Math 400 Outline
Section 101, Fall, 2012

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Focus: This class concentrates on analytic methods to solve partial differential equations (PDE's) coming from physical applications.

Topics:

1. Review: linearity; ordinary differential equations; wave, heat and Laplace equations; separation of variables. Introduction to: maximum principle; D'Alembert's solution; scaling and non-dimensionalization; well-posedness; weak solutions; asymptotic methods; numerical methods.
2. Classification of equations
3. Linear and quasi-linear first order equations, shock waves
4. Eigenfunction expansions and Sturm-Liouville theory
5. Parabolic (heat), elliptic (Laplace) and hyperbolic (wave) equations
6. Solution by integral transforms

Text: No required text. Optional text: "Elementary Applied Partial Differential Equations," by Richard Haberman. Handwritten notes for the course will be posted online.

Marks: 50% final, 20% midterm (Friday, October 19), 30% assignments.

Assignments: Challenging weekly assignments. Assigned Fridays, due Mondays (after ten days). Late homework will not be accepted after solutions are posted.

Math 400 Notes, Part I

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① A Note on Assignments.

Assigned Fridays, due Mondays (in 10 days). There is overlap on the assignments. Solutions will be posted in the afternoon of the due date. No late assignments will be accepted after this. Some assignment questions will be challenging, do not leave them to the last minute. It is OK to work together, within reason.

② Big Picture.

Analytic methods to solve partial differential equations (PDE's) will be considered in this course. An unknown function of two or more variables, $u(x,t)$, satisfies a relationship involving its partial derivatives at every point x and t, ie.

$$F(u, \overset{\partial u/\partial x}{u_x}, u_t, x, t) = 0 \qquad \text{[first order PDE]}$$

$$G(u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, x, t) = 0$$

[second order PDE].

The PDE describes some law (physical, fiscal, chemical, biological...) that the solution must obey.

Typically, also initial and/or boundary data is given. The PDE and this data together is called a PDE <u>problem</u> (an initial value problem or a boundary value problem).

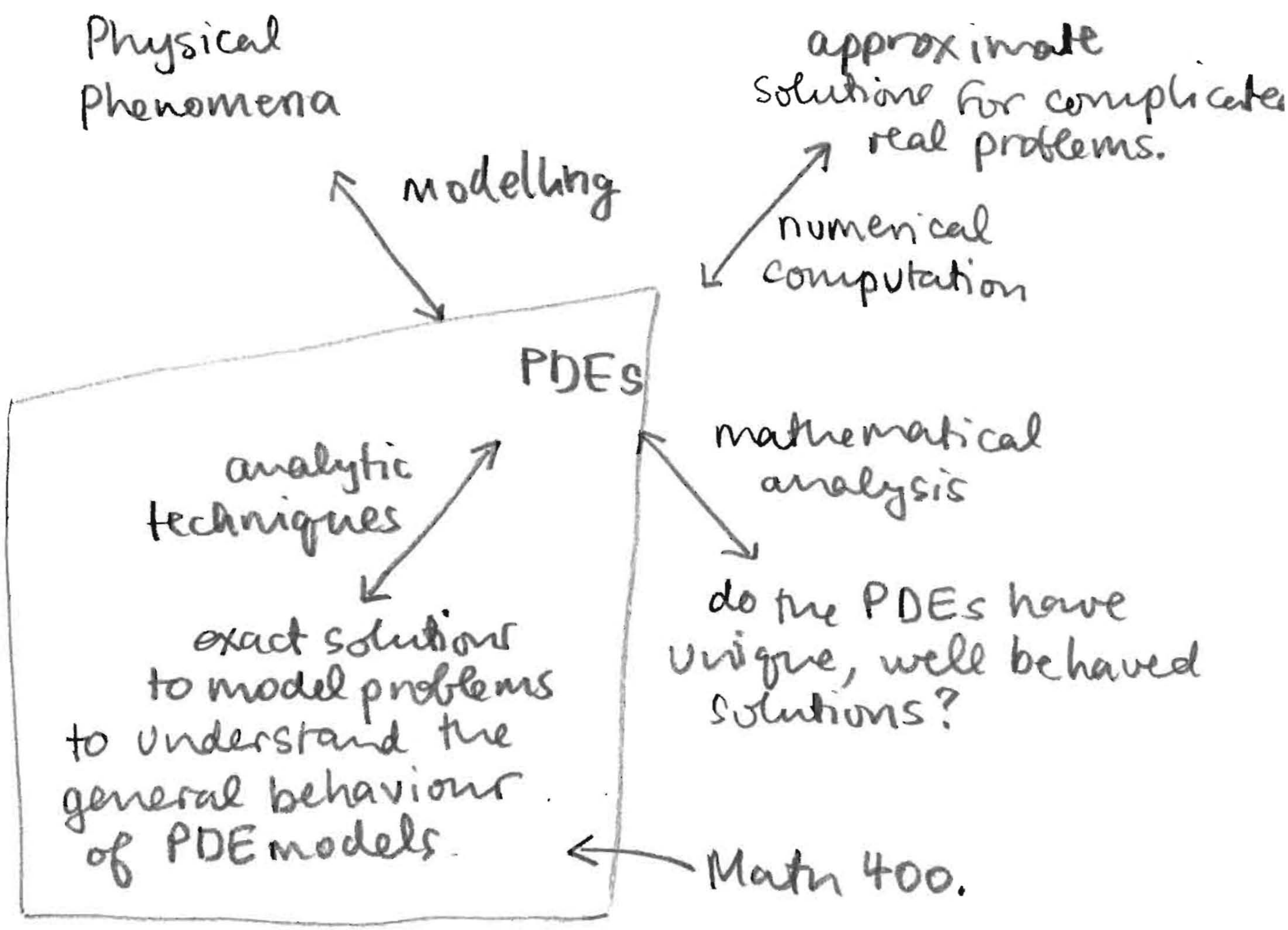
A function $u(x,t)$ that satisfies the PDE and the data is a <u>solution</u> of the PDE problem.

Our questions:

1. When do PDE problems have (unique) solutions?
2. How we find solutions when they exist (or at least approximate solutions or some properties of the solutions)?

③ Even Bigger Picture

Consider some physical phenomena of interest, like air flow around an airplane wing or temperature in a nuclear reactor or financial stock prices. The behaviour of these quantities can be described (approximately) by PDE's (that specify mass, momentum and energy conservation for example). This process is called Mathematical Modelling. Analytic or approximate numerical solution of these equations then gives insight into the phenomena. This is easier, cheaper, faster, and safer than doing experiments in many cases, and can give more complete information (ie. the pressure all around the airplane wing design, not just at experimental sensor locations).



Note that the arrows above do not go just one way. Understanding the properties of solutions with mathematical analysis can help develop suitable numerical approximation methods, properties of solutions to model problems can help in the modelling process, etc.

Let's <u>Review</u> some more basic problems you've seen over the years and remind ourselves of some concepts, like linearity.

4 <u>Scalar Problems</u>

So in Math 400, we want to solve for $u(x,t)$, an unknown <u>function</u> of <u>two</u> variables. Let's consider here the <u>much</u> simpler problem of finding a scalar unknown x.

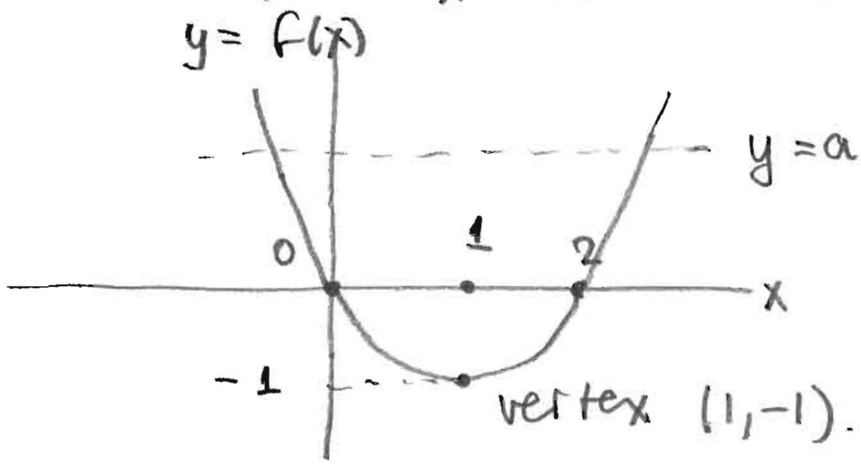
$$f(x) = a. \tag{1}$$

a: consider a to be the forcing term of a system

x is the unknown response to the forcing

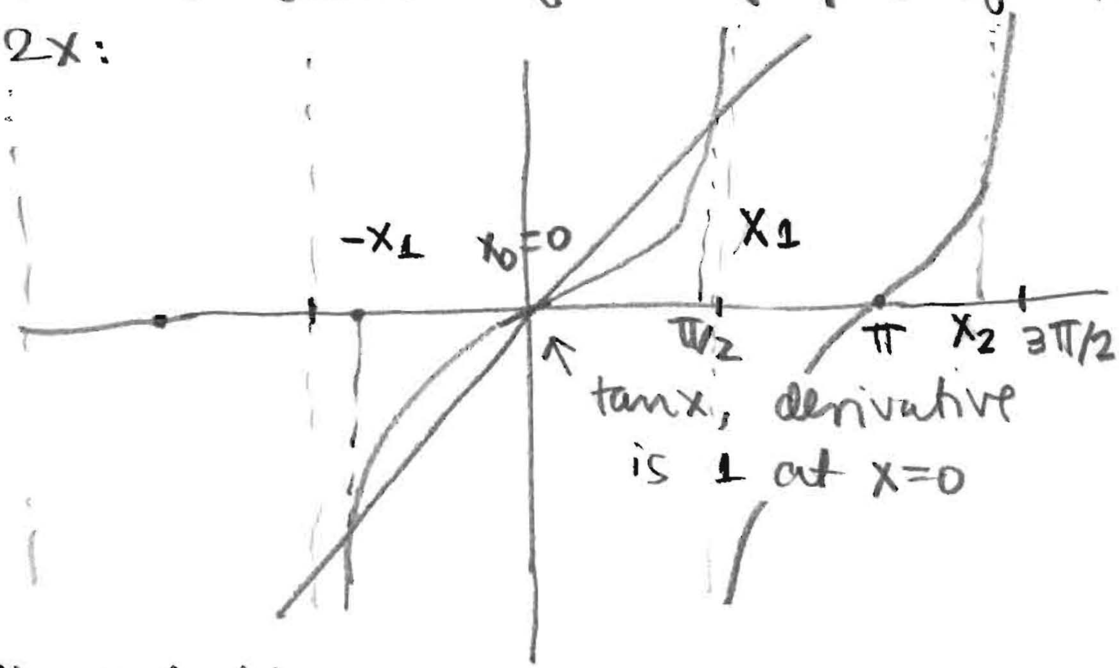
<u>Note</u>: If f is an invertible function, then $x = f^{-1}(a)$.

Example 1 $\quad f(x) = x^2 - 2x$. Can get both graphical and algebraic insight into (1) in this case.



If $\begin{cases} a < -1 & \text{there are no (real) solutions.} \\ a = -1 & \text{there is one solution, } x = 1 \\ a > -1 & \text{there are two solutions,} \\ & x = 1 \pm \sqrt{1+a} \end{cases}$

Example 2 $\tan x - 2x = 0$. Consider graphically as an intersection of the graphs of $\tan x$ and $2x$:



Infinite solutions, x_0, $\pm x_1$, $\pm x_2 \ldots$

$$\lim_{n \to \infty} x_n = (n - \tfrac{1}{2})\pi$$

To find x_1? Approximate with Newton's method.

In this setting (1), the idea of linearity is trivial, but let's introduce it here anyway. We say (1) is a <u>linear</u> system if the following is true for all x and y and scalar c.

(i) $f(x+y) = f(x) + f(y)$
(ii) $f(cx) = cf(x)$.

Think $f(x) = a$, $f(y) = b$.

Consider (ii) read backwards. It states that if you multiply the system input a by c then the response is also multiplied by c. (i) says that for linear systems the response to forcing $a+b$ is the response to a plus the response to b.

Note: (i) and (ii) are equivalent to

$$f(c_1 x + c_2 y) = c_1 f(x) + c_2 f(y)$$

for all x, y and scalars c_1 and c_2.

Note: (ii) with $c=0$ shows that linear systems necessarily satisfy $f(0) = 0$, i.e. no response is always consistent with no forcing.

For this scalar case, linearity restricts systems to the form $f(x) = dx$ for d a given scalar. (i) Has a unique solution for every a if $d \neq 0$.

Note: Scalar functions of the form $dx + e$ are called affine functions. Although their graphs are lines, they are not linear functions when $e \neq 0$.

5 Vector Problems

Consider now responses $\underline{x}$ (vector with n components) to inputs $\underline{a}$ (also vectors in $\mathbb{R}^n$), subject to n conditions

$$\underline{f}(\underline{x}) = \underline{a} \qquad \text{[consider } \underline{f}, \underline{x}, \text{ and } \underline{a} \text{ to be column vectors]} \qquad (2)$$

For the system to be linear, $\underline{f}(\underline{x}) = \mathbb{A}\underline{x}$ where $\mathbb{A}$ is an $n \times n$ matrix, and so (2) is a linear system in this case.

In what follows, assume $\mathbb{A}$ is invertible, so

$$\mathbb{A}\underline{x} = \underline{a} \qquad (3)$$

has a unique solution for every $\underline{a}$. Let $\underline{x} = \underline{b}_i$ be the response to the i'th unit vector, i.e.

$$\underline{a} = \underline{e}_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i\text{th column}$$

so $\mathbb{A}\underline{b}_i = \underline{e}_i$. For general $\underline{a} = (a_1, \cdots a_n)^T$, we have trivially

$$\underline{a} = \sum_{i=1}^{n} a_i \underline{e}_i$$

so by linearity, (3) is solved by

$$\underline{x} = \sum_{i=1}^{n} a_i \underline{b}_i$$

which can be written as

$$\underline{x} = \mathbb{B}\,\underline{a}.$$

matrix with vectors $\underline{b}_i$ in the columns

That is $\mathbb{B} = \mathbb{A}^{-1}$. Familiar? (I hope so)

Note: We can solve (3) for every $\underline{a}$ using just a finite number (n) of solutions.

Suppose now $\mathbb{A}$ is symmetric, still invertible.

Theorem: If $\mathbb{A}$ is $n \times n$ symmetric then there is an ortho-normal set of eigenvectors $\{\underline{v}_i\}$ with real eigenvalues of $\mathbb{A}$. That is,

$$\mathbb{A}\,\underline{v}_i = \lambda_i \underline{v}_i \qquad \text{(eigenvectors with eigenvalue } \lambda_i\text{)}$$

$$\|\underline{v}_i\| = 1 \qquad \text{(normalized)}$$

$$(\underline{v}_i, \underline{v}_j) = 0 \qquad i \neq j \text{ (orthogonal).}$$

Think of $\{\underline{v}_i\}$ as a new coordinate system. Let's write $\underline{a}$ in terms of this coordinate system,

$$\underline{a} = \sum_{i=1}^{n} c_i \underline{v}_i = \mathbb{V}\,\underline{c} \Rightarrow \underline{c} = \mathbb{V}^{-1}\underline{a}$$

The matrix $\mathbb{V}$ has the eigenvectors $\underline{v}_i$ in the columns. Since the columns are orthonormal (so linearly independent) $\mathbb{V}$ is invertible.

We assumed $\mathbb{A}$ was invertible so $\lambda_i \neq 0$ for all i

so $\quad \mathbb{A}\left[\left(\frac{1}{\lambda_i}\right)\underline{v}_i\right] = \underline{v}_i.$

Using linearity, (3) is solved by

$$\underline{X} = \sum_{i=1}^{n} \frac{c_i}{\lambda_i} \underline{v_i}$$

Note: Again, with this approach we can solve (3) for every $\underline{a}$ using just n solutions.

⑥ Summary of ④ and ⑤

Even for simple problems, questions of the existence and uniqueness of solutions is not straightforward. For linear problems, these questions are easier, and general solutions can be found with far less work.

⑦ Some review of Ordinary Differential Equations (ODEs)

Equations for an unknown function of one variable $[u(x)$ or $u(t)]$ and its derivatives.

$$F\left(u, \frac{du}{dt}, t\right) = 0$$ [general first order ODE]. to be satisfied at all t

$$G\left(u, \frac{du}{dt}, \frac{d^2u}{dt^2}, t\right) = 0$$ [general second order ODE].

$$\frac{du}{dt} + c(t)u = a(t)$$ [first order, linear]

$$\frac{du}{dt} + f(u,t) = 0$$ [firstorder, quasi-Linear] (4

Quasi-linear means that the nonlinearity does not enter into the highest derivative terms.

