qquad \hat{f} = \mathcal{L}f$$

Think of s as a complex variable.

If $|f(t)| < Ke^{at}$ for all t with some $K>0, a>0$. then $\tilde{f}(s)$ defined for all s with $Re(s) > \alpha$.

Defines $\hat{f}(s)$ as a complex analytic function for $Re(s) > \alpha$.

Then complex continuation to other s (with possible singularities & branch lines).

1. $f(t) = e^{at} \Leftrightarrow \hat{f} = \frac{1}{s-a}$. simple pole at $s=a$.
2. $f(t) = t^n \Leftrightarrow \tilde{f} = \frac{n!}{s^{n+1}}$ pole of order $n+1$ at $s=0$
3. $f(t) = \sin t \Rightarrow \frac{1}{1+s^2}$

   $f(t) = \cos t \Rightarrow \frac{s}{1+s^2}$
4. $f(t) = \operatorname{erfc}\left(\frac{1}{2\sqrt{t}}\right). \Rightarrow \underline{\frac{1}{s} e^{-\sqrt{s}}}!$

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-\tau^2}\,d\tau$$

~~Properties of the Laplace Transform.~~

~~Properties of the Laplace~~ Transform.

(i) $\mathcal{L}(f') = s\tilde{f} - f(0)$.

integration by parts.

(ii) $\mathcal{L}$ is linear and invertible on a set of functions.

(iii) $\mathcal{L}(f * g) = \tilde{f}(s)\,\hat{g}(s)$.

read backwards

$$\underline{\mathcal{L}}^{-1}\left(\tilde{f}(s)\,\tilde{g}(s)\right) = f * g.$$

look at some scalar examples.
look at a PDE example.
look at BC problem ← with time harmonic forcing.

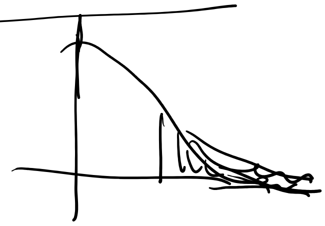
next time.

Laplace transform

$$\tilde{f}(s) = \int_0^\infty e^{-st} f(t)\,dt$$

Ex $\quad f(t) = e^{at} \Rightarrow \hat{f} = \dfrac{1}{s-a}.$

$$f(t) = \mathrm{erfc}\left(\frac{1}{2\sqrt{t}}\right) \Rightarrow \hat{f} = \frac{1}{s} e^{-\sqrt{s}}\,?$$



$$\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-\tau^2}\,d\tau$$

Note: $\mathrm{erfc}(0) = 1.$

$$\mathcal{L}(f') = s\hat{f} - f(0).$$

$$\mathcal{L}(f * g) = \tilde{f}(s)\,\tilde{g}(s).$$

Ex $\quad$ ODE example. $u(t)$

$$\ddot{u} + u = 0 \qquad u(0) = 1, \quad \dot{u}(0) = 0$$

$[\ u = \cos t\ ].$

Laplace transform.

$$s\,\mathcal{L}(\dot{u}) - \dot{u}(0) + \mathcal{L}(u) = 0.$$

$$s^2 \underbrace{\mathcal{L}(u)}_{\tilde{u}} - s\,\overset{1}{\cancel{u(0)}} - \overset{0}{\cancel{\dot{u}(0)}} + \underbrace{\mathcal{L}(u)}_{\tilde{u}} = 0$$

$$s^2 \tilde{u} + \tilde{u} - s = 0.$$

$$\tilde{u} = \frac{s}{1+s^2} \xrightarrow[\text{look in tables}]{} u(t) = \cos t.$$

Ex Use Laplace transform to get a formula for $u(t)$

$$\dot{u} = a u + f(t) \qquad u(0) = 0.$$

constant

$$\left[u(t) = \int_0^t e^{a(t-\tau)} f(\tau)\, d\tau \right]$$

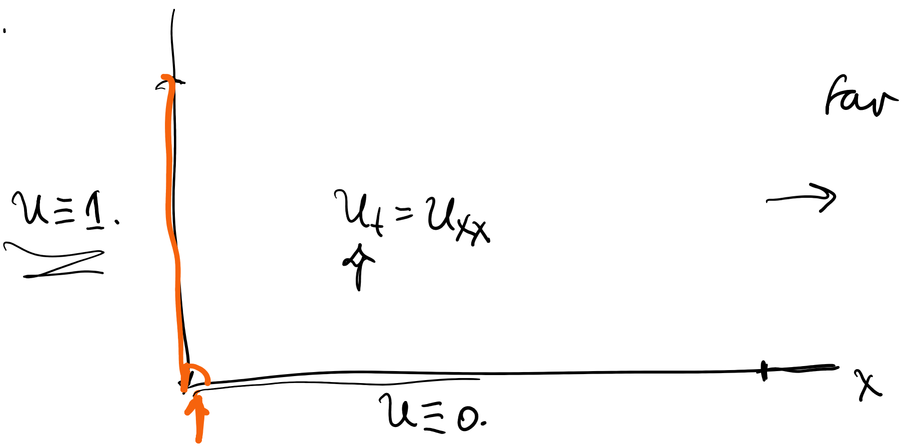
$$s\hat{u} - \cancel{u(0)} = a\tilde{u} + \hat{f}.$$

$$\hat{u}(s) = \frac{\hat{f}(s)}{s-a} = \hat{f}(s)\,\frac{1}{s-a}.$$

product $\Rightarrow$ u is a convolution.

$$u(t) = \int_0^t e^{a(t-\tau)} f(\tau)\, d\tau \quad \checkmark.$$

Ex.



far field condition $u(x,t) \to 0$ as $x \to \infty$.

space time corner. discontinuity here. (weak solution).

Take the Laplace transform (in t for every x).

Take the Laplace transform (in t for every x).

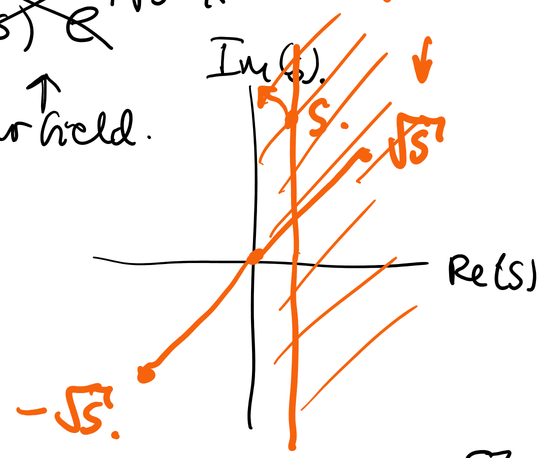
$$s\tilde{u} = \hat{u}_{xx} \quad \leftarrow \text{ ODE in } x \text{ for each } s.$$

$$\hat{u}(x,s) = A(s)\,e^{-\sqrt{s}\,x} + \cancel{B(s)\,e^{+\sqrt{s}\,x}}.$$

far field.

$$\hat{u}(0,s) = \frac{1}{s}$$

$\hat{1}$



$$\hat{u}(x,s) = \frac{1}{s}\,e^{-\sqrt{s}\,x}.$$

constant c.

$$\frac{1}{s}e^{-\sqrt{s}}$$

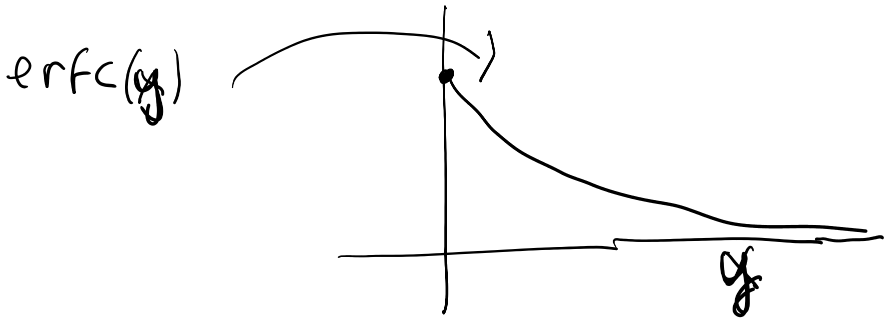
(property of $\mathcal{L}$): $\frac{1}{c} f\left(\frac{t}{c}\right) \Rightarrow \hat{f}(cs).$

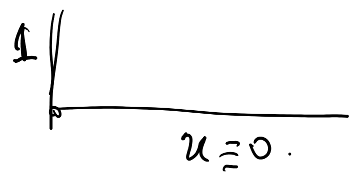
$$\hat{u}(x,s) = x^2\,\frac{1}{s x^2}\,e^{-\sqrt{s x^2}}$$

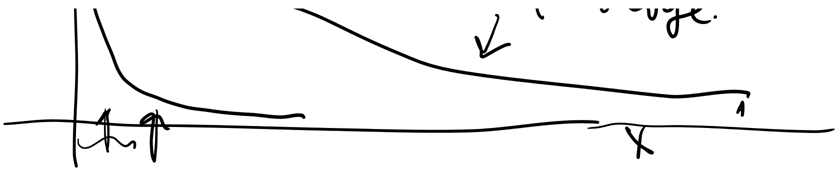
Use Table

$c = x^2$

$$u(x,t) = \operatorname{erfc}\left(\frac{x}{2\sqrt{t}}\right).$$







t small.

$f(t)$ defined $t \geq 0$ satisfies $|f(t)| < K e^{at}$ for all $t \geq 0$.

Consider $\eta > a$.

Real

$$g(t) = \begin{cases} f(t)\, e^{-\eta t} & t \geq 0. \\ 0 & t < 0. \end{cases}$$

g has a well defined Fourier transform.

$$\hat{g}(\alpha) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(t)\, e^{-i\alpha t}\, dt.$$

$$= \frac{1}{2\pi} \int_0^{\infty} f(t)\, e^{-(\eta + i\alpha)t}\, dt.$$

$$= \frac{1}{2\pi} \tilde{f}(s) \quad \text{with } s = \eta + i\alpha.$$

Inverse Fourier transform.

$$g(t) = f(t)\, e^{-\eta t} = \int_{-\infty}^{\infty} \hat{g}(\alpha)\, e^{i\alpha t}\, d\alpha.$$

$t \geq 0.$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(\eta + i\alpha)\, e^{(\eta + i\alpha)t}\, d\alpha.$$

$\frac{1}{2\pi} \int_{-\infty}$ ...

same complex argument.

$$f(t) = \frac{1}{2\pi i} \int_{\mathcal{C}} \hat{f}(z) e^{zt}\, dz$$

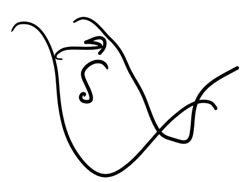
$dz = i\, d\alpha$.