<u>Theorem</u> The ODE problem (4) with initial conditions $u(0) = u_0$ has a unique solution in an open interval containing $t=0$ if $f(u,t)$ is continuously differentiable (C_1) in a neighborhood of $(u_0, 0)$.

<u>Note</u>: Conditions on f can be weakened somewhat.

<u>Example 3</u> Consider the following linear, constant coefficient, first order ODE problem for $u(t)$:

$$\left.\begin{aligned} &\frac{du}{dt} + u = \underbrace{2 + \sin t}_{\text{forcing}} \\ &u(0) = \underbrace{1}_{\text{initial data}}. \end{aligned}\right\} \quad (5)$$

Could describe for example $u(t)$ the amount of radioactive material at a site, with some there initially ($u(0) = 1$) and some added continuously at the time varying rate $(2 + \sin t)$. The ODE (5) describes the time rate of change of u, with the rate added and the rate of radioactive decay of u.

We could divide this problem into two problems for u_1 and u_2 with $u = u_1 + u_2$:

$$\frac{du_1}{dt} + u_1 = 0 \qquad\qquad \frac{du_2}{dt} + u_2 = 2 + \sin t$$

$$u_1(0) = 1 \qquad\qquad u_2(0) = 0.$$

Can verify directly that if u_1 and u_2 solve these problems, then $u = u_1 + u_2$ solves (5). We can find analytic solutions:

$$u_1(t) = e^{-t} \qquad \text{[trivial, linear homog cc, separable].}$$

$$u_2(t) = 2 + \tfrac{1}{2}\sin t - \tfrac{1}{2}\cos t - \tfrac{3}{2}e^{-t} \qquad \text{[linear, method of undetermined coefficients]}$$

In words, u_1 is the result of the initial amount with no further addition and u_2 is the result starting from no initial amount but with the deliveries. Solution

$$u(t) = 2 + \tfrac{1}{2}\sin t - \tfrac{1}{2}\cos t - \tfrac{1}{2}e^{-t}$$

solution because $u(t)$ satisfies the initial cond. and the ODE (5) at every t

(an example of the old rule "homogeneous plus particular").

Note that the division of the problem into two parts is similar in spirit to the first part of (5).

<u>Example 4</u> Consider the following linear, constant coefficient second order homogeneous initial value ODE for $u(t)$:

$$\left.\begin{array}{ll} \underset{\frac{d^2u}{dt^2}}{\ddot{u}} - u = 0 & u(0) = 1 \\ & \dot{u}(0) = 0. \end{array}\right\} (6)$$

Solution $u(t) = \frac{1}{2}\left(e^{t} + e^{-t}\right) = \cosh t$

<u>Recall</u> Analytic solutions can be found for any linear, c.c., homogeneous ODE. They are exponentials (including

trig functions), possibly multiplied by polynomials in special cases (resonance).

Note: We could also solve for u_2 that satisfied

$$\ddot{u}_2 - u = 0 \qquad u_2(0) = 0, \ \dot{u}_2(0) = 1$$

giving $u_2(t) = \frac{1}{2}(e^t - e^{-t}) = \sinh t$. Now if we wanted to solve for $y(t)$:

$$\ddot{y} - y = 0 \qquad y(0) = y_0, \ \dot{y}(0) = y_1$$

it is $y(t) = y_0 \cosh t + y_1 \sinh t$, again using ideas like the first part of 5.

Note that (6) and any higher order quasi-linear ODE can be easily converted to a first order ODE system. This is necessary in order to use standard numerical ODE approximation methods (which are designed for first order systems) and also allows basically the same arguments used to prove the Theorem on p. 10 stated for the scalar case to apply also to higher order equations.

To convert (6) to a first order system, introduce a new quantity $V(t)$ that we want to be $\frac{du}{dt}$ for each t. Now

$$\frac{du}{dt} - V = 0 \qquad \text{[makes } V \text{ what we want]}.$$

$$\frac{dV}{dt} - u = 0 \qquad \text{[original ODE]}$$

$$u(0) = 1, \quad V(0) = 0.$$

$$\uparrow \ \frac{d}{dt}\left(\frac{du}{dt}\right) = \ddot{u}$$

Consider now a vector, first order, c.c., homogeneous problem:

$$\dot{\underline{u}} = \mathbb{A}\,\underline{u} \qquad \text{(A)}$$

A an $n \times n$ matrix

$\underline{u}(t)$, a vector with n components

Reminder: The general solution of (A) is found by doing an eigenanalysis of $\mathbb{A}$. Suppose $\mathbb{A}$ has n linearly independent eigenvectors $\underline{v}_j$ with eigenvalues λ_j. [The formula below can be modified to handle the situation when $\mathbb{A}$ has fewer l.i. eigenvectors (resonance)]. Under the assumption, the general solution of (A) is

$$\underline{u}(t) = \sum_{j=1}^{n} c_j\,\underline{v}_j\,e^{\lambda_j t} \qquad \text{(B)}$$

It is clear that $\underline{u}(t)$ above solves (A) and the l.i. of $\underline{v}_j$ ensures that any initial data $\underline{u}(0) = \underline{u}_0$ can be matched, thus it is a general solution. In the case that the λ_j appear in complex conjugate pairs, a modified form of (B) with real entries involving trigonometric functions can be derived.

Example 5 Consider the following nonlinear (separable first order ODE problem for $u(t)$:

$$\frac{du}{dt} = u^2 \qquad u(0)=1.$$

Solution $u(t) = \frac{1}{1-t}$, defined only on $t \in (-\infty, 1)$, with $\lim_{t\to 1^-} u(t) = +\infty$. The formula for $t > 1$ (on the other side of the singularity from the initial data) has nothing to do with the ODE problem above.

Reminder: Nonlinear IVPs can have singularities in finite time.

Example 6 Consider the following linear, c.c., second order boundary value problem for $u(x)$.

Note: We're using x as the independent variable to make us think of a spatially varying rather than a time varying problem.

$$\left.\begin{array}{l} u(x), \quad x \in [0,1]. \\ \underbrace{u''}_{\frac{d^2u}{dx^2}} - u = f(x) \qquad u(0)=0, \ u(1)=0. \end{array}\right\} \quad (7)$$

Note: Can't use time-stepping numerical methods to approximate this problem, since $u'(0)$ is not given. These problems are approximated by values on a grid and a linear system is solved for these values, $\mathbb{A}\underline{U} = \underline{F}$, where $\mathbb{A}\underline{U}$ approximates $u''-u$ on the grid, using the boundary values.

Think of $\frac{d^2}{dx^2} - \mathbb{I}$, (with homogeneous boundary conditions) as an infinite dimensional matrix (operator).

$\uparrow$ identity

It is symmetric and has a basis of orthogonal eigenfunctions $\varphi_j(x)$. This follows from a general theory (Sturm Liouville) that we will consider in more detail later in the course. What would these eigenfunctions look like?

$$\varphi_j'' - \varphi_j = \lambda_j \varphi_j \qquad x \in [0,1], \quad \varphi_j(0) = \varphi_j(1) = 0.$$

$$\varphi_j'' - (1+\lambda_j)\varphi_j = 0, \quad \varphi_j'' = (1+\lambda_j)\varphi_j$$

This is a problem you've seen before, when doing separation of variables in a previous class. Solutions are

$$\varphi_j(x) = \sin(\pi j x) \qquad j = 1, 2, \dots$$

then $-(1+\lambda_j) = j^2\pi^2 \Rightarrow \lambda_j = -j^2\pi^2 - 1$.

$\{\varphi_j\}$ is an orthogonal set in the sense that

$$\int_0^1 \varphi_i(x)\varphi_j(x)\,dx = \begin{cases} 0 & \text{if } i \neq j \\ \frac{1}{2} & \text{if } i = j \end{cases} \qquad (8)$$

We can now write a series solution of (7) as follows: first write $f(x)$ in terms of the eigenfunction basis (i.e. expand $f(x)$ in a sine series)

$$F(x) = \sum_{j=1}^{\infty} f_j \sin(j\pi x) \qquad (8b)$$

with $f_j = 2\int_0^1 F(x)\sin(j\pi x)\,dx$

< continued on p.15B

Theorem: If $f(x)$ is continuous, then the series in (8b) converges pointwise for every x except possibly $x=0,1$ (where the series always sums to zero).

Theorem: if $f \in L_2$ (measureable, square integrable) then equality holds in (8b) in the L_2 sense.

Note: These theorems show that $\{\varphi_j\}$ spans the set of functions of interest. This is a harder question in the infinite dimensional setting.

Note: The procedure above is analagous to the finite dimensional idea in (5).

(8) Wave Equations and D'Alembert's Solution.

Example 7 Consider $u(x,t)$ (finally, a PDE!) that satisfies the homogeneous, linear, c.c., first order PDE:

$$u_t + u_x = 0 \quad , \quad u(x,0) = u_0(x), \text{ given } \quad (9, W1)$$

Consider $x \in \mathbb{R}$ (no spatial boundaries) for now. This is known as a Cauchy problem.

The solution is $u(x,t) = u_0(x-t)$, a wave moving to the right. We call (9) the one-way wave equation (W1).

[after (8b) on p. 15]

Now if we write $u(x) = \sum_{j=1}^{\infty} u_j \sin(j\pi x)$ with the u_j to be determined and put this form into (7) we have

$$\sum_{j=1}^{\infty} u_j \underbrace{(-j^2\pi^2 - 1)}_{\lambda_j} \underbrace{\sin(j\pi x)}_{\varphi_j} = \sum_{j=1}^{\infty} f_j \sin(j\pi x)$$

(for all x). Since $\{\varphi_j\}$ are orthogonal we have

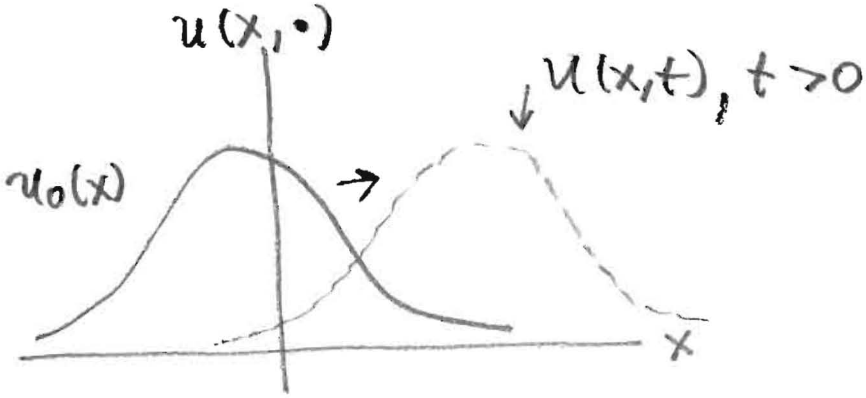
$$\lambda_j u_j = f_j \quad \Rightarrow \quad u_j = f_j/\lambda_j .$$

So now the series solution of (7) is

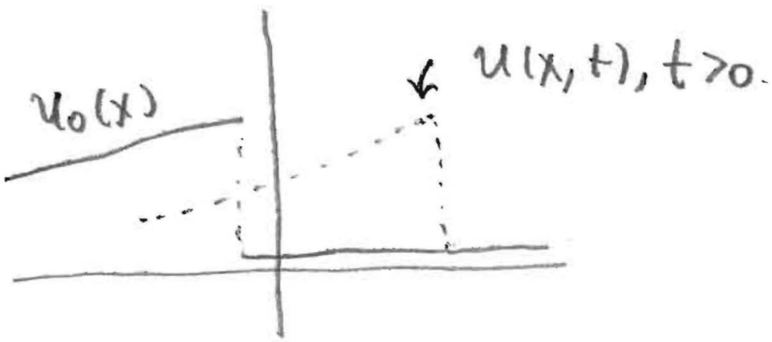
$$u(x) = -\sum_{j=1}^{\infty} \frac{f_j}{(1+j^2\pi^2)} \sin(j\pi x)$$

where f_j are given in (8b).

[return to p. 15].



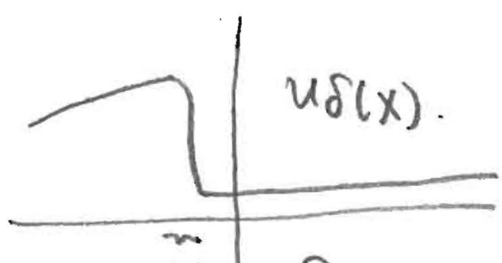
Note that we could have $u_0(x)$ discontinuous and make sense of the solution in this case.



☆

Such solutions that "make sense" but don't satisfy the PDE exactly (the solution above is not differentiable at all x,t so cannot strictly satisfy (9)) is called a weak solution.

Here, we can make sense of this solution in a limiting sense, by replacing $u_0(x)$ by a "smoothed" (mollified) version $u_\delta(x)$:



The solution $u_\delta(x,t)$ starting from initial data $u_\delta(x)$ does satisfy (9) at every point (a strong solution) and as $\delta \to 0$, $u_\delta(x) \to u(x)$ and $u_\delta(x,t) \to u(x,t)$ in figure ★ on the previous page. In this sense we make sense of the weak solution ★.

<u>Note</u>: Implicit in this discussion is the notion of well-posedness.

<u>Defn</u> A problem is said to be well-posed if all small changes in data (initial or boundary or forcing) lead to small changes in the solution.

otherwise, a problem is ill-posed, some small changes in data can lead to large changes in the solution. This is not reasonable.

<u>Note</u>: Part of the work to do rigorous PDE analysis is to figure out what norms to use for well-posedness results.

<u>Example 8</u> Consider the true (2 way) wave equation for $u(x,t)$, $x \in \mathbb{R}$ (no boundaries) (10, W2)

$$u_{tt} - u_{xx} = 0 \qquad \left.\begin{aligned} u(x,0) &= u_0(x) \text{ given} \\ u_t(x,0) &= 0 \end{aligned}\right\}$$

<u>Note</u>: Just part of a general problem in which we could have $u_t(x,0) \neq 0$ and also a non-zero RHS to the PDE.

Note that for any functions of one variable $f(\cdot)$ and $b(\cdot)$, $u(x,t) = f(x-t) + b(x+t)$ solves the PDE (10, W2). This is a general solution that is, we can find f & b to match any initial conditions $u_0(x)$:

$$u(x,0) = u_0(x) \Rightarrow f(x) + b(x) = u_0(x) \quad \text{(11a)} \quad \forall x$$

$$u_t(x,0) = 0 \Rightarrow -f'(x) + b'(x) = 0 \quad \forall x$$

integrate $\quad -f(x) + b(x) = C \quad$ — (11b)

(11) Can be solved at each x giving

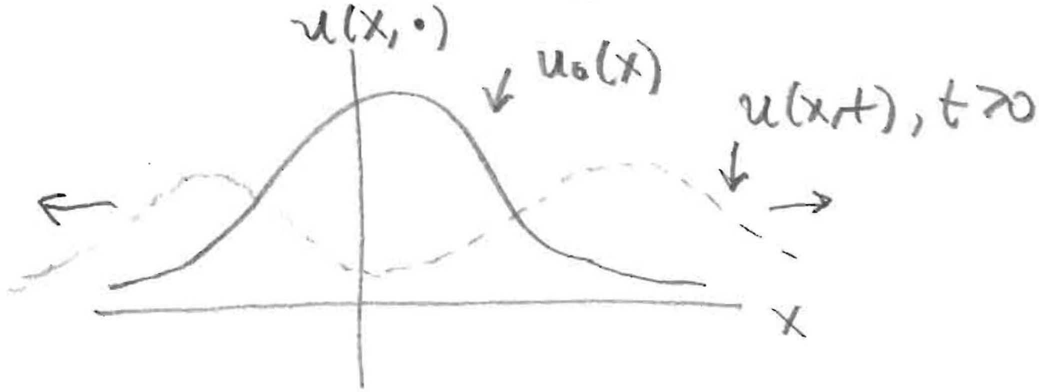
$$f(x) = \tfrac{1}{2}(u_0(x) - C)$$

$$b(x) = \tfrac{1}{2}(u_0(x) + C)$$

Note that the integration constant C does not appear in the final form

leading to $u(x,t) = \tfrac{1}{2}\left(u_0(x-t) + u_0(x+t)\right)$

That is, the initial condition $u_0(x)$ breaks into two equal pieces, one moving to the right and the other to the left.