---

time harmonic signals.

pure frequency (sinusoidal) periodic signal.
frequency ω. (angular frequency).

$$f(t) = \underline{c \cos \omega t + d \sin \omega t}.$$



An amplitude-phase form.

$$\boxed{f(t) = a \cos(\omega t - \delta)}.$$

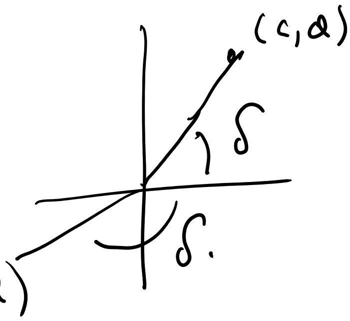
Use trig identity $\cos(\alpha - \beta) = \cos\alpha \cos\beta + \sin\alpha \sin\beta$.

~~a =~~ $c = a \cos\delta \qquad d = a \sin\delta.$

$a = \sqrt{c^2 + d^2}$ and δ is the angle from the positive x axis to the vector (c,d).

$$\delta = \operatorname{Arg}(c + id)$$

giving $\delta \in (-\pi, \pi]$.



Another form $f(t) = \operatorname{Re}\{A e^{i\omega t}\}$.

$A = c - id. \qquad |A| = a$ amplitude.

Ex Scalar ODE example.

Ex Scalar ODE example.

Find a particular solution $u(t)$ to.

$$\ddot{u} + \dot{u} + u = \sin(2t) = -\mathrm{Re}\left(ie^{i2t}\right).$$

Aside find homogeneous solutions

"Try" $u = e^{rt}$

$$r^2 + r + 1 = 0. \qquad r = -\tfrac{1}{2} \pm \sqrt{\tfrac{1}{4} - 1}$$

$$= -\tfrac{1}{2} \pm i\sqrt{3/4}$$

not interesting.

$\boxed{\text{decaying}}$, oscillatory homogeneous solutions

particular solution.

$$u(t) = \cancel{\mathrm{Re}\{} A e^{i2t} \cancel{\}}$$

solve $\ddot{u} + \dot{u} + u = -ie^{i2t}$

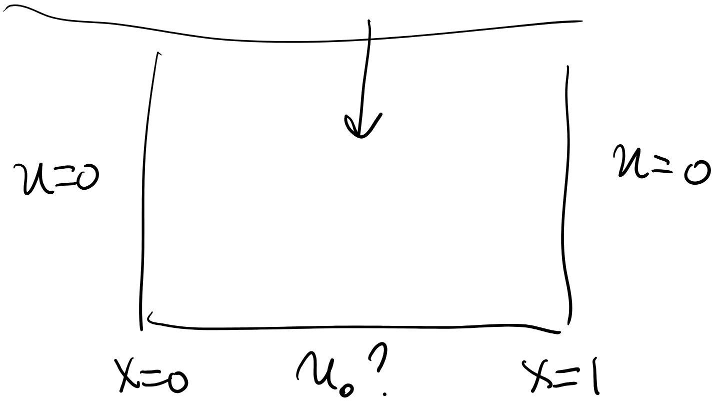
Real & Imaginary parts of the solution don't mix.

$$A e^{i2t}\left(-4 + 2i + 1\right) = -ie^{i2t}$$

$$A = \frac{-i}{-4+2i+1} =$$

Ex

$$u_t = u_{xx} + x(1-x)\sin t.$$

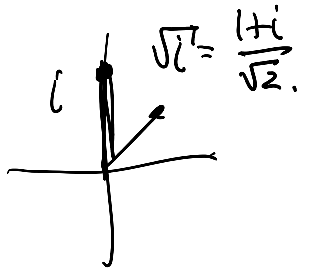


only interested in the periodic (in time) behaviour after the transient from u_0.

after the transient from u_0.

$$u = \underline{A(x) e^{it}}$$ ← want imaginary component is $u(x,t)$.

↑ complex A $\qquad$ $\sin t$.



$$iAe^{it} = A''e^{it} + x(1-x)e^{it}$$

$$-x + x^2.$$

$$\boxed{\underbrace{A'' - iA} = -x(1-x).} \qquad A(0)=0, \quad A(1)=0.$$

$$A(x) = e^{rx}.$$

$$r^2 - i = 0, \Rightarrow r = \pm\sqrt{i}.$$

$$r = \pm \frac{1+i}{\sqrt{2}},$$

$$A(x) = \underline{C}\, e^{\left(\frac{1+i}{\sqrt{2}}\right)x} + \underline{D}\, e^{-\left(\frac{1+i}{\sqrt{2}}\right)x}$$ ← homogeneous

$$\underline{+ a + bx + cx^2}.$$ ← particular.

Find a, b, c first to match RHS.

After pick C & D to match BCS.

↓

Then $A(x)e^{it}$ ← Find the imaginary component.

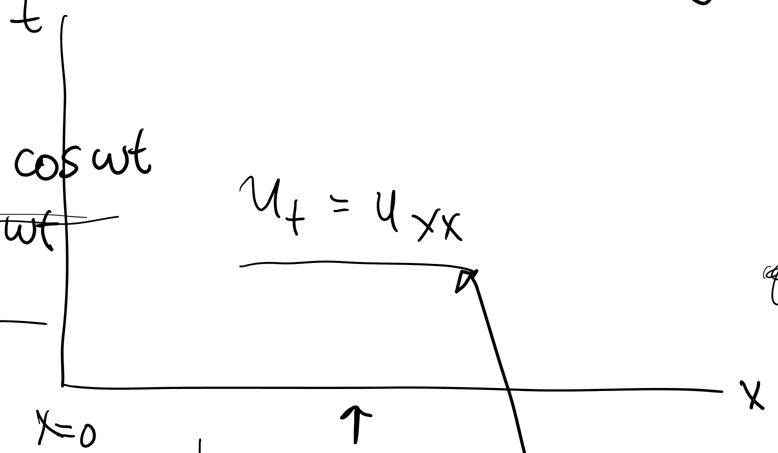
⋮

Details are in the notes.

Next time — time harmonic forcing on the boundary.

Then start on free boundary problems (Stefan problem).

Continue time harmonic forcing.



$u(0,t) = \cos \omega t$

$= e^{i\omega t}$

$u_t = u_{xx}$

u bounded as $x \to \infty$ at each t. ⟶ u

Initial conditions ← not interesting give a transient we don't care about.

Look for $u(x,t)$ in the form

$$u(x,t) = A(x)\, e^{i\omega t}$$

↑ complex

→ far field condition.

$$i\omega A = A''$$

$$A(x) = C\, e^{-\frac{1}{\sqrt{2}}(1+i)\sqrt{\omega}\, x} + \cancel{D\, e^{\frac{1}{\sqrt{2}}(1+i)\sqrt{\omega}\, x}}$$

$C = 1$ by the $x=0$ BC.

$$u(x) = \mathrm{Re}\left\{ A(x)\, e^{i\omega t} \right\} \quad \text{time harmonic.}$$

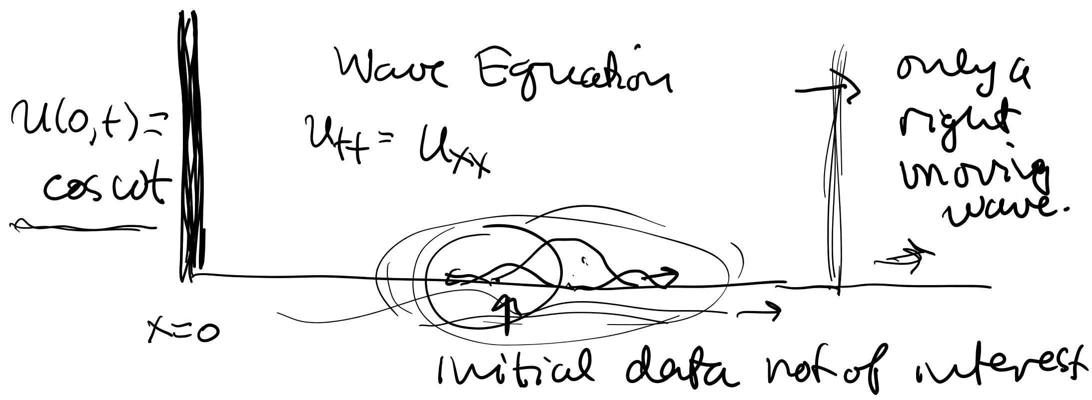
$$= e^{-\sqrt{\omega/2}\, x} \cos\left(\omega t - \sqrt{\frac{\omega}{2}}\, x\right).$$

↑ exponential decay, faster for larger ω.

π

$x = \pi\sqrt{\frac{2}{\omega}}$, the temperature is 180° out of phase with the surface signal. The temperature is with the surface signal. The temperature is coldest when the surface is hottest & vice versa.

Ex



Same assumed form of the solution (Ansatz).

$$u(x,t) = A(x)\, e^{i\omega t}$$

$$-\omega^2 A = A''$$

$$A(x) = C e^{i\omega x} + D e^{-i\omega x}$$

$$u(x,t) = \cancel{C e^{i\omega(x+t)}} + D e^{i\omega(-x+t)}.$$

Re. backward moving wave.

$$u(x,t) = \cos(\omega(x-t)) \leftarrow \text{wave moving to the right.}$$

Ex consider this wave travel through a defect in $x \in [0,1]$.

$$u_{tt} = c^2(x)\, u_{xx}$$

$$c^2 = \begin{cases} 1 & x<0,\ x>1. \\ f(x) & x \in [0,1] \end{cases} \leftarrow \text{manufacturing defect.}$$

Same Ansatz.

complex reflection coefficient to be determined

incoming wave. reflected wave.

$$u(x,t) = e^{i\omega t} \begin{cases} e^{-i\omega x} + R e^{i\omega x} & x<0. \end{cases}$$

$$u(x,t) = e^{i\omega t}\begin{cases} e^{-i\omega x} + Re^{i\omega x} & x<0 \\ A(x) & x \in [0,1]. \\ Te^{-i\omega(x-1)}. & x>1. \end{cases}$$

complex transmission coefficient to be determined.

u and u_x are continuous at $x=0,1$.

$$-\omega^2 A = f(x) A'' \qquad x \in [0,1]. \qquad (1).$$

$$\boxed{f(x) A'' + \omega^2 A = 0.}$$

Continuity of u & u_x at $x=0$.

$$A(0) = 1+R.$$

$$A'(0) = -i\omega + i\omega R.$$

eliminate R.

$$\boxed{A'(0) - i\omega A(0) = -2i\omega.} \qquad (2).$$

Continuity of u & u_x at $x=1$.

$$A(1) = T$$

$$A'(1) = -i\omega T$$

eliminate T

$$\boxed{A'(1) + i\omega A(1) = 0} \qquad (3).$$

Now (1), (2) & (3) are the BVP to solve for $A(x)$ [numerical solve], then recover R & T.