(WE2) can be converted to a first order system for $v = u_t$ and $w = u_x$ as follows:

$$w_t - v_x = 0 \quad \text{(equality of mixed partials)}$$

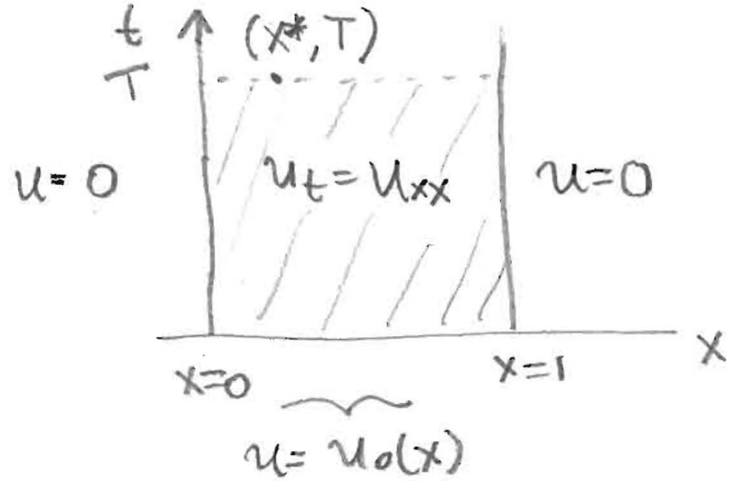
$$v_t - w_x = 0 \quad \text{(WE2)}$$

In this way it can be considered a vector analogy to (WE1).

9 Heat Equation

Example 9 $\quad u_t = u_{xx}, \quad x \in [0,1], \ t \geq 0$ with $\qquad$ (12, H)

$u(0,t) = 0$ & $u(1,t) = 0 \ \ \forall t, \quad u(x,0) = u_0(x). \ \forall x.$



Hint: Draw a diagram of PDE problems as above, showing initial and boundary data. This is a useful idea.

Problem (H) describes the evolution of temperature u in a rod of length 1 that is held at zero temperature (ice-water bath) at the ends.

We will write a series solution of problem (H) but even without knowing the solution we can prove some properties of it

Theorem (maximum principle) Let $M = \max\limits_{x \in [0,1]} \{u_0(x), 0\}$

The solution $u(x,t)$ to H satisfies

$$u(x,t) \leq M \quad \text{for all } x \text{ and } t.$$

Makes physical sense for the temperature evolution problem.

<u>Proof:</u> Consider $V = u + \varepsilon x^2/2$ in the shaded region of the diagram + on p. 19.

$$V_t - V_{xx} = \underbrace{u_t - u_{xx}}_{0 \text{ by } (12).} - \varepsilon < 0.$$

So the maximum of V cannot occur in the interior of the region since there $V_t = 0$, $V_{xx} \le 0$ (for a maximum), nor can it occur on any line $t = T > 0$ since there $V_{xx} \le 0$ and $V_t \ge 0$.

Thus the maximum of V must occur at either

(i) the left boundary where $V = 0$

(ii) the right boundary where $V = \varepsilon/2$

(iii) the initial line where $V \le M + \varepsilon/2$.

In all cases, $V \le M + \varepsilon/2$ $\quad (M \ge 0)$.

Now $u = V - \varepsilon x^2/2 \le V \le M + \varepsilon/2$. Letting $\varepsilon \to 0$ we have $u(x,t) \le M$ as desired.

<u>Note:</u> A similar argument shows that

$$u(x,t) \ge \min_{x \in [0,1]} \{ u_0(x), 0 \}$$

and so
$$|u(x,t)| \le \max_{x \in [0,1]} |u_0(x)|$$

<u>Cor:</u> Problem (H) is well-posed. Consider the solution $u(x,t)$ with initial data $u_0(x)$ and $V(x,t)$ with initial data $V_0(x)$. Then $W(x,t) = u - V$ solves (H) with initial data $u_0 - V_0$ and the

argument above then shows that

$$\left.\begin{aligned} |w(x,t)| &\le \max_{x\in[0,1]} |w_0(x)| \\ \Rightarrow\ |u(x,t)-v(x,t)| &\le \max_{x\in[0,1]} |u_0(x)-v_0(x)| \end{aligned}\right\} \quad \forall x\in[0,1],\ t\ge 0.$$

In words, the equation above says that small changes in initial data leads to small changes in the solution (well-posedness).

Cor: Solutions to (H) are unique.

Note: We haven't shown that (H) has a solution yet, only that if a solution does exist then it is unique.

Let's find a series solution for (H).

$$u_t = u_{xx} \qquad u(0,t) = u(1,t) = 0, \ u(x,0) = u_0(x)$$

If this were a problem for a vector $\underline{u}$ this stuff would be $A\underline{u}$ and we would proceed by finding the eigenvalues and eigenvectors of A. Instead of being a matrix it is a linear operator on an infinite dimensional space of functions. We know the eigenvalues and eigenfunctions of this operator

$$\lambda_j = -j^2\pi^2, \quad \varphi_j(x) = \sin j\pi x$$

and so the general solution to the PDE part of (H) is $u(x,t) = \sum_{j=1}^{\infty} C_j \varphi_j(x) e^{\lambda_j t}$

[check by plugging into (H)]

$$u(x,t) = \sum_{j=1}^{\infty} C_j \sin(j\pi x)\, e^{-j^2\pi^2 t} \qquad (13)$$

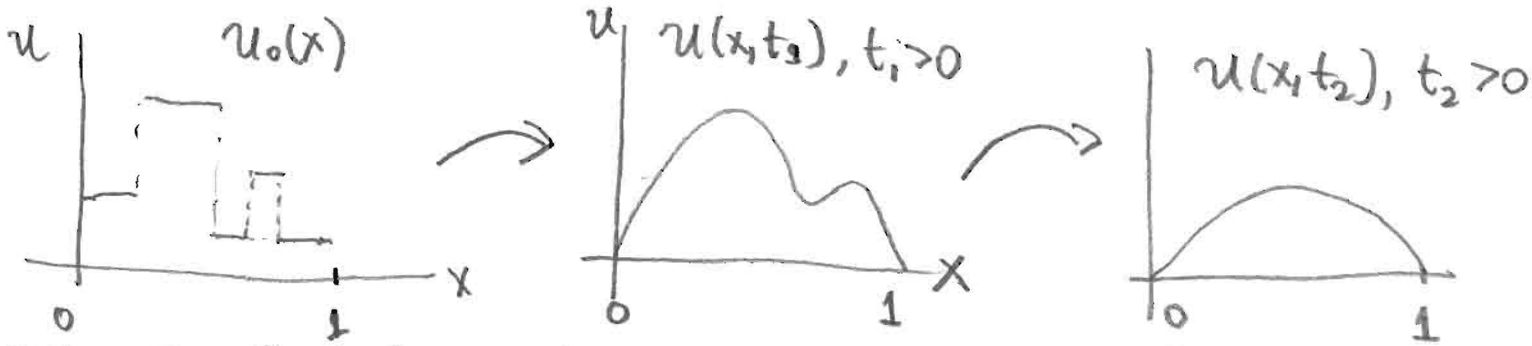
To match the initial conditions,

$$u(x,0) = \sum_{j=1}^{\infty} C_j \sin(j\pi x) = u_0(x)$$

The separation of variables approach you know presented in a new way

and so $C_j = 2\int_0^1 \sin(j\pi x)\, u_0(x)\, dx$, the sine transform coefficients of $u_0(x)$.

Note: The solution makes sense for u_0 discontinuous, so can handle weak solutions at $t=0$. Note that (13) shows that more oscillatory components (larger j, "higher wavenumber") of the solution decay more rapidly. Thus, solutions of (13) become smoother in time. This is shown qualitatively in the figure below. Analytically, it can be shown that $u(x,t)$ is infinitely differentiable (C_∞) for $t>0$.



Note: The first term of (13) is dominant as $t\to\infty$.

Example 10 Consider now the backward heat equation for $u(x,t)$, $x \in [0,1]$, $t \geq 0$:

$$u_t = -u_{xx} \qquad \underbrace{u(x,0) = u_0(x)}_{\forall x}, \ \underbrace{u(0,t) = u(1,t) = 0}_{\forall t} \tag{14}$$

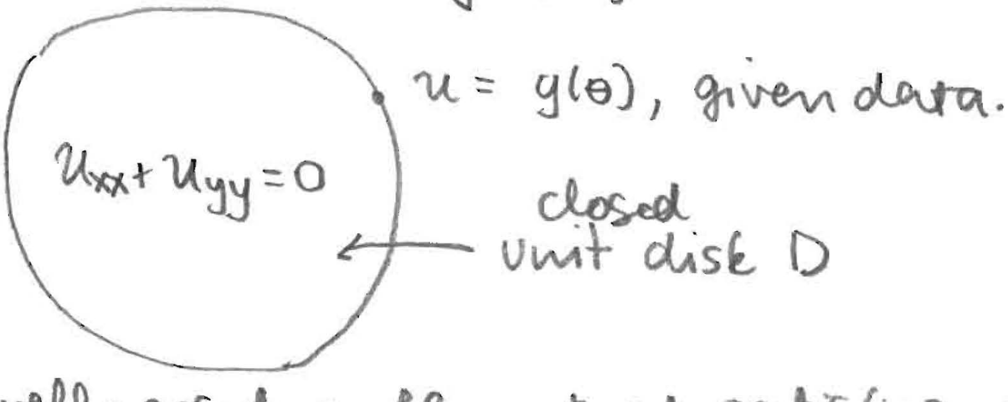
Using the same reasoning as above we can write the series solution

$$u(x,t) = \sum_{j=1}^{\infty} c_j \sin(j\pi x)\, e^{j^2\pi^2 t}$$

where $\{c_j\}$ are the sine series coefficients of $u_0(x)$ as before. Arbitarily small changes in $u_0(x)$ (in c_j for large j) lead to arbitrarily large changes in $u(x,t)$ for $t>0$. Thus, the problem (14) is <u>ill-posed</u>. It is not solvable for general $u_0(x)$ and does not correspond to a physically reasonable model.

10 Laplace Equation.

<u>Example 10</u> Consider the problem for $u(x,y)$ in the unit disk ($x^2+y^2 \leq 1$):

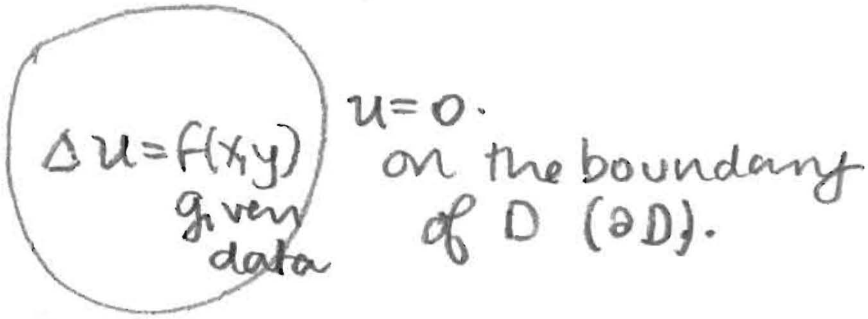


This is a well-posed problem that satisfies a maximum principle ($\max\limits_{(x,y)\in D} u(x,y) = \max\limits_{\theta} g(\theta)$).

Note: $u_{xx} + u_{yy}$ is a combination of spatial partial derivatives that often occurs and is denoted by Δu (the Laplacian of u). In 3D problems with $u(x,y,z)$, $\Delta u = u_{xx} + u_{yy} + u_{zz}$.

The problem in Example 10 can arise in a number of applications. In electrostatics, u can describe the voltage in a vacuum with g the applied exterior voltage. u could also describe the steady state temperature in an object with applied temperature $g(\theta)$.

Example 11 (Poisson problem) $u(x,y)$, $(x,y) \in D$



Also well-posed.

Note that 10 & 11 can be combined to solve a more complete problem with boundary data $g(\theta)$ and interior data $f(x,y)$ using linearity.

(11) Summary of Part I notes.

A PDE is a relationship between an unknown function and its partial derivatives that must be satisfied at every point in a domain. It models a physical law that the solution must satisfy.

A complete PDE problem also involves initial value data, boundary value data, and/or interior forcing data.

Review: Newton's method, linearity, eigen-analysis, sine series, c.c. linear ODEs, Method of Undetermined Coefficients, separable ODEs.

D'Alembert's solution for 1D wave equation.

Preliminary discussion of weak solutions and well-posedness.

Properties of PDEs that can be found without finding the solution, i.e. maximum principle and long-time behaviour of the heat equation.

Presented a simple separation of variables solution in a different way using the idea of eigenanalysis of boundary value problems (more on this later, when we discuss general Sturm-Liouville problems).

Introduced the wave, heat and Laplace equations. These are generic equations for functions of two variables (Notes, part III).

Part II will include a derivation of the heat equation and wave equation.

Math 400 Notes, Part II
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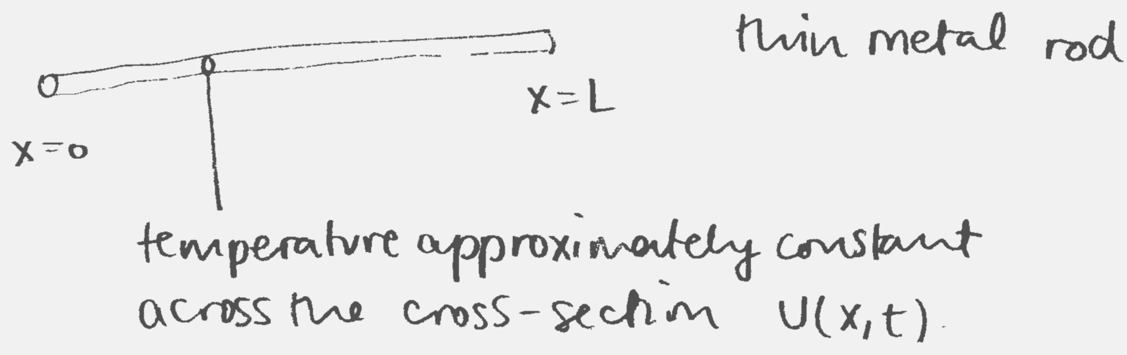
Sept 14, 2012

(1) Thermal conduction in a thin rod.



Temperature is held constant at the ends $U(0,t) = U_0$, $U(L,t) = U_1$. The length of the rod is insulated so that heat only moves along the length of the rod. We will reduce this problem to one for $u(y,T)$, $0 \le y \le 1$ and

$$u_T = u_{yy}, \quad u(0) = 0, \quad u(1) = 0. \qquad (1)$$

Two steps

(A) Mathematical Modelling, that is describing some physical mechanisms as mathematical equations.

(B.) Scale the equations (non-dimensionalization).

(2) Two experimental observations

Material sample mass m, initially at temperature U_0, heat Q (in J) is added and the temperature is raised to U_1. It is found that $U_1 - U_0$ is approximately proportional to Q/m.

$$\frac{Q}{m} = c(U_1 - U_0).$$

c does not depend on m and not strongly on T_0 in a range. c has units $J/_{kg\,°C}$ and is called specific heat capacity.

Note: C_p (constant pressure), C_v (constant volume), quite different for gasses, not so different for solids.

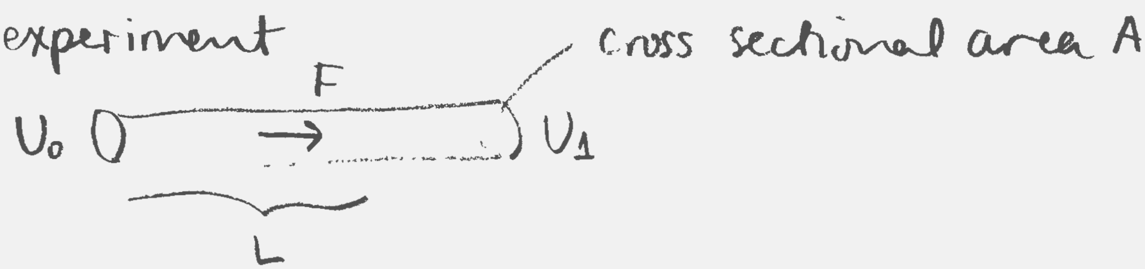
c is the energy required in J to heat on kg of the substance up by $1°C$.

let ρ be the density (kg/m^3) of the substance, approximately constant with U_1. Then

$$c\rho \quad (\text{units } J/_{m^3\,°C}) \tag{2}$$

is the volumetric heat capacity, the energy required to heat $1m^3$ of the substance by $1°C$.

Second experiment



At steady state, the heat flow F (J/s) through the object satisfies

$$F = \frac{KA}{L}(U_0 - U_1), \tag{3}$$

K independent of A and L, units $\frac{J}{s\,m\,°C}$

Consider the heat flux per unit area

$$j = \frac{F}{A}$$

and take the limit of (3) as $L \to 0$,

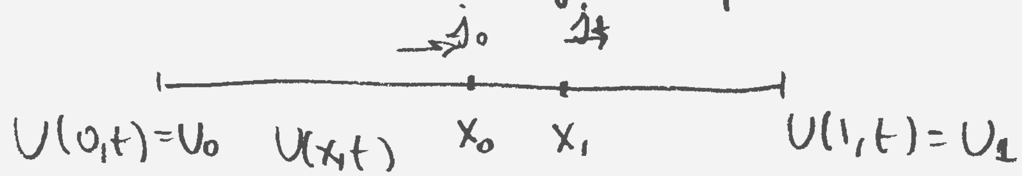
$$j = -K \frac{\partial U}{\partial x}$$

Note: This implies that at steady state (j is independent of x by conservation of energy)

$$U(x) = U_0 + (x/L)(U_1 - U_0)$$

3 Step A, Modelling

Now consider the original problem in part 1



Consider a small piece of the rod, between x_0 and x_1 at between times t_0 and t_1.

At t_0, heat flow in at the left

$$j_0 A = -kA \left.\frac{\partial U}{\partial x}\right|_{x_0}.$$

out at the right

$$j_1 A = -kA \left.\frac{\partial U}{\partial x}\right|_{x_1}.$$

So the net heat flow into the small piece is

$$j_0 A - j_1 A = kA \left(\left.\frac{\partial U}{\partial x}\right|_{x_1} - \left.\frac{\partial U}{\partial x}\right|_{x_0} \right)$$

This net heat flow must change the average temperature in the piece between times t_0 and t_1.

$$\underbrace{A(x_1-x_0)}_{\text{volume of the piece}} c\rho \left(U_{ave}(t_1) - U_{ave}(t_0)\right) \qquad (4).$$

$$\approx (t_1-t_0) KA \left(\frac{\partial U}{\partial x}\Big|_{x_1} - \frac{\partial U}{\partial x}\Big|_{x_0} \right).$$

Note: This is approximate because we are using the heat flows at t_0 for the interval $[t_0, t_1]$.

The equation above can be rewritten as

$$\frac{U_{ave}(t_1) - U_{ave}(t_0)}{t_1 - t_0} = \frac{K}{c\rho} \frac{\left(U_x|_{x_1} - U_x|_{x_0}\right)}{x_1 - x_0}.$$

Taking the limits as $t_1 \to t_0$, $x_1 \to x_0$ we obtain

$$\frac{\partial U}{\partial t} = \underbrace{\frac{K}{c\rho}}_{k\text{, thermal diffusivity.}} \frac{\partial^2 U}{\partial x^2}. \qquad (5).$$

Units of k: $\frac{J}{s\,m\,{}^\circ C} \frac{kg\,{}^\circ C}{J} \cdot \frac{m^3}{kg} = \frac{m^2}{s}$ ✓

Note: Keeping track of units is a good way to check modelling steps.

④ Step Ⓑ, Scaling and Nondimensionalization.

Let $u(x,t) = U(x,t) - U_0 - \frac{x}{L}(U_1 - U_0)$

Now $u_t = k u_{xx}$ and $u(0)=0$, $u(L)=0$.

Now let $y = x/L$ and consider $u(y,t)$.

$$x \in [0,L] \Rightarrow y \in [0,1]$$

and y is dimensionless

$$u(y=0,t) = 0, \quad u(y=1,t) = 0 \quad \checkmark.$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{L^2}\frac{\partial^2 u}{\partial y^2}$$ by the chain rule.

so $$\frac{\partial u}{\partial t} = \frac{k}{L^2}\frac{\partial^2 u}{\partial y^2}.$$

Now scale time, $\tau = t/T$, with T a constant to be determined

$$\frac{\partial u}{\partial t} = \frac{1}{T}\frac{\partial u}{\partial \tau}$$

so $$\frac{\partial u}{\partial \tau} = \frac{Tk}{L^2}\frac{\partial^2 u}{\partial y^2}.$$

Take $T = L^2/k$ (units s ✓) and then

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial y^2}, \quad u(0,\tau)=0, \quad u(y=1,\tau)=0$$

So we have arrived at (1) as desired.

Note: For free, we have identified $T = L^2/k$ as the natural time scale of the temperature distribution in the rod. After solving (1) we can recover the dimensional solution as

$$U(x,t) = \underbrace{u\left(\frac{x}{L}, \frac{t}{T}\right)}_{u(y,\tau)} + U_0 + \frac{x}{L}(U_1 - U_0).$$

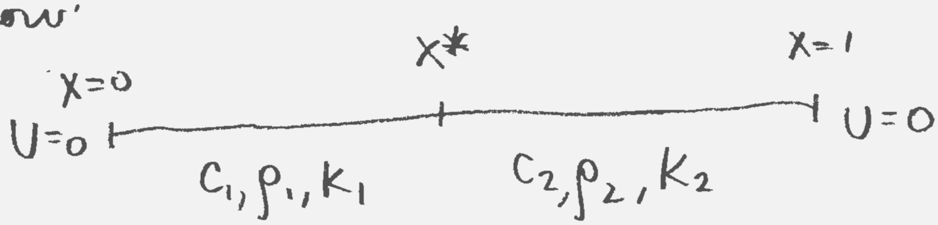
⑤ Variable coefficient problem.

Now consider the case where c, ρ, A and K can vary with x. Careful review of the previous argument leads to the unscaled equation

$$c(x)\,\rho(x)\,A(x)\,\frac{\partial U}{\partial t} = \frac{\partial}{\partial x}\left(K(x)\,A(x)\,\frac{\partial U}{\partial x}\right)$$

Note that there are physical, theoretical and computational reasons to leave the RHS term in this form (not expand it as a product rule) or to divide by the factor on the LHS.

Suppose c, ρ, and K are not continuous at a point x^* (for example, there is a join at x^* of two different metals in the rod). Consider c, ρ, and K to be different constants to the left and right of x^* as shown below.



For $x < x^*$, $\quad c_1\rho_1 \dfrac{\partial U}{\partial t} = \dfrac{\partial}{\partial x}\left(K_1 \dfrac{\partial U}{\partial x}\right)$

For $x > x^*$, $\quad c_2\rho_2 \dfrac{\partial U}{\partial t} = \dfrac{\partial}{\partial x}\left(K_2 \dfrac{\partial U}{\partial x}\right)$

left in this form even though K_1, K_2 are constant

At $x = x^*$ interface conditions are applied:

$$U(x^*_-) = U(x^*_+) \quad \text{[that is, } U \text{ is continuous at } x = x^*\text{]} \qquad (6)$$

and $$K_1 \frac{\partial U}{\partial x}(x^*_-) = K_2 \frac{\partial U}{\partial x}(x^*_+) \quad \text{[that is, the heat flux is continuous at } x = x^*\text{]} \qquad (7).$$

<u>Notation:</u> If x^* is a point at which quantities $Q(x)$ can change possibly discontinuously, we write

$$[Q] \underset{\substack{\uparrow \\ \text{defined to be}}}{:=} Q(x^*_+) - Q(x^*_-),$$

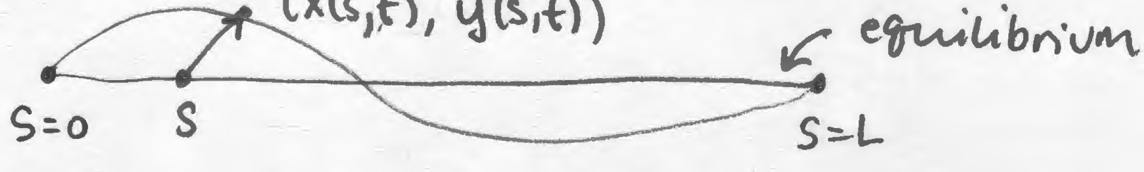
the "jump" in Q at x^*. Thus (6) and (7) can be written as

$$[U] = 0, \quad \left[K \frac{\partial U}{\partial x}\right] = 0$$

Relationships (6) and (7) can be shown to be physically correct. In addition, they are a minimal set of conditions that lead to a unique solution of the discontinuous coefficient problem.

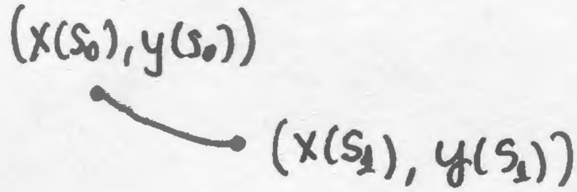
(6) <u>Derivation of the Wave Equation</u>

Consider a wire under tension held fixed at both ends. We'll neglect gravity in this discussion.



Mass per unit length of the wire μ, s is a material coordinate in the wire at equilibrium with uniform tension T_0. Let $x(s,t), y(s,t)$ be the displacement of the wire from equilibrium as shown.

Consider a small piece of the wire between s_0 and s_1 at some fixed time t.



The equilibrium length is $s_1 - s_0$. The current length is

$$l = \sqrt{(x(s_1) - x(s_0))^2 + (y(s_1) - y(s_0))^2}$$

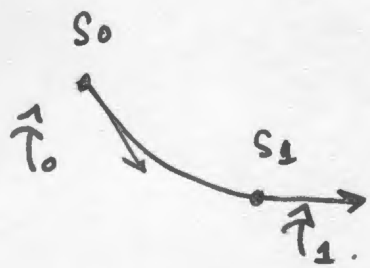
The ratio of current length to equilibrium length is

$$\frac{l}{(s_1 - s_0)} \underset{\text{as } s_1 \to s_0}{\longrightarrow} f(s) := \sqrt{x_s^2 + y_s^2}.$$

Material assumption: the local tension in the wire is a function of f, i.e.

$$T(f), \quad \text{note that } T(1) = T_0, \ T'(f) > 0.$$

Now let's look at the forces on a piece of wire, neglecting friction and gravity, leaving only the tension forces at the ends:



Unit tangent vector $\hat{\tau} = \frac{(x_s, y_s)}{f(s)}$. Net force is

$$T(f_1)\hat{\tau}_1 - T(f_0)\hat{\tau}_0$$

The piece has mass $\mu(s_1 - s_0)$. Apply Newton's second law $(F = ma)$ to obtain

$$\mu(s_1 - s_0)\,(x_{tt}, y_{tt}) = T(f_1)\hat{T}_1 - T(f_0)\hat{T}_0$$

of centre of mass
of the piece

Now divide by $s_1 - s_0$ and take the limit as $s_1 \to s_0$ to obtain

$$\mu\,(x_{tt}, y_{tt}) = \frac{\partial}{\partial s}\left(T(f)\,\hat{T}\right) \qquad (8)$$

with $f = \sqrt{x_s^2 + y_s^2}$, $\hat{T} = (x_s, y_s)/f$.

This is a nonlinear, coupled equation for $x(s,t)$, $y(s,t)$. To recover the scalar, linear wave equation, we need to <u>linearize</u> (8) around the equilibrium state.

Start by formally assuming that the solutions of (8) do not deviate far from the rest state:

$$\left.\begin{aligned} x(s,t) &= s + \varepsilon\, x_1(s,t) \\ y(s,t) &= \varepsilon\, y_1(s,t). \end{aligned}\right\} \quad \varepsilon \text{ is small, a book-keeping symbol.}$$

Plan: put these expressions into (8), neglect any terms of size ε^2, ε^3 etc. This should result in an equation that x_1 and y_1 satisfy approximately, also expect that the solutions of the new equation approximate x_1 and y_1 (hard analysis work). In the following, the "1" subscript will be dropped.

$$f(s) = \sqrt{(1+\varepsilon x_s)^2 + \cancel{\varepsilon^2 y_s^2}}$$

$$\approx \sqrt{1 + 2\varepsilon x_s} \approx 1 + \varepsilon x_s$$

$\sqrt{1+z} \approx 1 + z/2 + O(z^2)$ for z small

$$\hat{\tau} = \frac{(1+\varepsilon x_s,\ \varepsilon y_s)}{1+\varepsilon x_s} \qquad \frac{1}{1-\varepsilon x_s} \approx 1+\varepsilon x_s$$

$$\approx (1,\ \varepsilon y_s)$$

$$T(f) \approx T(1+\varepsilon x_s) \approx \underset{\substack{\uparrow \\ T(1)}}{T_0} + \varepsilon T'(1)\, x_s$$

$$T\hat{\tau} \approx (T_0 + \varepsilon T'(1) x_s,\ \varepsilon T_0 y_s)$$

$$(T\hat{\tau})_s \approx \varepsilon(T'(1) x_{ss},\ T_0 y_{ss}).$$

So now the PDE $\mu(x_{tt}, y_{tt}) = (T(f)\hat{\tau})_s$ becomes

$$\cancel{\varepsilon}\mu(x_{tt}, y_{tt}) \approx \cancel{\varepsilon}(T'(1) x_{ss},\ T_0 y_{ss})$$

Components decouple, have different wave speeds

$$x_{tt} = \frac{T'(1)}{\mu} x_{ss}, \qquad y_{tt} = \frac{T_0}{\mu} y_{ss}$$

Speed $c_x = \sqrt{\frac{T'(1)}{\mu}}$ Speed $c_y = \sqrt{T_0/\mu}$

Math 401 Notes - III

Classification of 2D, 2nd order PDE's

General, linear constant coefficient PDE in 2 space variables $\underline{x} = (x_1, x_2)$:

$$a\frac{\partial^2 u}{\partial x_1^2} + 2b\frac{\partial^2 u}{\partial x_1 \partial x_2} + c\frac{\partial^2 u}{\partial x_2 \partial x_2} + d\frac{\partial u}{\partial x_1} + e\frac{\partial u}{\partial x_2} = f(\underline{x})$$

or
$$\sum_{i,j=1}^{2} \underbrace{A_{ij}}_{} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{2} L_i \frac{\partial u}{\partial x_i} = f(\underline{x}) \qquad (1)$$

assume not invertible

where $A_{11} = a$, $A_{12} = A_{21} = b$, $A_{22} = c$

$L_1 = d$, $L_2 = e$.

orthonormal change of coordinates

$$\underline{y} = \mathbb{P}\underline{x} \qquad \mathbb{P}^T = \mathbb{P}^{-1}$$

$$\frac{\partial u}{\partial x_i} = \sum_{k=1}^{2} \frac{\partial u}{\partial y_k} \frac{\partial y_k}{\partial x_i} \quad P_{ki}$$

$$\frac{\partial^2 u}{\partial x_i \partial x_j} = \sum_{k=1}^{2} \sum_{l=1}^{2} \frac{\partial^2 u}{\partial y_k \partial y_l} P_{ki} P_{lk}$$

So now (1) reads (2)

$$\sum_{i,j} \sum_{k,l} A_{ij} P_{ki} P_{lj} \frac{\partial^2 u}{\partial y_k \partial y_l} + \sum_{i=1}^{2}\sum_{k=1}^{2} P_{ki} L_i \frac{\partial u}{\partial y_k} = f(\underline{x})$$

Note that if $\mathbb{C} = \mathbb{A}\mathbb{B}$,

$$C_{ik} = \sum_j A_{ij} B_{jk}.$$

so $$\sum_{i,j} A_{ij} P_{ki} P_{\ell j} = C_{k\ell} \quad \text{where } \mathbb{C} = \mathbb{P}\mathbb{A}\mathbb{P}^T = \mathbb{P}\mathbb{A}\mathbb{P}^{-1}.$$

$\mathbb{A}$ is symmetric and so can be diagonalized by an orthogonal matrix:

$\mathbb{P}$ = matrix of normalized eigenvectors in columns, then

$$\mathbb{P}\mathbb{A}\mathbb{P}^{-1} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \qquad \lambda\text{'s eigenvalues of } \mathbb{A}.$$

Thus (2) becomes (3)

$$\lambda_1 \frac{\partial^2 u}{\partial y_1^2} + \lambda_2 \frac{\partial^2 u}{\partial y_2^2} + \hat{a} \frac{\partial u}{\partial y_1} + \hat{b} \frac{\partial u}{\partial y_2} = \underline{f}(\mathbb{P}\underline{x}).$$

$$\hat{a} = \sum_{i=1}^{2} P_{1i} L_i \qquad \hat{b} = \sum_{i=1}^{2} P_{2i} L_i.$$

Three cases:

(i) λ_1 & λ_2 have the same sign, (1) is called elliptic. Can further scale y_1 & y_2 to x & y so that

$$u_{xx} + u_{yy} + a u_x + b u_y = g(x,y).$$

When $a=b=0$ this is called Poisson's equation.