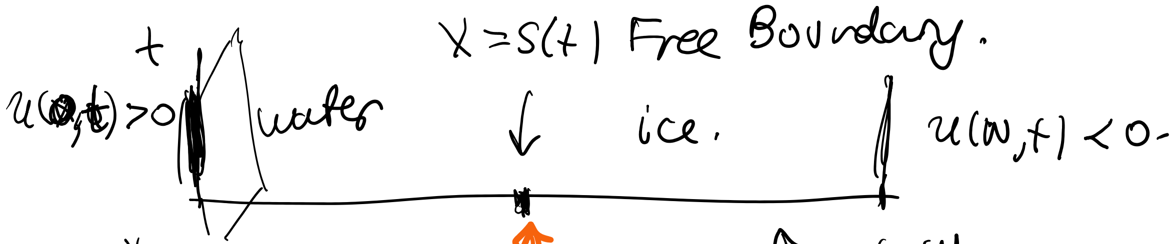
I believe this problem for $A(x)$ is not always solvable.

(I think you can have a resonant defect in the transmission line).

& moving

Free & moving boundary Value Problems

$x = s(t)$ Free Boundary.



water temperature $u(x,t)$ $x < s(t)$

$u_t = k_w u_{xx}$

thermal diffusivity m^2/s.

3 conditions needed here.

ice temperature $u(x,t)$. $x > s(t)$.

$u_t = k_i u_{xx}$ $\quad \alpha = \frac{k_i}{k_w} \approx 6.$

heat flux per unit cross sectional area $\quad -K_w \frac{\partial u}{\partial x} \qquad -K_i \frac{\partial u}{\partial x} \qquad \frac{K_i}{K_w} \approx 3\text{-}4.$

$\frac{J}{s\,m^2}$ $\qquad \frac{J}{s\,m\,°C}$ thermal conductivity

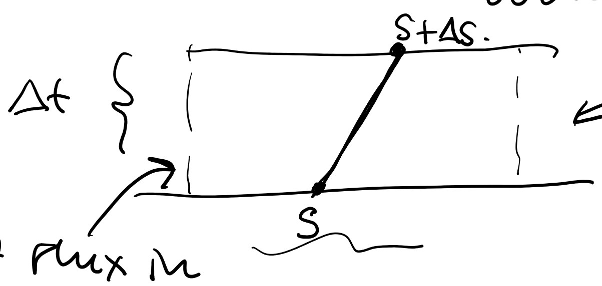
At $x = s(t)$ ← count suggests we need 3 conditions here - 1 each for heat equations on either side & one for determining $s(t)$.

$u_+ = u_- = 0$

(ice freezes at $u=0$).

Need latent heat L (J/kg) energy needed to melt one kg of ice at 0°C.

Zoom in on the free boundary over a time Δt:



heat flux out into ice.

$-K_i \frac{\partial u}{\partial x}\Big|_{x=s(t)_+}$.

heat flux in from water

from water

$$-K_w \frac{\partial u}{\partial x}\Big|_{x=s(t)-}$$ per unit cross sectional area.

Net heat in is

$$\left(K_i \frac{\partial u}{\partial x}\Big|_{+} - K_w \frac{\partial u}{\partial x}\Big|_{-} \right) \Delta t.$$

does not change the temperature (stuck at $u=0$).
So it must melt or freeze the ice.

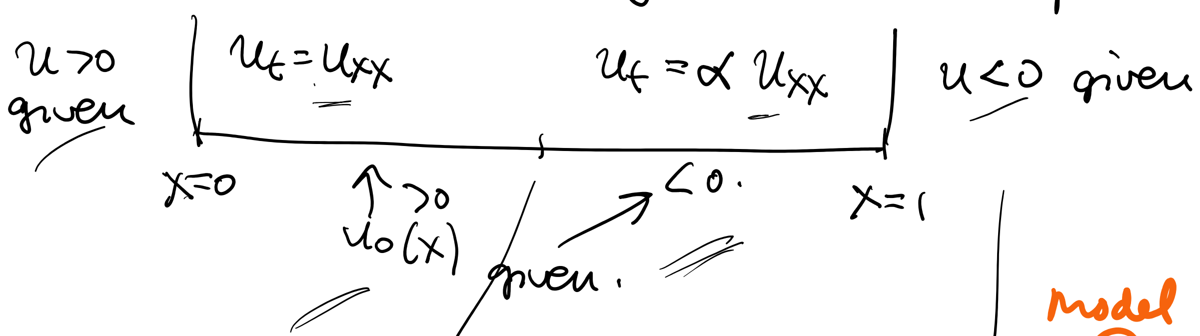
$$\rho \Delta s L =$$

Equate and take limits $\frac{\Delta s}{\Delta t} \to \frac{ds}{dt}$.

$$\frac{ds}{dt} = \frac{1}{\rho L}\left(K_i \frac{\partial u}{\partial x}\Big|_{+} - K_w \frac{\partial u}{\partial x}\Big|_{-} \right)$$ **Stefan condition.**

free boundary velocity

After scaling in x, t, and separately $u_{+/-}$, also s.
we can get the following dimensionless problem:



$s(0)$ given

model (T).

$u_+ = 0$.

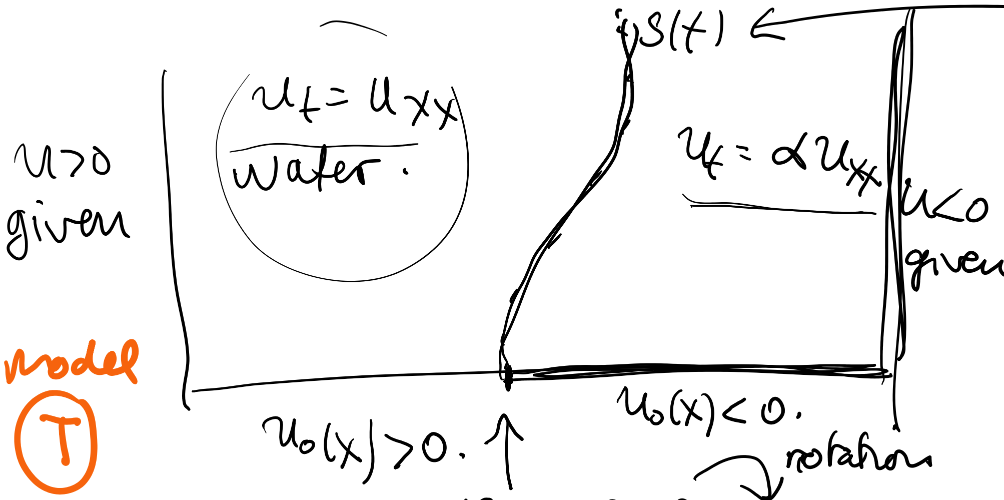
notation for

$$\frac{ds}{dt} = \left[\frac{\partial u}{\partial x}\right] = \frac{\partial u}{\partial x}\Big|_{+} - \frac{\partial u}{\partial x}\Big|_{-}.$$

Next time: some simplifications that lead to analytical results.

analytical results;

Last time - 1D Stefan problem.



boundary (interface) between water & ice: to be determined as part of the problem (free or moving boundary value problem).

rotation

$$\frac{ds}{dt} = \left[\frac{\partial u}{\partial x}\right] = \left.\frac{\partial u}{\partial x}\right|_{+} - \left.\frac{\partial u}{\partial x}\right|_{-}.$$

$$u|_{\pm} = 0.$$

This is a problem you can solve for $u_\pm(x,t)$ and $s(t)$.

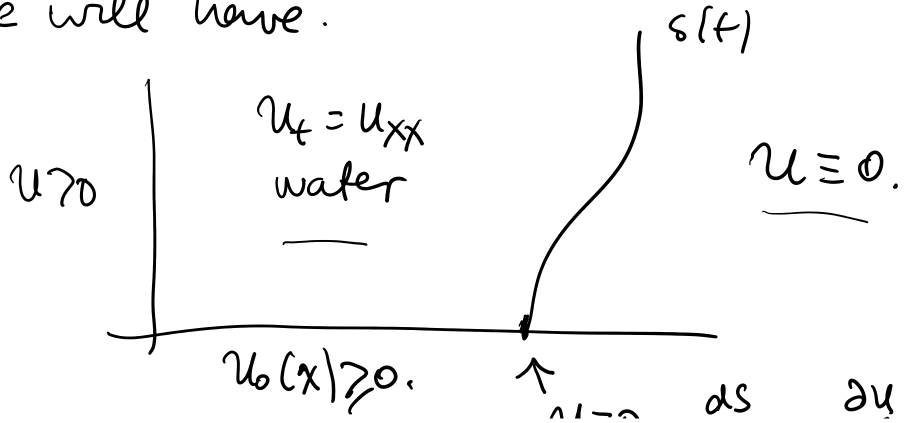
But the influence of $s(t)$ on $u_\pm(x,t)$ is nonlinear, so the whole problem is nonlinear.

In fact, all free/moving boundary value problems are nonlinear.

Simplifications.

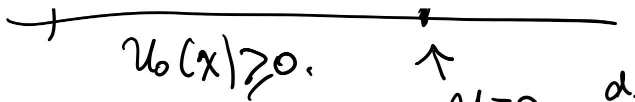
It is consistent to make a "one-sided" problem with $u \equiv 0$ in the ice region.

We will have.



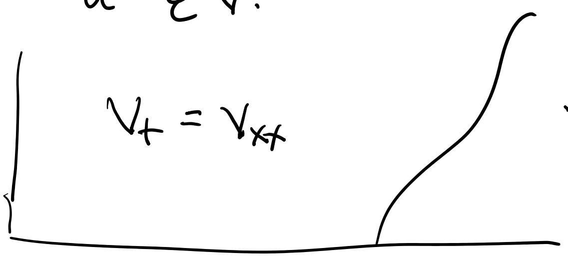
model
1
one sided.

$u|=0$ $\quad ds \quad \partial u$



In the scaling of u, u will end up "small" in many application problems.