(ii) λ_1 & λ_2 are opposite signs, (1) is called hyperbolic. Again scaling,

$$u_{xx} - u_{yy} + a u_x + b u_y = g(x,y)$$

$\uparrow$ x time $\quad$ $\uparrow$ y space.

When $a=b=0$ (and $g\equiv 0$) this is called the (unforced) wave equation. Further change of variables $\eta = x+y$, $\xi = x-y$ $\Rightarrow$

$$u_{\eta\xi} = 0 \quad \text{(unforced)}$$

This form leads to D'Alembert's solution.

(iii) If $\lambda_2 = 0$ (exclude also $\lambda_1 = 0$ since then (1) is not second order) assume $\hat{b} \neq 0$ (otherwise there is no differentiation in y_2 and the problem is a family of ODEs). Then (1) is called parabolic. Again scaling

$$-u_{xx} + u_y + a u_x = g(x,y).$$

When $a=0$ this is called the heat equation.

<u>Note:</u> $\lambda_1 \lambda_2 = \det \mathbb{A} = \begin{vmatrix} a & b \\ b & c \end{vmatrix} = ac - b^2$

remember, b is half the coefficient of the $\frac{\partial^2 u}{\partial x_1 \partial x_2}$ term

so it's easy to determine the class from the original equation (1).

<u>Note:</u> The sign of $\det \mathbb{A}$ determines the local behaviour even when $a(\underline{x}), b(\underline{x}), c(\underline{x})$. Equations that change type in $\underline{x}$ are tricky (potential in transonic flow).

<u>Ex</u> Classify the PDE

$$\frac{\partial^2 u}{\partial x^2} + 4 \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = f.$$

$2b$; so $b = 2$.

$\det \mathbb{A} = 1 - 4 = -3$ so hyperbolic.

Let's look at Green's functions for these 3 important equations.

Math 400 Notes, Part IV

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Sept 18, 2012

The first PDE we saw in the course was the one-way wave equation.

$$u_t + u_x = 0. \qquad (W1)$$

We saw in assignment #2, Q9 that the general solution of this PDE is $f(x-t)$, for arbitrary scalar function f.

Def<u>n</u> Consider a function $u(\underline{x}, t)$ to be determined for all $\underline{x}$ and $t \geq 0$. A Cauchy problem is one for which u satisfies a PDE of the form:

$\frac{\partial^n u}{\partial t^n}$ = a function of f, u, t, partial derivatives with respect to the $\underline{x}$ variables, partial derivatives with respect to t of order less than n.

plus initial data for $u(\underline{x}, 0)$, $u_t(\underline{x}, 0)$, … $\frac{\partial^{n-1} u}{\partial t^{n-1}}(\underline{x}, 0)$.

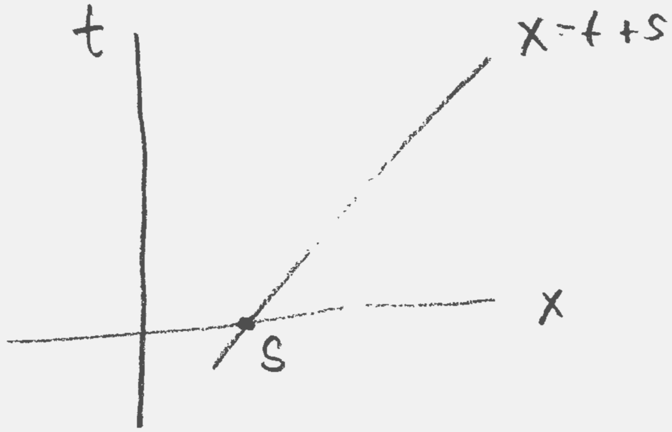
<u>Notes</u>: The initial data <u>can</u> be given on more general smooth surfaces than the hyperplane $t=0$, and the problem <u>can</u> involve finding solutions on both sides of the data surface ($t<0$ in our case).

Example: The Cauchy problem for (W1) adds the initial data $u(x,0) = u_0(x)$, given. We have seen the solution, it is $u(x,t) = u_0(x-t)$.

We saw that solutions of (WI) were constant along characteristics

$$x = t + s. \qquad (x = s \text{ when } t=0)$$

Consider s to be the coordinate x on the initial line, so s is where the initial data determines the value of u on this characteristic.



Let's try and generalize this idea to more complicated problem.

Consider the Cauchy problem for

$$u_t + c(x,t)\, u_x = 0, \qquad c(\cdot,\cdot) \text{ given} \qquad (1).$$

We will see that c represents a varying speed of propagation. Let's look for characteristics for this problem, that is curves $x(t)$ with $x(0) = s$ such that u is constant on these curves, i.e. $u(x(t), t) = u_0(s)$. take the derivative with respect to t.

$$\frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial t} = 0.$$

want this zero by choice of $\frac{dx}{dt}$.

Comparing to (1), we see that we will have the desired result if we take

$$\frac{dx}{dt} = c(x(t), t), \quad x(0) = s \qquad (2)$$

So (2) is an ODE for $X(t)$ for every s. We will write the solutions as $X(t,s)$. We know that

$$u(X(t,s),t) = u_0(s). \tag{3}$$

So (3) is an implicit form of the desired solution - this may be as far as we can get with this technique. In some cases, however, we may be able to invert the relationship $X(t,s)$ to $s(t,x)$ and then an explicit solution $u(x,t)$ can be found.

<u>Example</u>: solve $u_t + (\cos t)\cdot u_x = 0$ with initial data $u(x,0) = u_0(x)$ given (Cauchy problem).

Solve for the characteristics $X(t)$.

$$\frac{dX}{dt} = \cos t \qquad X(0) = s.$$

integrate $X = \sin t + C$, $X(0) = s \Rightarrow C = s$.

So we have $X = \sin t + s$.

By construction, along these curves u is constant

$$u(\underbrace{\sin t + s}_{X}, t) = u_0(s).$$

Here it is easy to find $s(t,x) = x - \sin t$ so the explicit solution is

$$u(x,t) = u_0(x - \sin t).$$

Example: Solve the Cauchy problem for

$$u_t + \frac{1}{1+x^2} u_x = 0.$$

Characteristics $x(t)$ satisfy

$$\frac{dx}{dt} = \frac{1}{1+x^2} \qquad x(0) = s. \tag{4}$$

separable "$dx\,(1+x^2) = dt$" (mnemonic for a more rigorous process).

integrate $x + \frac{x^3}{3} = t + c$ ← when $t=0$, $x=s$, so $c = s + \frac{s^3}{3}$.

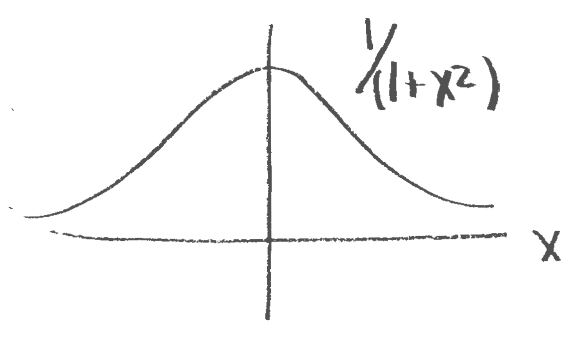
Since $s + s^3/3$ is increasing,

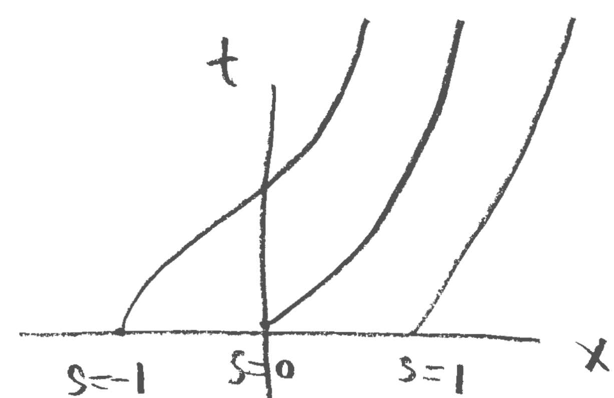
$$x + \frac{x^3}{3} - t = s + s^3/3$$

determines $s(x,t)$ uniquely (ugly cubic root formula) and then

$$u(x,t) = u_0(s(x,t)).$$

Even without the formula, we can see that the initial data is propagated to the right with speed $\frac{1}{1+x^2}$ (max at $x=0$).





Also, we notice that the values of $u(x,t)$ are the same as the values of $u_0(s)$ [just transformed non-uniformly). So the solution obeys a maximum principle:

if $M = \sup u_0(x)$ then $u(x,t) \leq M$ for all x and $t \geq 0$.

maximum if attained

Idea: even if exact solutions can't be found, we can often get a great deal of insight into the behaviour of solutions from analytic techniques.

Example Solve $u_t + u_x = -u + xt$ (still linear) with $u(x,0) = u_0(x)$ given. Written in a suggestive way. The LHS will be a derivative along characteristics $x(t)$ with $x(0) = s$.

$$u(t) = u(x(t), t). \qquad (5)$$

u on the characteristic line only depends on t.
Take the derivative of (5) using the chain rule

$$\frac{du}{dt} = u_x \frac{dx}{dt} + u_t = u_t + u_x = -u + xt$$

Take $\frac{dx}{dt} = 1$, so $x = s + t$ as expected

So $\frac{du}{dt} = -u + (t+s)t$ along characteristics (s constant). Solve (homogeneous plus the method of undetermined coefficients)

$$u(t) = [u_0(s) + s - 2] e^{-t} + t^2 + (s-2)(t-1)$$

This gives $u(s,t)$. Using $s = x - t$ we obtain

$$u(x,t) = \left[u_0(x-t) + x - t - 2\right] e^{-t} + t^2 + (x-t-2)(t-1)$$

<u>General</u> <u>Theory</u> for linear first order scalar PDE's with a time-like variable

$$u_t + a(x,t)\, u_x = b(x,t)\, u + f(x,t) \qquad (6)$$

<u>step 1</u> Identify characteristic curves.

$$\frac{dx}{dt} = a(x(t),t) \qquad x(0) = s$$

can be solved for $x(s,t)$.

<u>step 2</u> Consider the solution on these curves

$u(t) = u(x(t),t)$.

$$\frac{du}{dt} = u_x \frac{dx}{dt} + u_t = u_t + a\, u_x = b(x(t),t)\, u(t) + f(x(t),t)$$

$$u(0) = u_0(s)$$

For every s, this is an ODE that can be solved for $u(s,t)$.

<u>step 3</u> (if possible) Solve the expression $x(s,t)$ for $s(x,t)$, then write $u(s,t)$ as $u(x,t)$.

Consider now a slight generalization of (6) for $u(x,y)$:

$$a(x,y)\, u_x + c(x,y)\, u_y = b(x,y)\, u + f(x,y). \qquad (7)$$

Note: if $a(x,y) > 0$ we could divide (7) by a and then the equation would be in the form (6) with x a time-like variable. If $c(x,y) > 0$ we can similarly make y a time-like variable. If neither is true then the characteristics have to be represented as parametrized curves

$$x(T), \ y(T), \quad T \text{ is a parameter.}$$

Example Solve $y u_x - x u_y = 0$ with initial data $u(x,0) = e^x$ on the line segment $y=0$, $1 \le x \le 2$. Describe the region on which your solution is defined. Consider

$$u(T) = u(x(T), y(T)) \qquad \begin{array}{l} x(0) = s \quad (1 \le s \le 2) \\ y(0) = 0. \\ u(0) = e^s \end{array}$$

$$\frac{du}{dT} = u_x \frac{dx}{dT} + u_y \frac{dy}{dT}$$

To have the characteristics match the PDE form, we take

$$\frac{dx}{dT} = y, \quad \frac{dy}{dT} = -x. \quad [x(0) = s, \ y(0) = 0] \qquad (8)$$

On curves that solve (8), $u(T)$ is constant, so $u(T) = e^s$ for all T.

To solve (8), we could write it out as a system

$$\frac{d}{dT}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

and proceed by doing the eigenanalysis of the matrix. However, in this case there is an easier way to proceed.

$$\frac{dx}{dT} = y \Rightarrow \frac{d^2x}{dT^2} = \frac{dy}{dT} = -x.$$

$$\ddot{x} + x = 0 \Rightarrow x = A\sin T + B\cos T$$

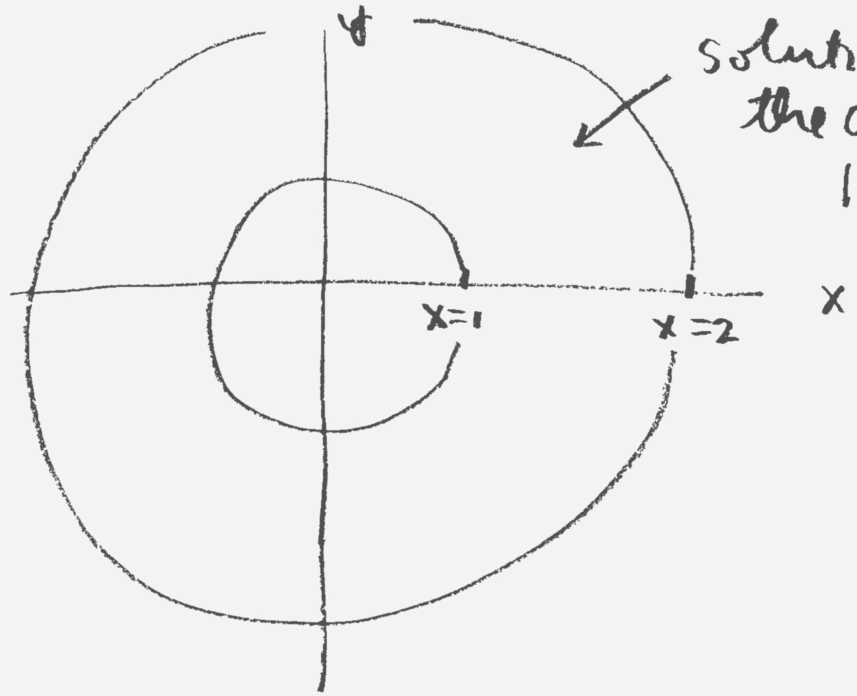
$$y = -\dot{x} = -A\cos T + B\sin T$$

with $x(0) = s$, $y(0) = 0$ we have

$$x(T) = s\cos T, \quad y = s\sin T \qquad (9)$$

and on these curves, $u(s,T) = e^s$. Note that $s = \sqrt{x^2+y^2}$, so $u(x,y) = e^{\sqrt{x^2+y^2}}$.

From (9) we see that the characteristics are circles with radius s



solution defined in the annulus $1 \le r \le 2$ with $u = e^r$

<u>Note</u>: If you transform the original PDE into polar coordinates it becomes $\frac{\partial u}{\partial \theta} = 0$, makes sense.

General Theory of linear, first order, scalar PDEs in two variables.

$$a(x,y)\,u_x + c(x,y)\,u_y = b(x,y)\,u + f(x,y). \quad (10)$$

Initial data on a curve $x_0(s), y_0(s)$,

$$u(x_0(s), y_0(s)) = u_0(s).$$

characteristics for each s, $x(\tau), y(\tau)$,

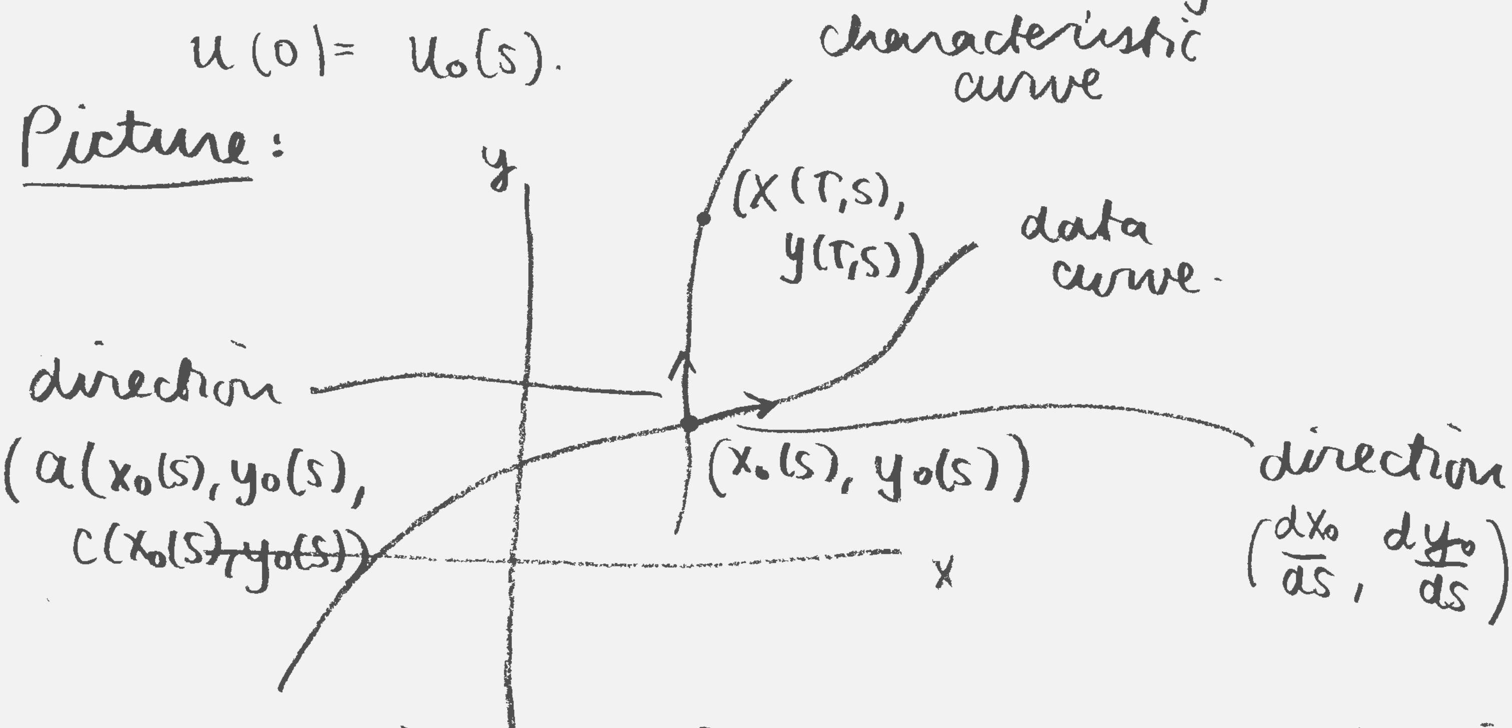
$$\left.\begin{aligned} \frac{dx}{d\tau} &= a(x,y), & \frac{dy}{d\tau} &= c(x,y) \\ x(0) &= x_0(s) & y(0) &= y_0(s) \end{aligned}\right\} \quad (11)$$

On these curves $u(\tau) = u(x(\tau), y(\tau))$ satisfies the ODE (linear)

$$\frac{du}{d\tau} = b(x(\tau), y(\tau))\,u(\tau) + f(x(\tau), y(\tau))$$

$$u(0) = u_0(s).$$

Picture:



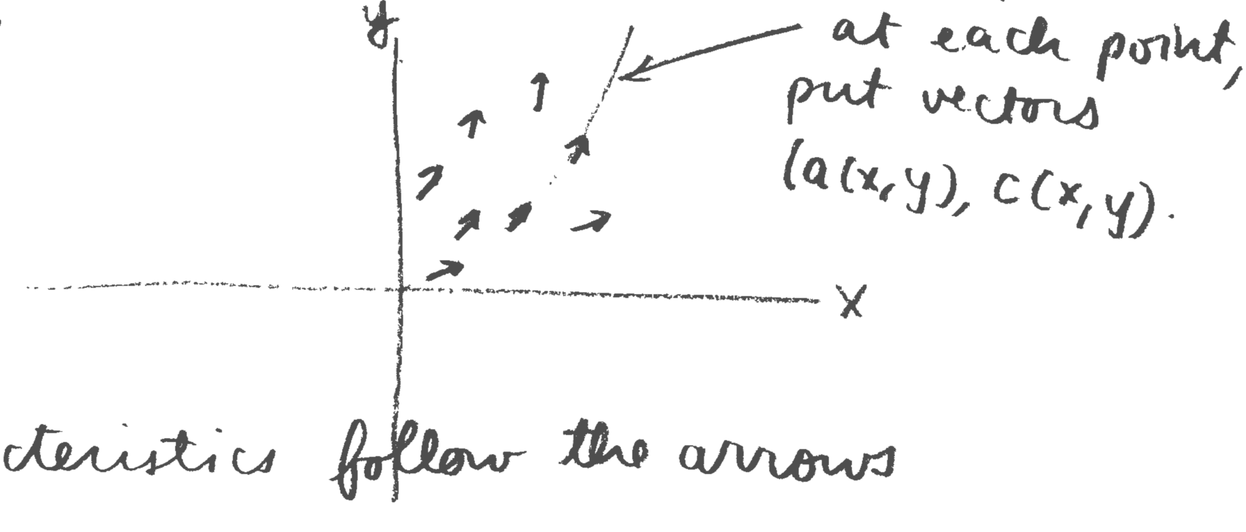
Can't prescribe data in a characteristic direction since in this direction, the solution is determined by the PDE. Thus, the two directions above can't be parallel for any s. That is

$$\det \begin{vmatrix} a(x_0, y_0) & c(x_0, y_0) \\ \frac{dx_0}{ds} & \frac{dy_0}{ds} \end{vmatrix} \neq 0,$$

$a y_0' - c\, x_0' \neq 0$ for every s. Under these conditions, (10) is guaranteed to have a unique solution in a neighborhood of the data curve (a local solution).

Note: The condition above is the determinant of the Jacobian matrix $\frac{\partial(x,y)}{\partial(s,t)}$. This makes the determinant nonzero and so guarantees a local invertibility $x(s,t)$, $y(s,t)$ and so the solution $u(s,t)$ can be written as $u(x,y)$.

Consider the system (11). It is an autonomous system for $x(t)$ and $y(t)$. Insight can be gained into the system with a phase plane portrait:



characteristics follow the arrows

Consider now some nonlinear first order PDE's. Go back to ones with a time-like direction, and restrict ourselves to quasi-linear problems (nonlinear in u but not in u_x).

General form

$$u_t + g(u,x,t)\, u_x = h(u,x,t)$$

Many models have the following form

$$u_t + (F(u))_x = 0$$

(conservation form).

Example $\quad u_t + \left(\frac{1}{2}u^2\right)_x = 0 \quad$ (Burger's equation) (1)

When solutions $u(x,t)$ are differentiable,

$$u_t + u\, u_x = 0 \qquad (2).$$

Solutions of Burger's equation can naturally develop discontinuities. We can make sense of weak solutions using the conservation form (1). This is related to considering the desired solution of (1) as the limit as $\nu \to 0$ of

$$u_t + \left(\tfrac{1}{2}u^2\right)_x = \nu u_{xx}$$

(viscous Burger's equation). Burger's equation is a model of 1D gas dynamics (which is a vector sysem of three variables: density, velocity and energy - in this case, the equations represent conservation of mass, momentum, and energy).

Consider (2) with smooth Cauchy data $u(x,0) = u_0(x)$, given. We can look for characteristics $x(t)$ with $x(0) = s$ as before. Consider

$$u(t) = u(x(t), t) \qquad u(0) = u_0(s) \text{ as before.}$$

$$\frac{du}{dt} = u_x \frac{dx}{dt} + u_t$$

This would match up with the form $u_t + u u_x = 0$ if we took

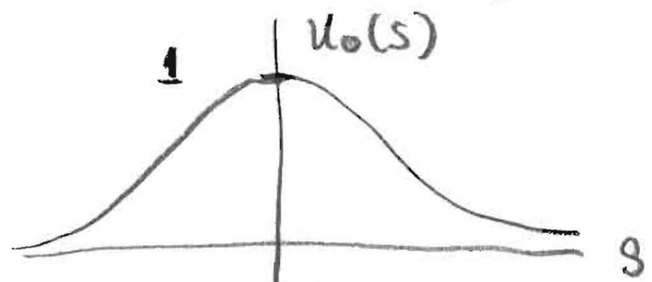
$$\frac{dx}{dt} = u(t) \tag{3}$$

and then on these characteristics $\frac{du}{dt} = 0$, so $u(t) = u_0(s)$. With u constant on characteristics, (3) shows that characteristics are straight lines with slope $u_0(s)$.

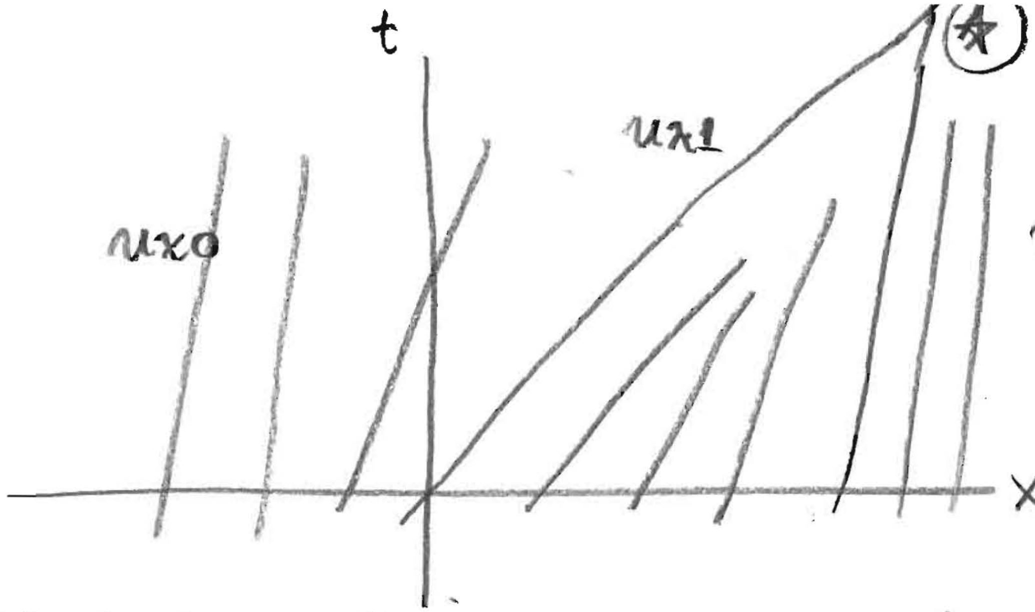
$$x(t) = u_0(s)\, t + s$$

$$u(u_0(s)t + s, t) = u_0(s) \tag{4}$$

(4) is an implicit form of the solution $u(s,t)$. Consider initial data of the following form:



Along characteristics, u is constant (value $u_0(s)$) and for Burger's equation, speeds are also $u_0(s)$.



Note that characteristics can cross! At the first time at which characteristics cross, the solution is not differentiable and after that, the solution is discontinuous (a shock wave forms)

Q: At what time t^* does the shock form?
For $t<t^*$ the solution is smooth (continuous and differentiable).

At time t^*, two s values lead to the same x value. Therefore, the map $s \to x$ at t^* is no longer invertible. Look for values of t and s where $\frac{\partial x}{\partial s}=0$. In fact,

$$t^* = \min_t \left\{ t : \frac{\partial x}{\partial s}(t,s)=0 \text{ for some } s \right\}.$$

In our case, (Burger's Eq) $x = u_0(s)\,t + s$, so

$$\frac{\partial x}{\partial s} = 1 + \underbrace{u_0'(s)}\,t.$$

Let $M = \inf_s u_0'(s)$. If $M>0$, all $u_0'(s)$ values are positive and a shock never forms. Consider the figure on the top of p. 3. Note that it is the negative values of $u_0'(s)$ that lead to shock formation. So if $M<0$, a shock forms at $t^* = -1/M$.

Suppose $u_0(x) = e^{-x^2}$, as in the picture at the bottom of p. 2. To find t^*, we need to find the minimum M of u_0',

$$u_0' = -2xe^{-x^2}$$

To find the minimum, take the derivative, set equal to zero:

$$u_0''(x) = (4x^2-2)e^{-x^2}$$

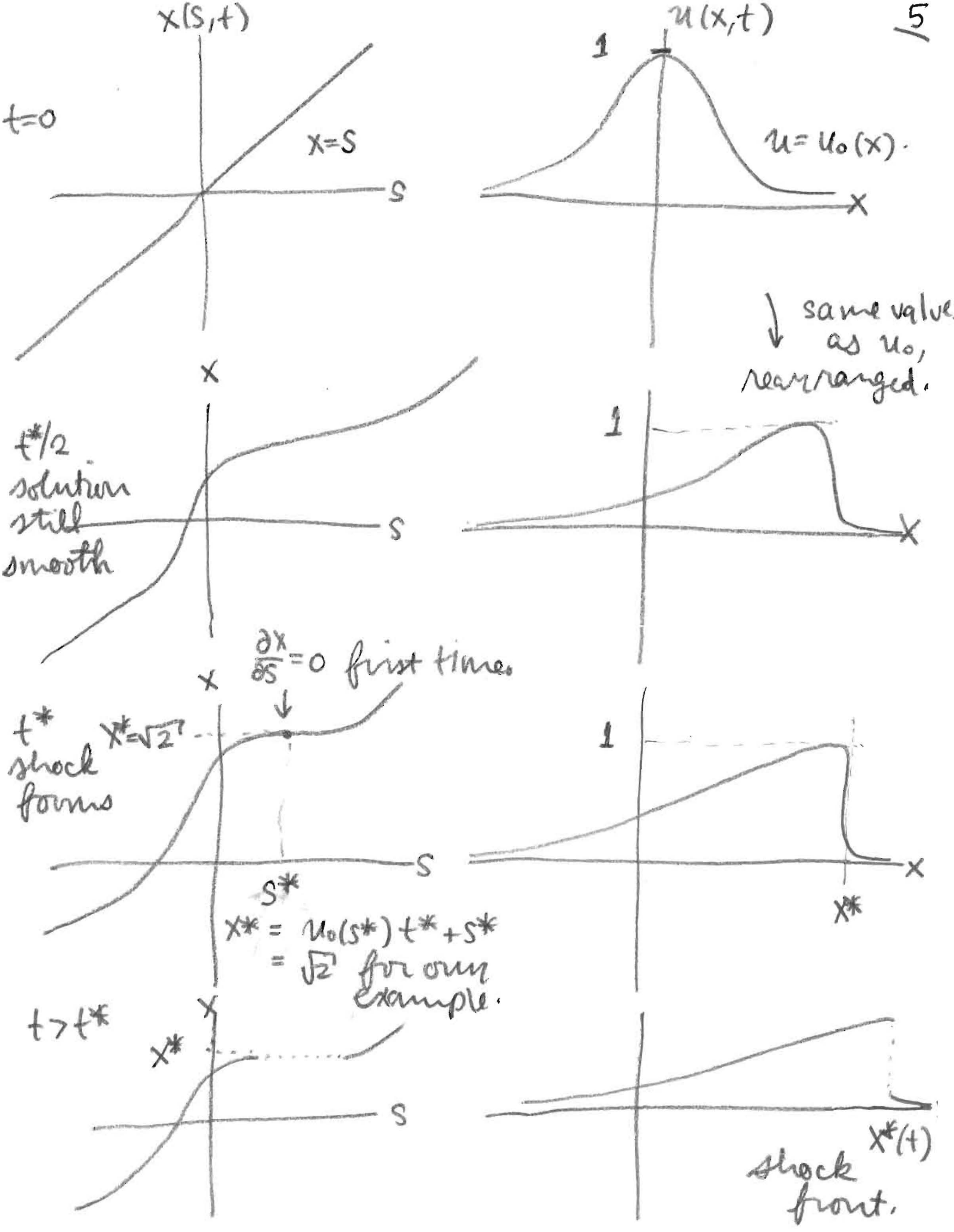
$$u_0''(x) = 0 \quad \text{when} \quad 4x^2-2=0, \; x = \pm \frac{1}{\sqrt{2}}.$$

(maximum at $x = -1/\sqrt{2}$),

$$M = u_0'\left(\frac{1}{\sqrt{2}}\right) = -\sqrt{2}\,e^{-1/2}$$

so a shock will form at $t^* = \frac{-1}{M} = \frac{1}{\sqrt{2}}e^{1/2}$.

How do we continue the solution after t^*? A discontinuity forms, moves with a speed determined by the conservation form of the PDE. We'll get to that in a few pages. First, let's look at a graphical picture of shock formation on the next page, using the example above.


x(s,t)
u(x,t)
t=0
x=s
1
u = u₀(x).
s
x
same values as u₀, rearranged.
x
t*/2 solution still smooth
1
s
x
∂x/∂s = 0 first time.
x
t* shock forms
x* = √2
1
s
s*
x
x*
x* = u₀(s*) t* + s* = √2 for our example.
t > t*
x
x*
s
x*(t)
shock front.