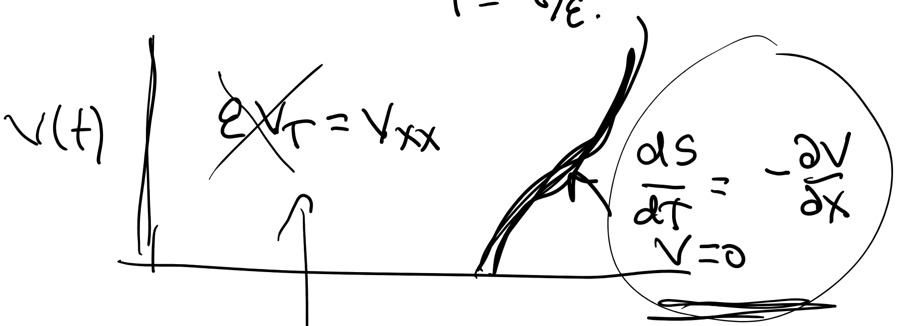
$u = \varepsilon V$.



$$\frac{ds}{dt} = -\varepsilon \frac{\partial V}{\partial x}.$$

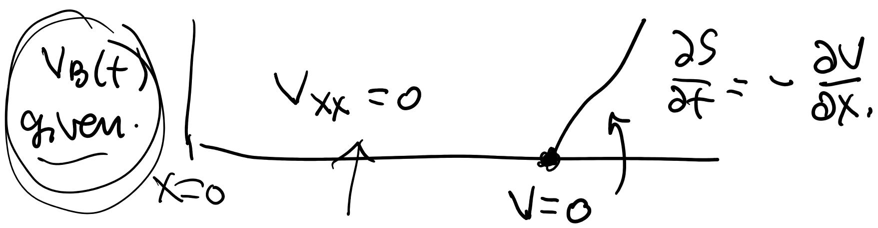
free boundary moves slowly.
scale time so we capture this slow motion.

$\tau = t/\varepsilon$.



At this time scale, this equation reaches steady state (quasi-steady).

change notation back to t.



Model
E1
epsilon?

solve analytically.

$$V(x,t) = V_B(t)\left(1 - x/s(t)\right).$$

$$v(x,t) = v_B(t)\,(1 - x/s(t)).$$

$$\frac{\partial v}{\partial x} = -v_B(t)/s.$$

$$\frac{ds}{dt} = -\frac{\partial v}{\partial x} = \frac{v_B(t)}{s}.$$ ← ODE for $s(t)$.

← nonlinear problem for s.

These can generalize to higher dimensions.

(T)

two D.

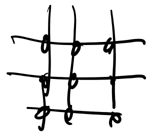
water

$u_t = \Delta u.$

$u_{xx} + u_{yy}$

ice.

$u_t = \alpha \Delta u.$



normal velocity

$$V = \left[\frac{\partial u}{\partial n}\right] = \frac{\partial u}{\partial n}\Big|_+ - \frac{\partial u}{\partial n}\Big|_-$$

$x(s,t),\ y(s,t)$

The Stefan problem emerges when solidification (melting) occurs due to boundary forcing.

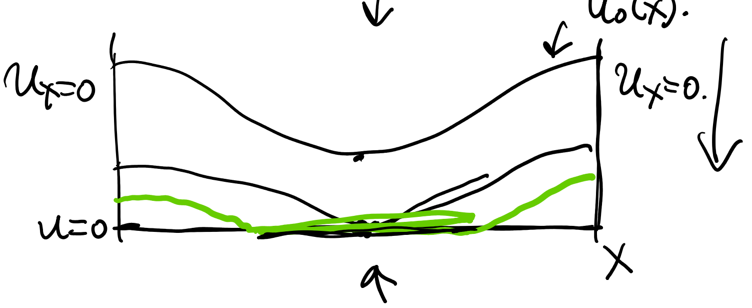
Now let's look at interior forcing.

start with domain all water $u > 0$.

$$u_t = u_{xx} + \cancel{F(x,t)} \quad \leftarrow \quad F(x,t) < 0. \quad (1)$$

$-1.$

(cooling down water).



t increasing.

freezing start here.

ice will form, fraction of

meeting start here.

ice will form, fraction ϕ

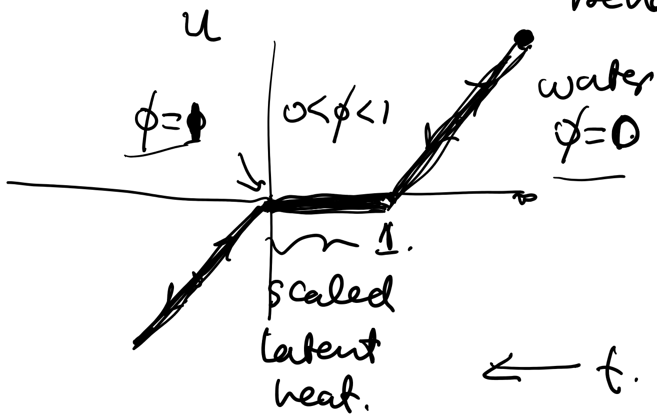
$0 < \phi < 1$

pure water

pure ice

"mushy" region mix of ice & water.

Thought experiment — start with water $u > 0$ and cool it uniformly. Stir it so it is uniform behaviour in space.



remove heat (scale to be at a unit rate). in time.

Enthalpy. (Scaled) $E = u + \phi$.

post process

$$\phi(E) = \begin{cases} 1 & E < 0 \\ 1-E & 0 < E < 1 \\ 0 & E > 1. \end{cases}$$

Modify the temperature equation.

$$E_t = u_{xx} + f(x,t) \qquad (2).$$

To proceed, we need an expression for $u(E)$.

$$u(E) = \begin{cases} E & E < 0. \\ 0 & 0 < E < 1. \\ E-1 & E > 1. \end{cases}$$

We have $E_t = (u(E))_{xx} + f(x,t)$ (3).

Expect weak solutions since $u(E)$ is continuous but $u'(E)$ is not.

Expect weak solutions since $u(E)$ is continuous but $u'(E)$ is not.

Sensible solutions come from volumetric heating/cooling to mushy regions formation & transitions to regions of pure ice / water.

Capturing computational method based on this formulation.

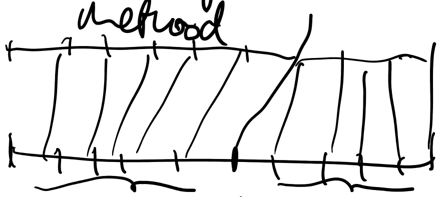
This formulation can also handle our original Stefan problem.

water $\qquad\qquad\qquad\qquad$ ice.

$E \geq 1 \qquad E=0 \mid E_t = 0 \qquad E \leq 0.$

---

tracking method



$s(t)$.

↑

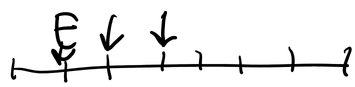
an unknown.

$$\frac{ds}{dt} = \left[\frac{\partial u}{\partial n}\right].$$

↑

implement this.

Capturing method.



$$E_t = (u(E))_{xx}$$

Stefan problem is an example of an explicit free boundary value problem.

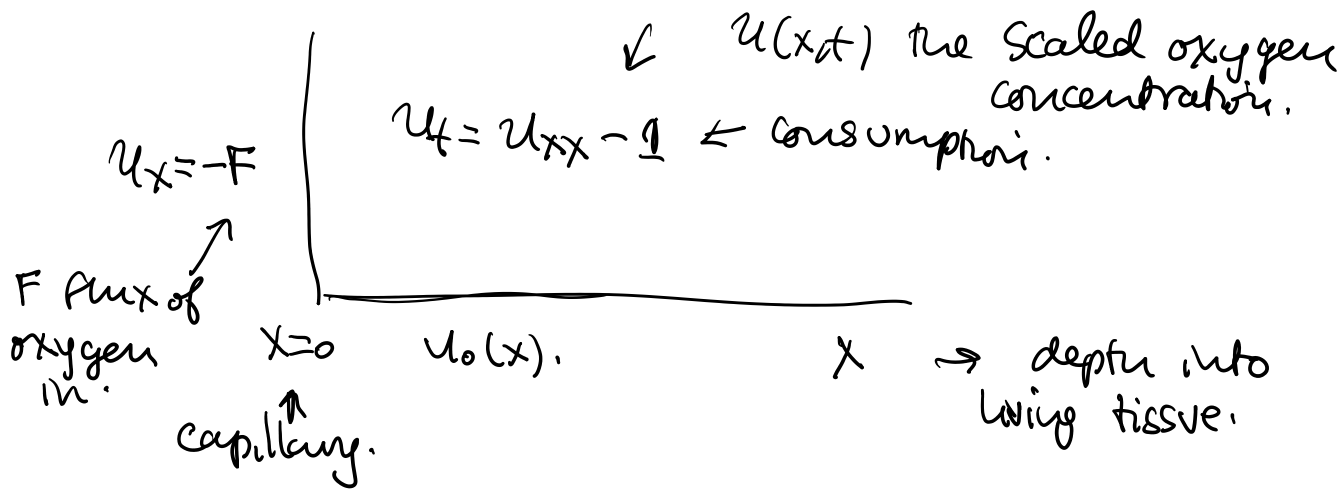
$V = \left[\frac{\partial u}{\partial n}\right]$. velocity given

$V = \left[\frac{\partial u'}{\partial u}\right]$. velocity given explicitly.

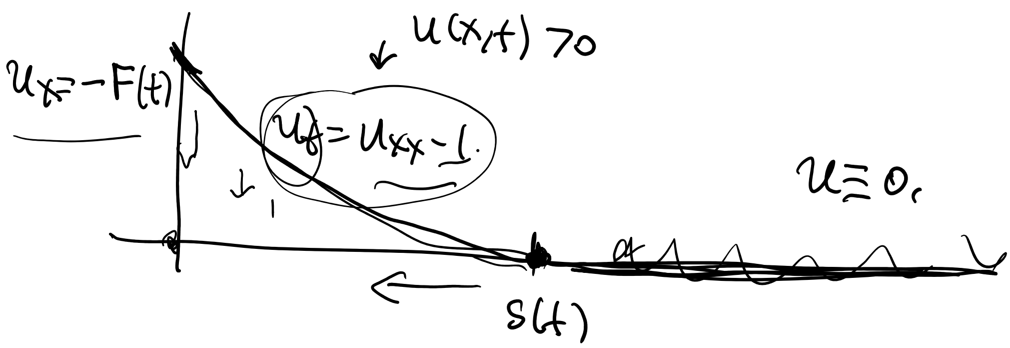
Implicit Free boundary value problem.

there is an extra condition at the free boundary that determines the motion implicitly.

Oxygen Depletion Problem.



u is constrained to be ≥ 0.



Here $u(s(t),t) = 0$. ✓ } one BC for u

$u_x(s(t),t) = 0$ ✓ } the other implicitly determines $s(t)$.

Ex Let's determine the steady state $u(x)$ when $F=1$.

$u'' - 1 = 0 \Rightarrow u = \frac{1}{2}x^2 + A + Bx$. / work out

$u'(0) = $

$$u'(0) = -1 \Rightarrow B = -1,$$

$$u(s)=0 \Rightarrow \frac{1}{2}s^2 + A - s = 0,$$

$$u'(s)=0 \Rightarrow s-1=0$$

$s=1$ $A=1/2$

work out A & s

$$u(x) = \frac{1}{2}(x-1)^2.$$

Lecture #21

March 30, 2021 9:18 AM

Oxygen Depletion problem.

Implicit free boundary problem - no interface velocity expression.