To make sense of the shock front, consider a general problem in conservation form.

$$u_t + (f(u))_x = 0. \qquad \text{(for Burger's equation, } f(u) = \tfrac{1}{2}u^2\text{)}.$$

$f(u)$ can be interpreted as the flux of the quantity u to the right (if negative, to the left) in the following way:

Consider

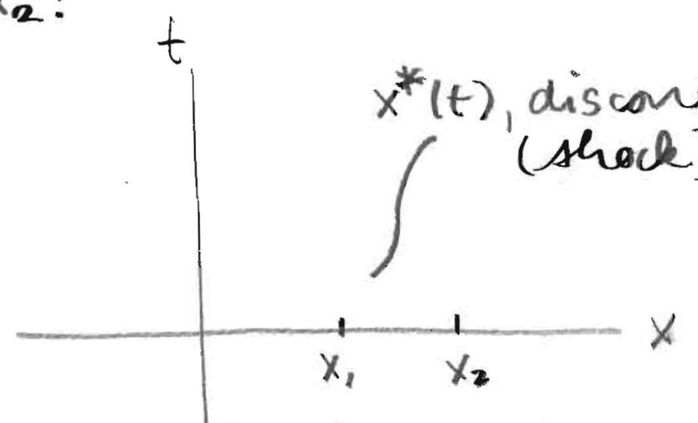
$$M(t) = \int_{x_1}^{x_2} u(x,t)\,dx$$

The rate of change of u should be equal to the net flux at the ends,

$$\frac{dM}{dt} = \int_{x_1}^{x_2} \frac{\partial u}{\partial t}(x,t)\,dx = \int_{x_1}^{x_2} -(f(u))_x\,dx$$

$$= f(u(x_1,t)) - f(u(x_2,t)) \qquad (5)$$

This validates the interpretation of f as a flux. Now the derivation of (5) was done assuming the solution was smooth between x_1 and x_2. What if there is a discontinuity (shock) at $x^*(t)$ between x_1 and x_2:



assume the solution is smooth to the left & right of x^*

Now (5) should still be satisfied, but now

$$M(t) = \int_{x_1}^{x_*(t)} u(x,t)\,dx + \int_{x^*(t)}^{x_2} u(x,t)\,dx$$

solution just to the left of the shock.

$$\frac{dM}{dt} = \int_{x_1}^{x_*(t)} u_t\,dx + \frac{dx_*}{dt}\, u(x^*_-, t)$$

shock speed S

$$-S\, u(x^*_+, t) + \int_{x^*}^{x_2} u_t\,dx$$

solution just to the left

$$= -S(\underbrace{u_+ - u_-}_{\text{the amount of the discontinuity}}) - \int_{x_1}^{x_*} (f(u))_x\,dx - \int_{x^*}^{x_2} (f(u))_x\,dx$$

$$= -S[u] - f(u(x^*_-,t)) + f(u(x_1,t)) - f(u(x_2,t)) + f(u(x^*_+,t))$$

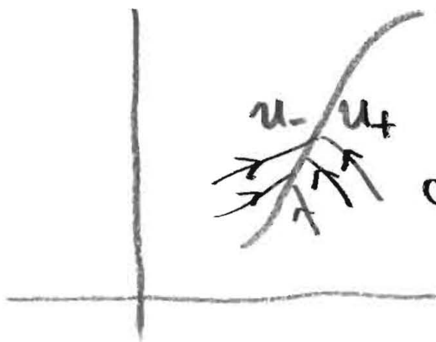
Now this form matches (5), so u is properly conserved, only if

$$-S[u] + f(u(x^*_+,t)) - f(u(u^*_-,t)) = 0,$$

or

$$-S[u] + [f(u)] = 0, \quad S = \frac{[f]}{[u]}. \qquad (6)$$

Equation (6) is known as the Rankine-Hugoniot condition. Let's consider (6) for Burger's equation.



for Burger's equation, $f(u) = \frac{1}{2}u^2$; $f'(u) = u$,

$$\dot{s} = \frac{[f]}{[u]} = \frac{1/2\,(u_+^2 - u_-^2)}{u_+ - u_-} = \frac{1}{2}(u_+ + u_-),$$

the average of the characteristic speeds on either side of the shock.

Back to the general case. Recall that the shock appeared because characteristics crossed. To retain this property as in the picture at the top of the page we need

$$\underbrace{f'(u_-)}_{\text{left side characteristic speed}} > \dot{s} > \underbrace{f'(u_+)}_{\text{right side characteristic speed.}} \qquad (7).$$

(7) are known as the Lax entropy conditions. In the gas dynamic equations, they correspond to a shock which increases entropy (a physical requirement). They arise naturally when considering solutions of

$$u_t + (f(u))_x = \nu u_{xx}$$

as $\nu \to 0$ (viscosity solutions). We'll see another way in which (7) makes sense.

For Burger's equation, (7) becomes

$$u_- > \tfrac{1}{2}(u_+ + u_-) > u_+$$

which occurs when $u_- > u_+$. This confirms our first observation that shocks only occur when u is a decreasing function of x.

Note: (7) reduces to $u_- > u_+$ whenever $f(u)$ is a strictly convex function ($f''(u) > 0$ for all u).

We know that shocks (discontinuities) form, why don't we introduce them at $t=0$.

Defn A Riemann problem is a Cauchy problem for $u_t + (f(u))_x = 0$ with initial data

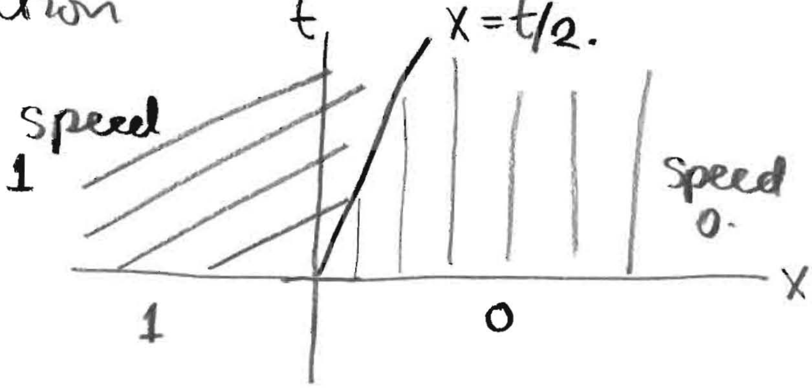
$$u_0(x) = \begin{cases} U_L & x < 0 \\ U_R & x > 0. \end{cases}$$

(U_L and U_R given constants).

Example Solve the Riemann problem for Burger's equation with $U_L = 1$, $U_R = 0$. ($U_L > U_R$ so a shock forms).

$$S = \tfrac{1}{2}(U_L + U_R) = 1/2.$$

Solution



recall

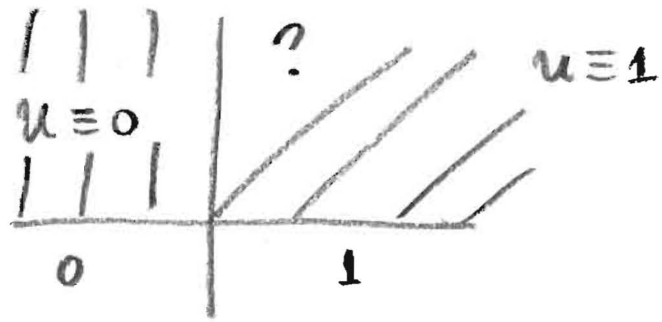
$f(u) = u^2/2$

$f'(u) = u.$

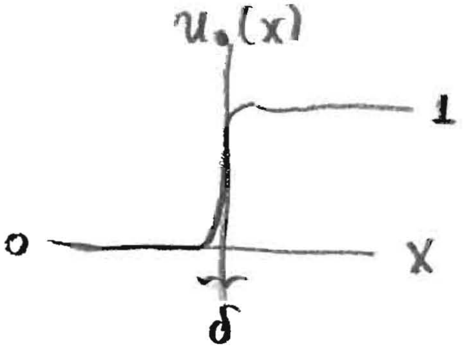
characteristic speed.

so $u(x,t) = \begin{cases} 1 & x < t/2 \\ 0 & x > t/2. \end{cases}$

Example Solve the Riemann problem for Burger's equation with $u_R = 0$, $u_L = 1$ ($u_R < u_L$ so a shock should not form.

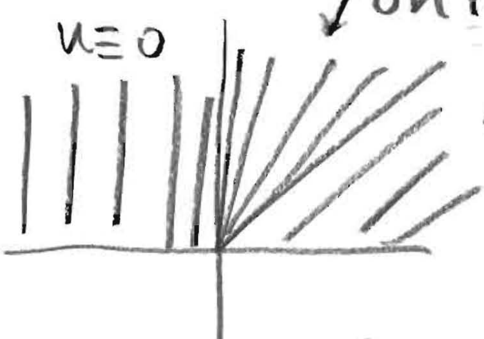


Imagine smoothing out the discontinuity in the initial data over a small distance δ.



u goes continuously from zero to 1. So characteristics with speed from 0 to 1 come out from near the origin with the value of u equal to the speed ($f'(u) = u$). Let $\delta \to 0$.

on the line $x = ct$, $u = c$.



$u \equiv 1$ rarefaction wave.

$$u(x,t) = \begin{cases} 0 & x < 0 \\ c & 0 < x/t < 1 \\ 1 & x > t \end{cases}$$

Note: If f is concave up or concave down, all Riemann problems are a single shock or a single rarefaction coming from the origin.

Example Simple traffic flow model. Let $\rho(x,t)$ be the density of cars and $v(x,t)$ their speed along the length x of a single lane highway (one direction $v \geq 0$). Suppose

$$0 \leq \rho \leq \rho_{max} \quad (\text{bumper to bumper})$$

$$0 \leq v \leq V_{max} \quad (\text{obeyed speed limit}).$$

ρv is the flux of cars, so the conservation law for cars in this setting is

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho v)}{\partial x} = 0.$$

To proceed, we make the following modelling assumption that $v(\rho)$. We could take for example

$$v(\rho) = V_{max}(1 - \rho/\rho_{max})$$

speed $v(0) = V_{max}$ (low traffic density moves at maximum speed) decreases to $v(\rho_{max}) = 0$ (maximum density traffic jam, no movement). Now we have flux $f(\rho) = \rho v = \rho V_{max}(1 - \rho/\rho_{max})$. It is possible to nondimensionalise this problem to one for $u(x,t)$, $0 \leq u \leq 1$, ← maximum density scaled to 1

$$u_t + \big(F(u)\big)_x = 0$$

with $F(u) = u(1-u) = u - u^2$. ← maximum speed scaled to 1.

Let's look at the Rankine-Hugoniot and the Lax Entropy condition for this problem.

$$\text{(RH)} \qquad s = \frac{[f]}{[u]} = \frac{u_+ - u_+^2 - u_- - u_-^2}{u_+ - u_-} = 1 - (u_+ + u_-)$$

$$\text{(LE)} \qquad f'(u_-) > s > f'(u_+) \qquad\qquad f'(u) = 1 - 2u$$

$$1 - 2u_- > 1 - (u_+ + u_-) > 1 - 2u_+ \qquad -1, \div -2$$

switch signs of inequalities

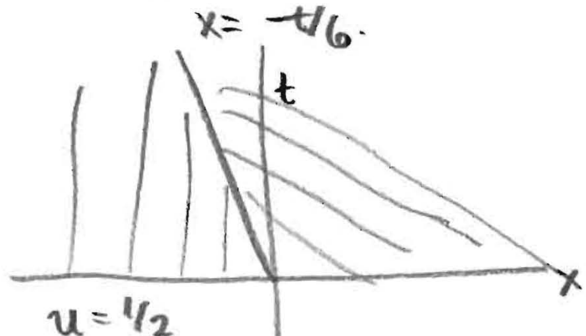
$$\text{or} \qquad u_- < \tfrac{1}{2}(u_+ + u_-) < u_+$$

Satisfied if $u_- < u_+$. In words, shocks form in traffic flow (this model) only when the traffic ahead is denser than the traffic behind.

Consider two Riemann problems for this model:

(i) $U_L = \frac{1}{2}$, $U_R = \frac{2}{3}$ (shock will form)

$$s = 1 - \left(\tfrac{1}{2} + \tfrac{1}{3}\right) = -1/6.$$



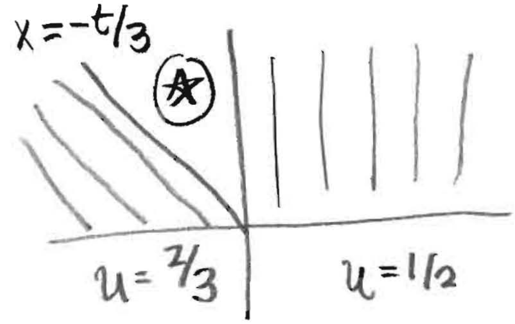
$f'(u) = 1 - 2u$

$f'(U_L) = 0$

$f'(U_R) = -1/3.$

$$u(x,t) = \begin{cases} 2/3 & x > -t/6 \\ 1/2 & x < -t/6. \end{cases}$$

(ii) $U_L = 2/3$, $U_R = \frac{1}{2}$ (no shock forms)



In the region ⊛, a rarefaction wave. On the lines $x = ct$ (characteristics with $c = f'(u)$) the solution has value u as before.

$$c = f'(u) = 1 - 2u \,, \quad \text{so} \quad u = (1-c)/2.$$

Note: In ⊛ as c goes from $-1/3$ to 0, u goes from $2/3$ to $1/2$ as in the diagram above.

$$u(x,t) = \begin{cases} 1/2 & x > 0 \\ (1-c)/2 & \text{if } -\frac{t}{3} < x < 0 \\ 2/3 & x < -t/3. \end{cases}$$

Consider now a rarefaction wave in the general setting for $u_t + (f(u))_x = 0$. It has a left and right state with

$$f'(U_L) < f'(U_R). \tag{8}$$

It is a solution where u is constant on lines $x = ct$ and $f'(u) = c$ determines the value of u on the line with speed c. If f is strictly convex ($f''(u) > 0$ for all u) then f' is increasing and so (8) requires $U_L < U_R$ as in Burger's equation. In this case $f'(u) = c$ uniquely determines $u(c)$ increasing from $c = f'(U_L)$ to $c = f'(U_R)$.

If f is strictly concave ($f''(u) < 0$ for all u) then $U_L > U_R$ as in the traffic flow case and $f'(u)$ is decreasing. $f'(u) = c$ in the rarefaction uniquely determines $u(c)$, decreasing from $c = f'(U_L)$ to $c = f'(U_R)$.

<u>Note</u>: When $f(u)$ is not convex, the solution to Riemann problems can be a mix of shocks and rarefaction waves. We will not consider this case in Math 400.

Math 400, Notes part VI.
Fall, 2012, Brian Wetton.

Consider the heat equation with zero temperature boundary conditions

$u(x,t)$, $0 \le x \le L$, $t \ge 0$.

$u(0,t) = 0$, $u(L,t) = 0$ (boundary conditions) (1)

$u(x,0) = u_0(x)$ given (initial conditions)

$$C(x)\rho(x)A(x)u_t = (K(x)A(x)u_x)_x \quad (2)$$

When C, ρ, A, K are constant, we wrote the nondimensional equation.

$$u_t = u_{xx}, \quad x \in [0,1]. \quad (3)$$

$$u(0,t) = 0, \quad u(1,t) = 0.$$

Which we wrote as $u_t = \mathcal{L}u$, where $\mathcal{L}$ was the spatial differential operator defined on functions $\varphi(x)$, $x \in [0,1]$ with $\varphi(0) = 0$, $\varphi(1) = 0$, and $\mathcal{L}\varphi = \varphi''$. Remember that $\mathcal{L}$ includes the spatial derivatives <u>and</u> the boundary conditions. We wrote a series solution for (3) using the eigenanalysis of $\mathcal{L}$ (which led to the Fourier sine series).

Consider the vertical displacements in the stretched wire, linearized equation

$u(x,t)$, $0 \le x \le L$, $t \ge 0$

$u(0,t) = 0$, $u(L,t) = 0$ (boundary conditions)

$$\left.\begin{aligned} u(x,0) &= u_0(x) \\ u_t(x,0) &= u_1(x) \end{aligned}\right\} \text{given}$$

$$\mu(x)u_{tt} = T u_{xx} \quad (4)$$

When $\mu(x)$ [wire mass density per unit length] is constant, we could also scale this problem to

$$u_{tt} = u_{xx} \qquad x \in [0,1] \qquad (5).$$

$$u(0,t) = 0, \quad u(1,t) = 0.$$

Note that here, $u_{tt} = \mathcal{L} u$ with the same $\mathcal{L}$ as above. Recall,

$$\mathcal{L}\, \varphi_n(x) = \lambda_n\, \varphi_n(x)$$

with $\varphi_n(x) = \sin n\pi x$, $\lambda_n = -n^2\pi^2$. We could consider solutions of (5) of the form

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t) \sin n\pi x, \quad u_n(t) \text{ to be determined.}$$

put this form into (5),

$$\sum_{n=1}^{\infty} \ddot{u}_n \sin n\pi x = \sum_{n=1}^{\infty} -n^2\pi^2 u_n \sin n\pi x$$

using orthogonality,

$$\ddot{u}_n = -n^2\pi^2 u_n \quad \text{for each } n.$$

Solutions $u_n(t) = A_n \sin n\pi t + B_n \cos n\pi t$.
so $(A_n, B_n \text{ constants})$

$$u(x,t) = \sum_{n=1}^{\infty} \left(A_n \sin n\pi t + B_n \cos n\pi t\right) \sin n\pi x. \qquad (6)$$

This is the general solution to the wave equation in the interval $x \in [0,1]$ with these boundary conditions.