$u=0,\ u_x=0.$

consumed.

$u_x=-F$ $\quad u_t = u_{xx} - 1$

$x=S(t)$ free (moving.) boundary.

$u\equiv 0.$

$x=0$ $\quad u_0(x).$ $\quad x \to$ tissue depth

F flux of oxygen into tissue

Ex 1 $F=1$ found steady state $u(x)=\frac{1}{2}(x-1)^2$

$S(0)=1.$

Consider the 0D solution $u(x,t)$.

Look at $V(x,t)=u_t(x,t)$

problem that this solves?

Take $F(t)=F$ constant.

$V_x=0.$ $\quad V_t = V_{xx}$ $\quad V=0.$

$V(x,0)=u_0''-1.$ $\quad \underline{x=S(t)}.$

$u, \boxed{u_x=0} \to u(S(t),t)=0.$

$\frac{d}{dt}$: $\dot S\, \underbrace{u_x(S(t),t)}_{\text{zero } \checkmark} + \underbrace{u_t(S(t),t)}_{V(S(t),t)} = 0.$

$\Rightarrow$ $\underline{V=0}$ on free boundary.

$\underline{V} = u_t = u_{xx}-1.$

$u_{xx}=1$ on free boundary.

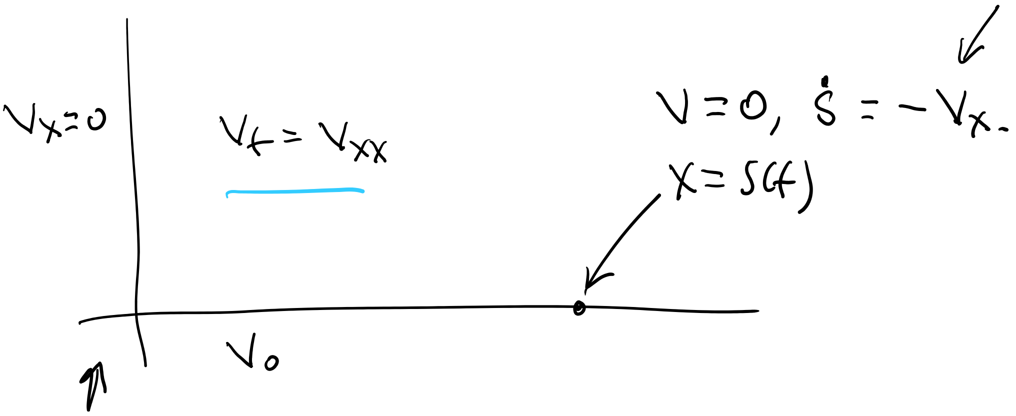
$\underline{u_x(S(t),t)=0}$

$\frac{d}{dt}$: $\underbrace{\dot S\, u_{xx}(S(t),t)} + u_{tx}(S(t),t)=0.$

$V_x(S(t),t).$

$$\underbrace{s\, u_{xx}(s(t),t)}_{1} + u_{tx}(s(t),t) = 0.$$ on free boundary.

$$\dot{s} = -v_x(s(t),t).$$

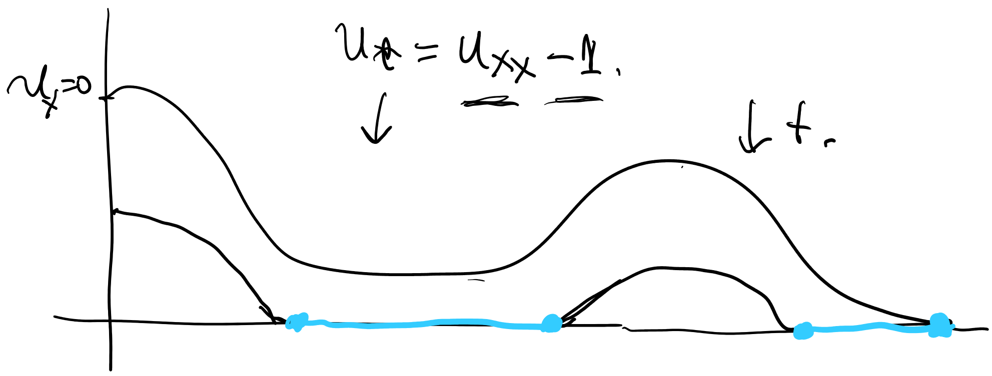


solve for $v(t)$, recover $u(t)$ with

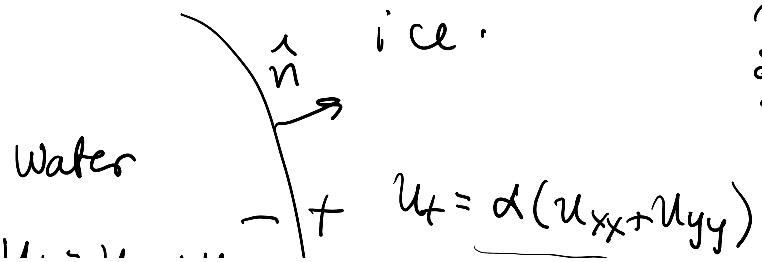
$$u(x,t) = u_0(x) + \int_0^t v(x,s)\,ds.$$

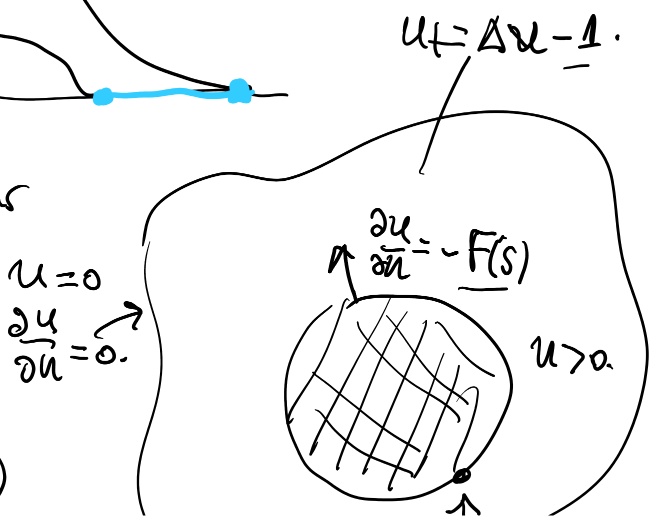
Mathematically interesting, but not a good computational formulation (in general).

Because it does not recognize changes in topology of the $u>0$ set.



Problems in higher dimensions





$u_t = u_{xx} + u_{yy}$

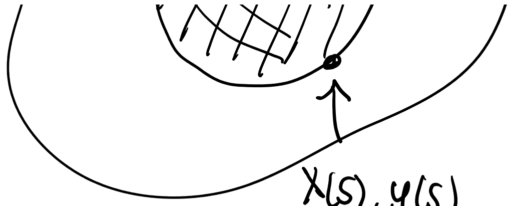
Δu.

$u_t = \alpha(u_{xx} + u_{yy})$

explicit normal velocity $V = \left[\frac{\partial u}{\partial n}\right]$.

$u_t = 0$.

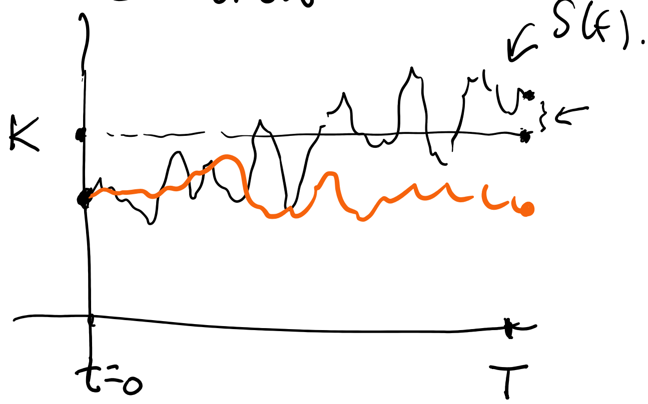
$\frac{\partial u}{\partial n}\Big|_+ - \frac{\partial u}{\partial n}\Big|_-$.



Wraps up Free Boundary Value Problems.

Black - Scholes Equations from Math Finance.
Used to price stock options.

Stock price $S(t)$ modelled as a stochastic process.
Modeled with geometric growth and geometric Brownian motion.



A stock option (European) gives the right but not the obligation to buy or sell shares at the strike price K at time T.

A call option buyer can buy shares at the strike price at time T

A put option buyer can sell shares at the strike price at time T.

Option sellers must execute the option if the buyer requests it.

Want to know $u(S,t)$.

↑
expected value over all random paths of the value of the option for given S and at time t.

Seller wants to know $u(S(0), 0)$. to value the option competitively.

Buyer wants to know $u(S(t), t)$ $\quad 0 < t < T$.
in case they want to sell the option before the execution time T.

Known information.

$$u(S,T) = \begin{cases} \max\{S-K, 0\}. & \text{call option.} \\ \max\{K-S, 0\}. & \text{put option} \end{cases}$$

let's only consider call options from now on.

$u(0,t) = 0 \quad \leftarrow$ company went Bankrupt.

u obeys the Black-Scholes Equation. $u(S,t)$.

$$u_t + \frac{1}{2}\sigma^2 S^2 u_{SS} + rSu_S - ru = 0$$

variance of the geometric Brownian motion.

risk free investment rate.

convective term.

$$u_t = -\frac{1}{2}\sigma^2 S^2 u_{SS} - rSu_S + ru$$

exponential growth

$u_t = -u_{SS}$ ill posed? (ill posed increasing time
well posed decreasing time)

Scale $S = sK \Rightarrow$ strike price ~~K=1~~. $s = 1$.
(don't change the equation).

Scale $t = \tau T \Rightarrow$ execution time is $T = 1$.

$\sigma^2 \to \sigma^2 T$

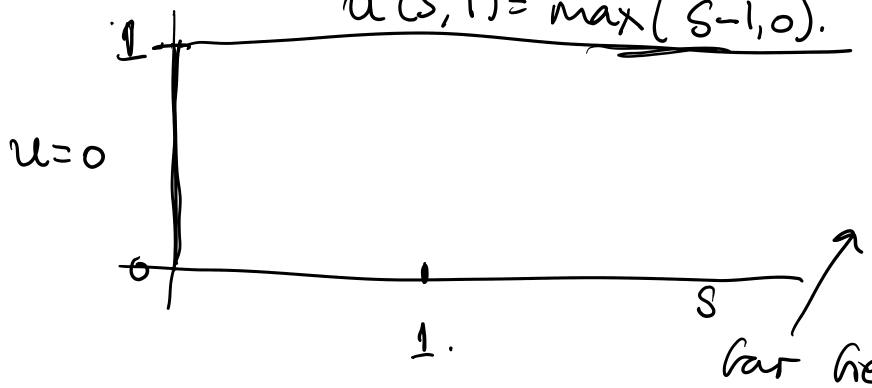
... is $T-1$.

$$\sigma^2 \to \sigma^2 T$$
$$r \to r\,T.$$

Scale $u \to u/K$.

$u(S,1) = \max(S-1, 0)$.

go back to t instead of T.



far field condition? $S \to \infty$.

I took $\frac{\partial u}{\partial S} = 1$.

Can make analytic progress on this problem.
(sequence of transformations).

I made a numerical approximation
(MATLAB code is posted).

Lecture #22

April 1, 2021 9:20 AM

Last topic of the term:

Reaction diffusion Equations.

Scalar $u(x,t)$

$$\underline{u_t} = \underbrace{u_{xx}}_{\text{diffusion}} + \underbrace{f(u)}_{\text{reaction term.}}$$

involves u but not derivatives of u. does not depend on x or t.

Nonlinear!

Cauchy problem $u(x,0) = u_0(x)$ given for all x.

We will later look at vector problems.

$$\underline{u}_t = \mathbb{D}\, \underline{u}_{xx} + \underline{f}(\underline{u})$$

symmetric, change coordinates & scale $\mathbb{D} = \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix}$.

write in components $\underline{u} = (u, v)$.

$$u_t = u_{xx} + f(u,v)$$
$$v_t = \underline{D v_{xx}} + g(u,v).$$

For scalar case, 2 generic examples.

I $f(u) = u(\underline{1-u})$. Fisher's Equation.

$f(u) = u(1-u)$ — exponential growth; $(1-u)$: Population limit due to resources.

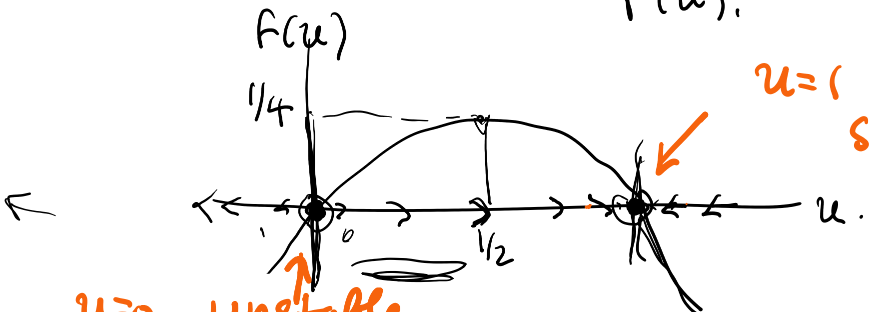
II $f(u) = -\frac{1}{\varepsilon^2}(u^3 - u)$ Allen-Cahn Equation.

Forget diffusion for now, just consider the reaction.

I $u(t)$ $\dot{u} = \underbrace{u(1-u)}_{f(u)}$ Logistic equation.

$f(u)$

$f(u)$.



$u=1$ equilibrium. stable.

$u=0$ unstable equilibrium.

$f'(u) = 1-2u.$

$f'(1) = \underline{-1} < 0$

stable.

exponential behaviour near $x=1$.

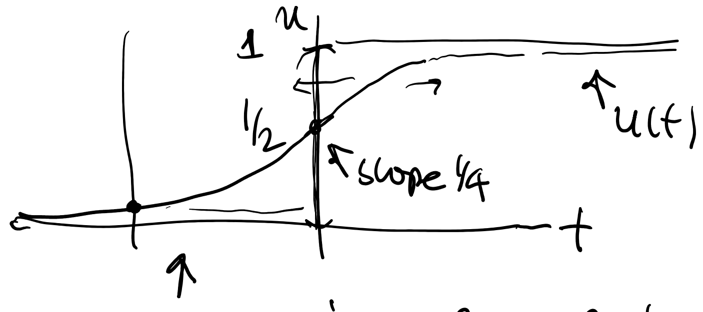
Near $u \approx 1$, $u \approx \underline{1 + ce^{-t}}$.

$f'(0) = 1$, near $u \simeq 0$, $u \approx Ce^{t}$.

Closed form solution $u(0) = u_0$.

$$u(t) = \frac{1}{\left(1 + \left(\frac{1}{u_0} - 1\right)e^{-t}\right)}$$

From the Biological application $u \in [0,1]$.



Because the system is Autonomous, $\underline{u(t-T)}$ is also a solution.

This one curve gives all solutions for $u_0 \in \underline{(0,1)} \Rightarrow u(t) \in (0,1)$ for all t.

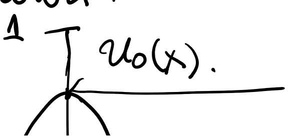
Let's put the diffusion back in.

$$\underline{u_t} = \underline{u_{xx}} + \underline{u(1-u)}. \quad u \in (0,1) \quad (3) \quad \cancel{f(u)=u^2}$$

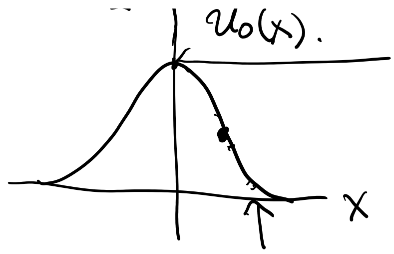
for all x & t.

population redistribution by diffusion.

local population growth.



$u_{xx} < 0$ here, limit the



$u_{xx} < 0$ here, limit the local growth

$u_{xx} > 0$, enhance the local growth.

Note: If $0 \le u_0(x) < 1$. for all x. and $u_0(x)$ is not identically zero. Then $0 < u(x,t) < 1$. for all x and $t > 0$.

At each time t, consider

$$M(t) = \max_x u(x,t).$$

attained * at $x(t)$. At $(x(t),t)$ $u_{xx} \le 0$.

$$\frac{dM}{dt} \le M(1-M).$$

so $M(t)$ lie below the logistic growth curve, so $M(t) < 1$.

Consider a travelling wave solution to ~~the~~ (3)

$t_2 > t_1$.

travelling wave?

$\downarrow$

profile $u(\xi)$

$u(x-ct)$ solves (3).

$$-cu' = u'' + \overbrace{u(1-u)}^{f(u)}.$$

du

solve with $u \to 1$ as $\xi \to -\infty$

$u \to 0$ as $\xi \to +\infty$.

To make the solution unique (translationally invariant)

add the condition $u(0) = 1/2$.

Note: we will only find solutions for $c \geq 2$.

To warm up, let's look at the solutions as $\xi \to \pm\infty$.

$\xi \to +\infty$ $\quad$ $u \to 0$. $\quad$ $f(u) \approx u$. $\quad$ $f'(0) = 1$.

$$u'' + cu' + u \approx 0. \qquad u = e^{r\xi}.$$

$$r^2 + cr + 1 = 0$$

$$r = -\frac{c}{2} \pm \sqrt{\frac{c^2}{4} - 1}.$$

$(c \geq 2)$.

2 decaying modes.

If $c < 2$. oscillations around $u=0$ (negative values) not physical.

slowest decaying mode

$$r_0 = -\frac{c}{2} + \sqrt{\frac{c^2}{4} - 1}.$$

$$u \approx D e^{r_0 \xi},$$

$\xi \to -\infty$ $\quad$ $u = 1 + v$, $v \to 0$. $\quad$ $f(u) \approx -v$.

$$v'' + cv' - v \approx 0 \qquad \text{linearization.}$$

$$r^2 + cr - 1 = 0 \qquad r = -\frac{c}{2} \pm \sqrt{\frac{c^2}{4} + 1}$$

$r^2 + cr - 1 = 0 \qquad r = -\frac{c}{2} \pm \sqrt{\frac{c^2}{4} + 1}$

one negative
one positive.

$$r_1 = -\frac{c}{2} + \sqrt{\frac{c^2}{4} + 1}.$$

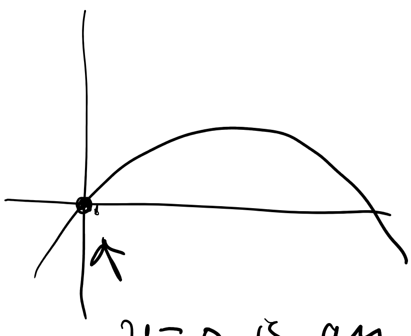
$$u \to 1 + E e^{r_1 \xi}.$$

To make progress, we introduce the phase plane.
Introduce $w = u'$.

$$u' = w$$
$$\underset{\substack{\uparrow \\ u''}}{w'} = -cw - u(1-u).$$

RHS does not depend on t.

plot in the u,w plane, the arrows that represent the vector direction RHS.



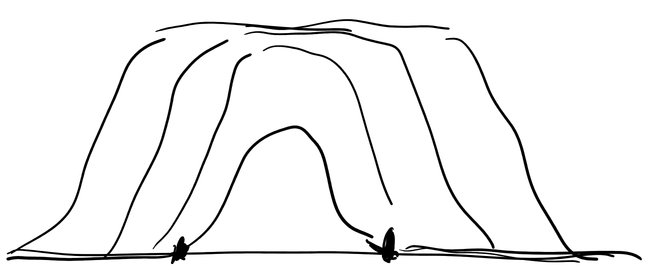
$u=0$ is an equilibrium.

$\underline{u \equiv 0.}$ ← for all time.

$$\dot{u} = \underbrace{u(1-u)}_{f(u)}.$$

"near" $u \equiv 0$. $\quad \underline{f(u) \approx \cancel{f(0)} + f'(0)\,u.}$ Equilibrium. 1.

$$\dot{u} \simeq u.$$
$$\Downarrow$$
$$u(t) \simeq \underbrace{Ce^t}.$$



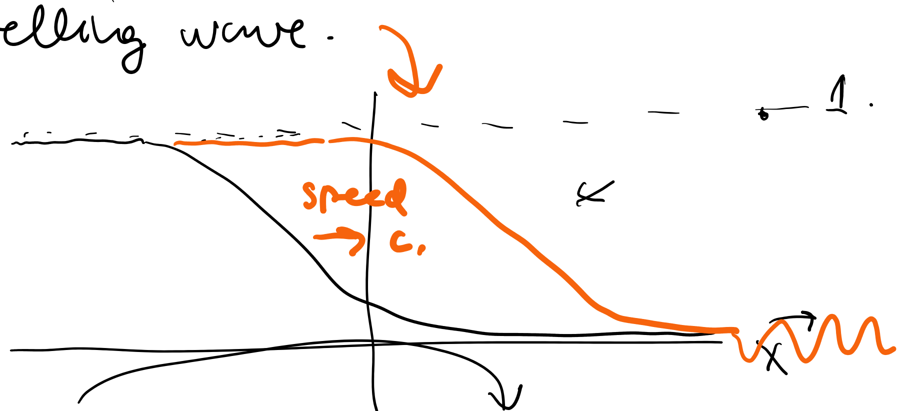
Lecture #23

April 6, 2021 9:11 AM

$u_t = u_{xx} + u(1-u)$. Fisher KPP Equation.

population redistribution by diffusion. local population growth

Travelling wave.



$u(\xi) \to u(x,t) = u(x-ct)$.

$-cu' = u'' + u(1-u)$.

with $u \to 1$ as $\xi \to -\infty$

$u \to 0$ as $\xi \to +\infty$

Last time, linearized $\xi \to \pm\infty$.

$-\infty$ $u \approx 1 + \boxed{De^{r_1\xi}}$

$+\infty$ $u \approx Ee^{r_0\xi}$

$r_1 = -\frac{c}{2} + \sqrt{\frac{c^2}{4}+1}$.

$r_0 = -\frac{c}{2} + \sqrt{\frac{c^2}{4}-1}$.

$c<2$. oscillations around 0 ← unphysical ($u<0$). someplaces.

As $c \to \infty$

$r_0 \to 0_-$

large c

Know $c \geq 2$.

Turn to phase plane analysis.

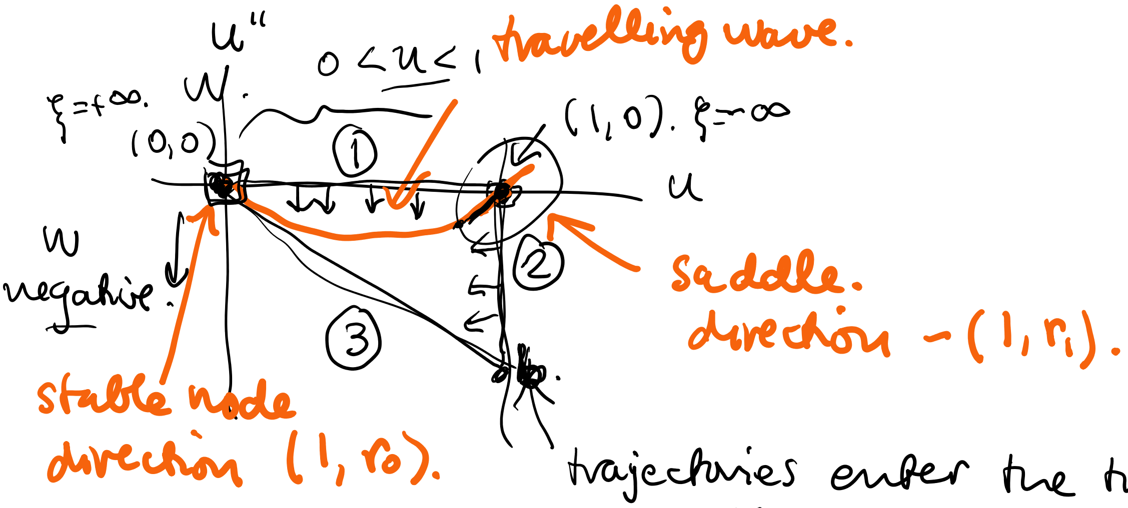
Introduce $w = u'$.

$$\begin{cases} u' = w \\ w' = -cw - u(1-u). \end{cases}$$

RHS does not depend on time.

$\{\ w' = -cw - u(1-u).$

u''

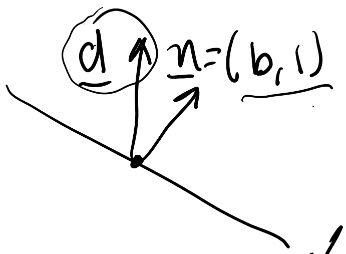


trajectories enter the triangle on all sides.

on ① $w = 0$ $\quad 0 < u < 1 \quad w' = -u(1-u) < 0.$

② $u = 1 \quad u' = w < 0$ on this side.

③ Taking $b > 0$. on the line normal direction $\underline{n} = (b, 1)$.



$-\left[\left(b - \frac{c}{2}\right)^2 + \frac{c^2}{4} - 1\right]u$

RHS of the ODE

$w = -bu.$ $\qquad -b^2u + cbu - u + \cancel{u^2}$

trajectory direction $\underline{d} = (-bu, cbu - u + u^2)$.

For the result that the trajectories point into the triangle, we need to show $\underline{n} \cdot \underline{d} \geq 0$

$$\underline{n} \cdot \underline{d} = \cdots > \left[-\left(b - \frac{c}{2}\right)^2 + \frac{c^2}{4} - 1\right]u \geq 0$$

if $b = \frac{c}{2}$ and $c \geq 2$.

So a travelling wave exists for every $c \geq 2$.

Some (deep) theory shows that these waves are stable (*).

Some (deeper) theory shows that if u_0 has compact support ($u_0 \equiv 0$ outside a finite interval) then $c = 2$ is the wave that results.

then $C=2$ is the wave that results.

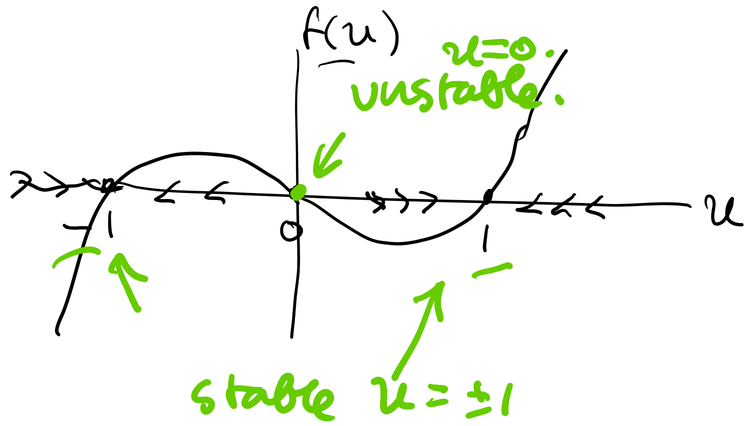
Allen - Cahn Equation.

$$u_t = u_{xx} - f(u) \qquad f(u) = \frac{1}{\varepsilon^2}(u^3 - u).$$

$$= \frac{1}{\varepsilon^2} u(u+1)(u-1).$$

Cauchy Problem
$u_0(x)$ given for all x

Reaction term $\dot{u} = -f(u)$.



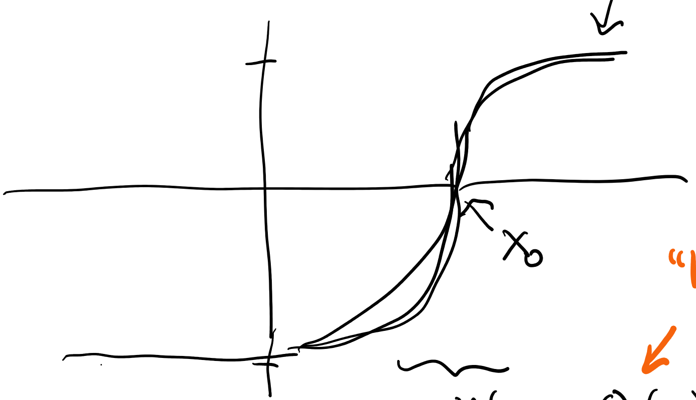
Now add diffusion back in.

No travelling waves, but there is a steady state solution.

parameter.

$$u(x) = \tanh\left(\frac{(x-x_0)}{2\varepsilon}\right).$$

u(x) steady state

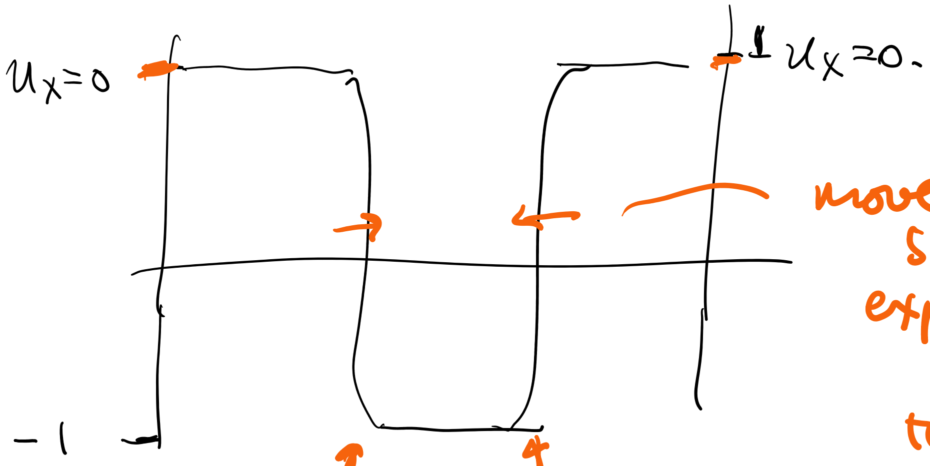


width $O(\varepsilon)$.

"big O" notation
$O(\varepsilon) \Leftarrow$ a quantity $\approx C\varepsilon$

Start from general u_0, u will quickly tend to regions of $u \approx \pm 1$ separated by diffuse interfaces of width $O(\varepsilon)$

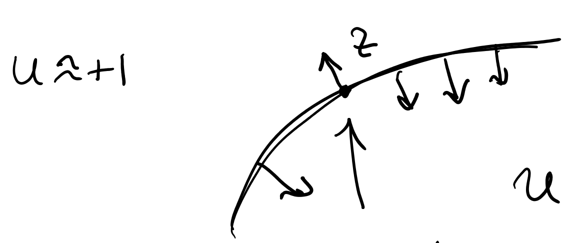
regions of $u \approx \pm 1$ separated by diffuse interfaces of width $O(\varepsilon)$.



move very slowly exponentially slowly time scale $O(e^{-c/\varepsilon})$.

approx tanh profiles.

In 2D & 3D, solutions



} diffuse interface width $O(\varepsilon)$.

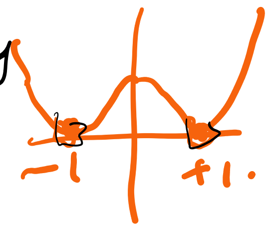
$u \approx -1$.

$u \approx \tanh\left(\frac{z}{2\varepsilon}\right)$

Interfaces move in $O(1)$ time, move with curvature motion (mean curvature in 3D).

A-C is a gradient flow on the Energy

$$\underbrace{\mathcal{E}(u)}_{u(x)} = \int_{-\infty}^{\infty} \left\{ \underbrace{\tfrac{1}{2}(u_x)^2}_{\text{interfacial energy}} + \underbrace{W(u)}_{\text{chemical energy.}} \right\}.$$



$W(u) = \frac{1}{4}(u-1)^2(u+1)^2$

$W'(u) = f(u) = u^3 - u$.

Consider $\underline{u(x,t)}$ solution of AC. $\underline{u(\cdot,t)}$

$\mathcal{E}(u(\cdot,t)) =$

$\dfrac{d\mathcal{E}}{dt} = \int_{-\infty}^{\infty} \{ u_x u_{xt} + W'(u) u_t \} \, dx$

$$\frac{d\mathcal{E}}{dt} = \int_{-\infty}^{\infty} \{ u_x u_{xt} + W'(u) u_t \} \, dx$$

Integrate by parts. assume $u_x \to 0$ at $x = \pm\infty$.

$$= \int_{-\infty}^{\infty} \{ -u_{xx} + W'(u) \} \underline{u_t} \, dx \qquad u_{xx} - f(u)$$

$$u_{xx} - W'(u).$$

$$= \int_{-\infty}^{\infty} (-u_{xx} + W'(u)) (u_{xx} - W'(u)) \, dx$$

$$= -\int_{-\infty}^{\infty} (u_{xx} - f(u))^2 dx \le 0.$$

So $\mathcal{E}(u(\cdot,t))$ decreases for solutions of AC.

This is an infinite dimensional version of finite dimensional gradient flows.

$\underline{u}(t) \in \mathbb{R}^n \qquad \underline{f}(\underline{u}) = -\nabla \underline{\Phi}(\underline{u})$ scalar function.

then if $\underline{u}$ solves $\dot{\underline{u}} = \underline{f}(\underline{u})$.

$$\frac{d\,\underline{\Phi}(\underline{u}(t))}{dt} = \nabla\underline{\Phi} \cdot \frac{d\underline{u}}{dt} = \nabla\underline{\Phi} \cdot (-\nabla\underline{\Phi}) = -|\nabla\underline{\Phi}|^2.$$

For such a system, $\underline{\Phi}(\underline{u}(t))$ decreases.

Other examples of energy estimates are in the notes.

$u_t = u_{xx}$ ← with homogeneous Dirichlet or Neumann BC's on a finite interval.

multiplying by u and integrating.

$$\int u u_t = \int^1 u u_{xx}$$

Add a Numerical

$$\int_0^1 u u_t = \int_0^1 u u_{xx}$$

Not a physical energy.

$$\frac{1}{2}\frac{d}{dt}\int_0^1 u^2 dx = -\int_0^1 (u_x)^2 dx.$$

Now I know $\boxed{\int_0^1 u^2(x,t)\,dx}$ decreasing in time.

Lecture #24

April 8, 2021 9:20 AM

Last lecture!

Turing instability

↑

Reaction Diffusion Systems (Pattern Formation).

Turing Observation: If $\underline{u} = \underline{0}$ is a stable (asymptotically) equilibrium of 2 dynamical systems M_1 & M_2. The origin is not necessarily stable for the system $M_1 + M_2$.

$\dot{\underline{u}} = M_1(\underline{u})$

$\lim_{t \to \infty} \underline{u}(t) = \underline{0}$

for all $\underline{u_0}$ in a neighborhood of $\underline{0}$.

$\dot{\underline{u}} = M_2(\underline{u})$

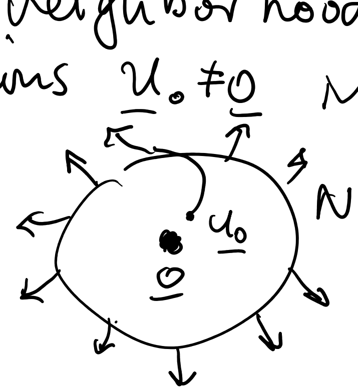
$\lim_{t \to \infty} \underline{u}(t) = \underline{0}$

for all $\underline{u_0}$ in a neighborhood of $\underline{0}$.

There are examples where solutions of

$$\dot{\underline{u}} = M_1(\underline{u}) + M_2(\underline{u})$$

all leave a neighborhood of $\underline{u} = \underline{0}$ for all initial conditions $\underline{u_0} \neq \underline{0}$ in that neighborhood.



Need at least 2 components.

Examples in Reaction Diffusion Equations

$$\left.\begin{aligned} u_t &= D_1 u_{xx} + f(u,v) \\ v_t &= D_2 v_{xx} + g(u,v). \end{aligned}\right\} \quad (1).$$

Diffusion coefficients $D_1, D_2 > 0$.

M_1.

Suppose reaction has a stable equilibrium at u_0, v_0.

$$f(u_0, v_0) = 0$$
$$g(u_0, v_0) = 0.$$

and $\mathbb{J} = \left.\begin{pmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \end{pmatrix}\right|_{u_0,v_0}$ ← has 2 eigenvalues with negative real parts.

Jacobian matrix

$\lambda_1 + \lambda_2 = \operatorname{trace}\mathbb{J} < 0$

$\lambda_1\lambda_2 = \det\mathbb{J} > 0.$

Note: Consider $\underline{u} - (u_0, v_0) \xrightarrow[\text{relabel}]{} \underline{u}$.

Equilibrium $\underline{u} = \underline{0}$.

After that shift, linearize (1).

$$u_t = D_1 u_{xx} + \left.\frac{\partial f}{\partial u}\right|_{\underline{u}=\underline{0}} u + \left.\frac{\partial f}{\partial v}\right| v.$$

$$v_t = D_2 v_{xx} + \left.\frac{\partial g}{\partial u}\right| u + \left.\frac{\partial g}{\partial v}\right| v.$$

or in vector form $\underline{u} = \begin{pmatrix} u \\ v \end{pmatrix}$.

$$\underline{u}_t = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix} \underline{u}_{xx} + \mathbb{J}\,\underline{u}.$$

$\uparrow \qquad M_1 \qquad M_2.$

Constant coefficient Cauchy problem - analyze with Fourier series.

" Consider solutions of the form"

$$\underline{u}(x,t) = e^{i\alpha x} e^{\lambda t} \underline{u}$$

constant vector (eigenvector).

$$\lambda \cancel{e^{\lambda t}} \cancel{e^{i\alpha x}} \underline{u} = \cancel{e^{\lambda t}} \cancel{e^{i\alpha x}} \begin{pmatrix} -D_1 \alpha^2 u \\ -D_2 \alpha^2 v \end{pmatrix} + \cancel{e^{\lambda t}} \cancel{e^{i\alpha x}} J \underline{u}.$$

$\begin{pmatrix} u \\ v \end{pmatrix}$.

$$\left[J - \begin{pmatrix} +D_1\alpha^2 & 0 \\ 0 & D_2\alpha^2 \end{pmatrix} \right] \underline{u} = \lambda \underline{u}.$$

$$A(\alpha) = \begin{bmatrix} f_u - D_1\alpha^2 & f_v \\ g_u & g_v - D_2\alpha^2 \end{bmatrix}.$$

If $\mathrm{Re}\,\lambda_1(\alpha) > 0$ or $\mathrm{Re}\,\lambda_2(\alpha) > 0$. Then system is unstable.

We are looking for a Turing instability example.

$A(0)$ should have eigenvalues with negative real parts.

$\uparrow$ original J

trace < 0

det > 0.

and $A(\alpha)$ should have at least one eigenvalue with positive real part for some α.

Need f_u & g_v to have opposite signs.

Need g_u & f_v also have to have opposite

Need g_u & f_v also have to have opposite signs.

$$J = \begin{pmatrix} 1 & 1 \\ -2.1 & -2 \end{pmatrix}.$$
$\uparrow$
$\mathbb{A}(0)$

$\lambda_1 \approx -0.1127$
$\lambda_2 \approx -0.8873.$

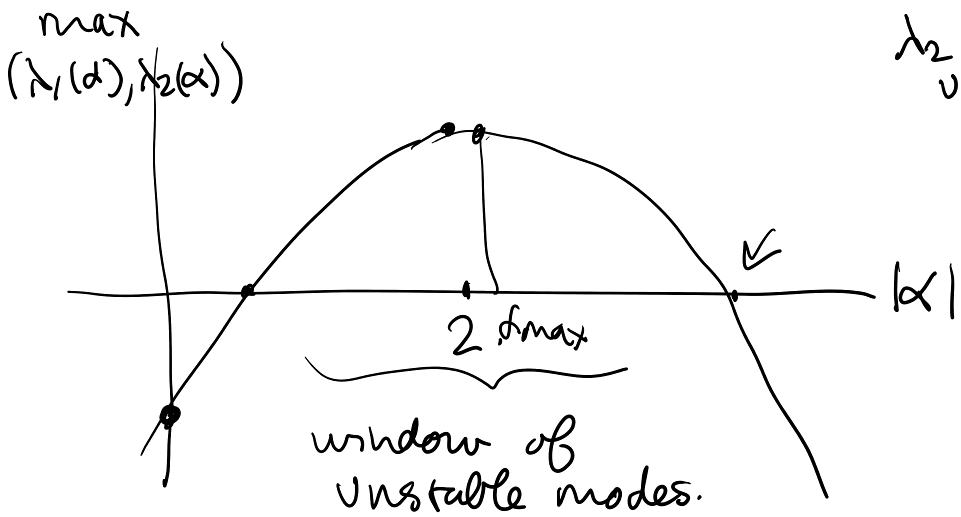
$D_1 = 0.1$, $D_2 = 2$, $\alpha = 2$

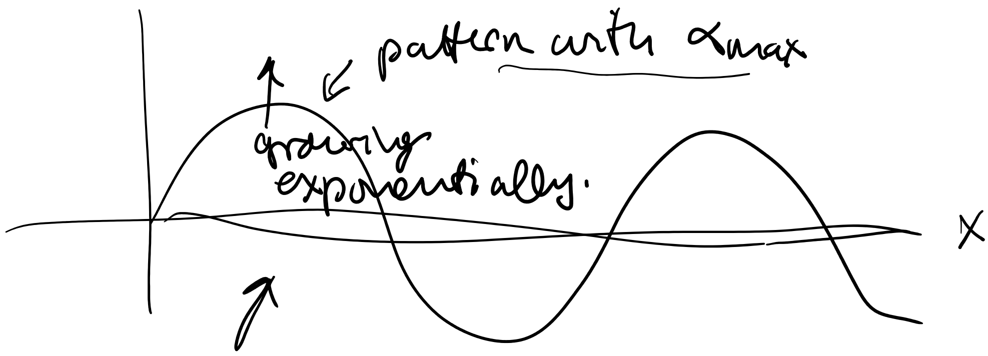
$\dot{u} = u + v$
$\dot{v} = -2.1u - 2v.$

linearized reaction.

$$\mathbb{A}(2) = \begin{pmatrix} 1 - 0.1(2)^2 & 1 \\ -2.1 & -2-2(2)^2 \end{pmatrix} = \begin{pmatrix} 0.6 & 1 \\ -2.1 & -10 \end{pmatrix}$$

$\lambda_1 \approx -9.7980$
$\lambda_2 \approx 0.3980 > 0$
unstable.





If you put the nonlinear terms back in, patterns saturate at a certain amplitude (analagous to the logistic equation).