$\boxed{Q:}$ Don't we have a general solution to the 1D wave equation, the D'Alembert solution? It can be shown that (6) can be written in the D'Alembert form. However, (6) is an alternate description as a sum of standing waves (Frequencies $2/n$).

If we wanted to find series solutions of the general problems (2) and (4)

$$u_t = \frac{1}{c(x)\rho(x)A(x)}\left(K(x)A(x)u_x\right)_x$$

$$u_{tt} = \frac{T}{\mu(x)}u_{xx}$$

We would naturally want to look at eigen values of other linear operators of the form

$$L\varphi = \frac{1}{a(x)}\left(b(x)\varphi'\right)' \qquad (7)$$

with $a(x) > 0$, $b(x) > 0$. Generalized operators and also generalized boundary conditions.

We considered in the derivation of the heat equation that $-u_x$ is proportional to heat flux in the $+x$ direction. So for the heat equation a boundary condition of $u_x = 0$ is for an insulated end. This would lead to boundary conditions $\varphi' = 0$ for L in (7). We could also consider boundary conditions at $x = 0$ (left end) of

$$u - \alpha u_x = 0 \qquad (\alpha u_x = u) \qquad \alpha > 0 \text{ constant}$$

In words this says that heat flux outwards (to the left) increases as the temperature increases. This is a radiation condition to zero temperature. At $x = L$ (right end),

$$u + \alpha u_x = 0$$

is appropriate.

These boundary conditions have corresponding conditions for $\mathcal{L}$ in (7)

$$u = 0 \quad \Rightarrow \quad \varphi = 0 \qquad \text{Dirichlet}$$

$$u_x = 0 \quad \Rightarrow \quad \varphi' = 0 \qquad \text{Neumann}$$

$$u \pm \alpha u_x = 0 \quad \Rightarrow \quad \varphi \pm \alpha \varphi' = 0 \qquad \text{Robin}$$

So we'd like to consider general linear operators (7) with general boundary conditions

$$\left.\begin{aligned} &c_1 \varphi(0) + c_2 \varphi'(0) = 0 \qquad c_1, c_2 \text{ constants not both zero.} \\ &d_1 \varphi(L) + d_2 \varphi'(L) = 0 \qquad d_1, d_2 \text{ constants not both zero} \end{aligned}\right\} \quad (8)$$

If $\mathcal{L}\varphi = \lambda \varphi$ in (7) then

$$(b(x)\varphi')' = \lambda a(x) \varphi \qquad (b(x) > 0,\ a(x) > 0). \quad (9)$$

Together with (8) this is known as a Sturm-Liouville problem. The general theory of these problems is summarized below:

[1] all eigenvalues λ are real

[2] all eigenvalues are simple, that is there is only one eigenfunction for each eigenvalue.

[3] If $\varphi_n(x)$ and $\varphi_m(x)$ are two eigenfunctions corresponding to different eigenvalues λ_n and λ_m then

$$\int_0^1 a(x)\, \varphi_n(x)\, \varphi_m(x)\, dx = 0.$$

(a kind of orthogonality). Here $[0, L]$ has been scaled to $[0, 1]$.

[4] The eigenvalues form an infinite sequence.

$$\lambda_1 > \lambda_2 > \lambda_2 > \cdots$$

and $\lambda_n \to -\infty$ as $n \to \infty$.

[5] Suppose f is C_1 (continuously differentiable) on $[0,1]$, then

$$\sum_{n=1}^{\infty} f_n \varphi_n(x)$$

Converges pointwise to $f(x)$ where

$$f_n = \frac{\int_0^1 a(x)\, \varphi_n(x)\, f(x)\, dx}{\int_0^1 a(x) [\varphi_n(x)]^2\, dx} \qquad (10).$$

Where does (10) come from? If we wanted to represent any function $f(x)$ as a linear combination of eigenfunctions, i.e.

$$f(x) = \sum_{n=1}^{\infty} f_n \varphi_n(x) \qquad (11)$$

Then we could use property [3] above, multiplying the relationship above by $a(x)\varphi_m(x)$ to get

$$\int_0^1 a(x)\, \varphi_m(x)\, f(x)\, dx = \sum_{n=1}^{\infty} f_n \int_0^1 a(x)\, \varphi_n(x)\, \varphi_m(x)\, dx$$

(some justification of the exchange of sum and integration is needed). The summand is only non-zero when $n=m$, so

$$\int_0^1 a(x)\, \varphi_m(x)\, f(x)\, dx = f_m \int_0^1 a(x) [\varphi_m(x)]^2\, dx$$

from which (10) follows. However, property [5] says more than this, it says that the set of eigenfunctions is always complete, i.e. that it spans the set of all functions in the set, that it is a basis.

We can prove property [1]. It is the same argument as in the assignment #1 question. We can also show property [3]. The others require more theoretical techniques.

Proof of [3]: Consider φ_n that satisfies

$$\lambda_n a(x)\varphi_n(x) = (b(x)\varphi_n')' \qquad (12).$$

with $c_1\varphi_n(0) + c_2\varphi_n'(0) = 0$

$$d_1\varphi_n(1) + d_2\varphi_n'(1) = 0.$$

multiply (12) by φ_m, which also satisfies the boundary conditions above and (12) with λ_m, and integrate from zero to one.

$$\lambda_n \int_0^1 a(x)\varphi_n(x)\varphi_m(x)\,dx = \int_0^1 (b(x)\varphi_n')'\varphi_m\,dx$$

$$\underset{\substack{\uparrow\\ \text{integrate by}\\ \text{parts.}}}{=} -\int_0^1 b(x)\varphi_n'\varphi_m'\,dx + \underbrace{b(x)\varphi_m\varphi_n'\Big|_{x=0}^{x=1}}_{\star} \qquad (13)$$

Consider $\star$ at $x=1$. If $d_2=0$ then $\varphi_m(1)=0$ and this boundary term is zero. Otherwise, $\varphi_n'(1) = -\frac{d_1}{d_2}\varphi_n(1)$ and the boundary term is $b(1)\varphi_m(1)\varphi_n'(1) = -\frac{b(1)d_1}{d_2}\varphi_m(1)\varphi_n(1)$. We can make a similar conclusion at the $x=0$

boundary, leading to (13) $\Rightarrow$

$$\lambda_n \int_0^1 a(x)\,\varphi_n(x)\,\varphi_m(x)\,dx = -\int_0^1 b(x)\,\varphi_n'\varphi_m'\,dx$$

$$\underbrace{-\frac{b(1)\,d_1}{d_2}\varphi_m(1)\,\varphi_n(1)}_{\text{or zero if } d_2=0} + \underbrace{\frac{b(1)\,c_1}{c_2}\varphi_m(0)\,\varphi_n(0)}_{\text{or zero if } c_2=0.}$$

To get this result, we just needed φ_m to satisfy (12) and both φ_n and φ_m to satisfy the boundary conditions. Thus, we could start with the expression for φ_m in (12), multiply by φ_n and proceed to get

$$\lambda_m \int_0^1 a(x)\,\varphi_n(x)\,\varphi_m(x)\,dx = -\int_0^1 b(x)\,\varphi_n'\varphi_m'\,dx$$

$$\underbrace{-\frac{b(1)\,d_1}{d_2}\varphi_m(1)\,\varphi_n(1)}_{\text{or zero if } d_2=0} + \underbrace{\frac{b(1)\,c_1}{c_2}\varphi_m(0)\,\varphi_n(0)}_{\text{or zero if } c_2=0.}$$

Subtracting this from the result at the top of the page gives

$$(\lambda_m - \lambda_n)\int_0^1 a(x)\,\varphi_n(x)\,\varphi_m(x)\,dx = 0$$

Thus, when $\lambda_n \neq \lambda_m$, the integral must be zero. This is the result [3] on p. 4.

So there are some strong theoretical properties of generalized operator eigenvalue problems (8), (9). We'll only be able to make limited progress on these problems analytically, but it is important to realize why we are interested in the results.

Getting even the first few eigenvalues would tell us the slowest decaying modes for heat conduction problems and the lowest frequency standing waves for wave equation problems. These are important quantities to applications.

Math 400, Notes VIb
Fall, 2012

Let's work out some Sturm-Liouville problems.

<u>Example</u> Find the eigenvalues and eigenvectors of

$\mathcal{L}\varphi = \varphi''$, where $\varphi(0)=0$, $\varphi(1)+\alpha\varphi'(1)=0$. $\alpha>0$ constant.

$\varphi''=\lambda\varphi$, can show that $\lambda>0$ leads only to trivial solutions, take $\lambda=-\mu^2$. Then

$$\varphi = A\sin\mu x + B\cos\mu x, \quad \varphi(0)=0 \Rightarrow B=0.$$

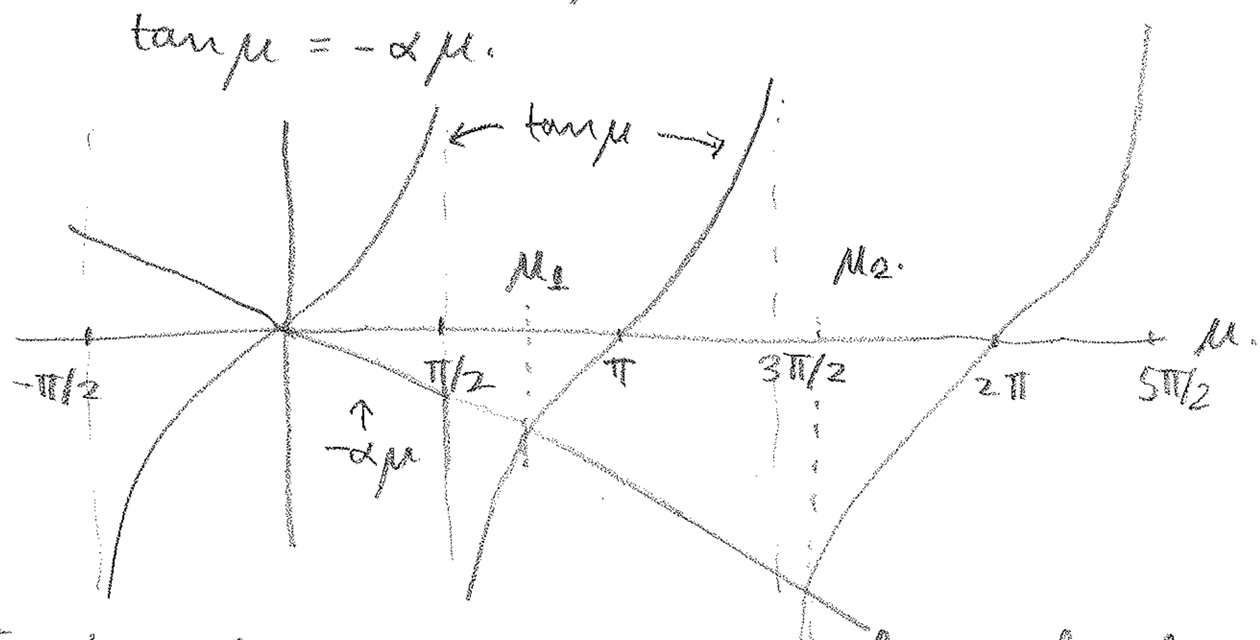
$$\varphi(1)+\alpha\varphi'(1) = A(\sin\mu + \alpha\mu\cos\mu)=0.$$

Need $A\neq 0$ for nontrivial solution, so

$$\sin\mu + \alpha\mu\cos\mu = 0$$

$\mu=0$ is a trivial root, otherwise

$$\tan\mu = -\alpha\mu.$$



Roots $\pm\mu_1, \pm\mu_2, \ldots$

No analytic formula for μ_n, but can find the first few using Newton's method.

Eigenvalues

$$\lambda_n = -\mu_n^2,$$

expected from property [4] of the S-L theory.

Also,

$$\int_0^1 \sin\mu_n x \sin\mu_m x\, dx = 0 \quad \text{if} \quad n\neq m \text{ and}$$

$\sum_{n=1}^{\infty} f_n \sin\mu_n x$ converges to $f(x)$ when

$$f_n = \frac{\int_0^1 f(x)\sin\mu_n x\,dx}{\int_0^1 \sin^2\mu_n x\,dx.}$$

We could call this the sine-α series.

Application $-\mu_1^2$ is the slowest decay rate in the heat conduction problem,

$$u_t = u_{xx}, \quad u(0,t)=0, \; u(0,t) + u_x(0,t) = 0$$

Also μ_1, μ_2, μ_3 are the three lowest frequencies of vibration in the wave equation problem

$$u_{tt} = u_{xx} \quad u(0,t)=0, \; u(0,t) + u_x(0,t) = 0.$$

Note that $\mu_1 < \pi$. This is expected from our understanding of heat conduction models – do you see why?

Example Consider heat conduction in a rod of unit length composed of two different materials, each of length $1/2$:

$$u_t = \begin{cases} \frac{1}{\rho_1 c_1}(K_1 u_x)_x & x < 1/2 \\ \frac{1}{\rho_2 c_2}(K_2 u_x)_x & x > 1/2. \end{cases}$$

Take Dirichlet conditions at the ends, $u(0,t)=0$, $u(1,t)=0$ and remember the junction conditions that we derived,

$$u(\tfrac{1}{2}_-, t) = u(\tfrac{1}{2}_+, t)$$

$$K_1 u_x(\tfrac{1}{2}_-, t) = K_2 u_x(\tfrac{1}{2}_+, t).$$

We are naturally led to the S-L problem for $\varphi(x)$ with $\varphi(0)=0$, $\varphi(1)=0$, $\varphi(\frac{1}{2}_-) = \varphi(\frac{1}{2}_+)$ and $K_1\varphi'(\frac{1}{2}_-) = K_2\varphi'(\frac{1}{2}_+)$. (1)

with $\mathcal{L}\varphi = \frac{1}{\rho c}(K\varphi')'$ $\quad x \neq 1/2$

so $\mathcal{L}\varphi = \lambda\varphi \Rightarrow (\underset{\sim}{K}\varphi')' = \underset{\sim}{\rho c}\,\lambda\varphi.$ $\quad$ (2)

piecewise constant, discontinuous at $x = 1/2$.

Let's work through the algebra with constant numbers, $c_1\rho_1 = 1$, $K_1 = 1$, $c_2\rho_2 = 2$, $K_2 = 3$. On each side of $1/2$, K is constant, so (2) can be written as

$$\varphi'' = \begin{cases} \lambda\varphi & x < 1/2 \\ \frac{2}{3}\lambda\varphi & x > 1/2 \end{cases} \qquad (3)$$

It can be shown that (3) with boundary and interface conditions (1) has only the trivial solution for $\lambda > 0$. Consider $\lambda = -\mu^2$, solve (3) to get

$$\varphi(x) = \begin{cases} A\sin\mu x + B\sin\mu x, & x < 1/2 \\ C\sin\left(\sqrt{\tfrac{2}{3}}\,\mu(x-1)\right) + D\cos\left(\sqrt{\tfrac{2}{3}}\,\mu(x-1)\right) & x > 1/2. \end{cases}$$

The $x > 1/2$ form is convenient for the application of $\varphi(1) = 0 \Rightarrow D = 0$, like $\varphi(0) = 0 \Rightarrow B = 0$.
We still have to satisfy the interface conditions at $x = 1/2$:

$$\varphi(\tfrac{1}{2}_-) = \varphi(\tfrac{1}{2}_+) \Rightarrow A\sin(\mu/2) = -C\sin\left(\sqrt{\tfrac{2}{3}}\,\tfrac{\mu}{2}\right) \quad (4)$$

$$K_1\varphi'(\tfrac{1}{2}_-) = K_2\varphi'(\tfrac{1}{2}_-) \Rightarrow A\mu\cos\mu/2 = 3C\sqrt{\tfrac{2}{3}}\,\mu\cos\left(\sqrt{\tfrac{2}{3}}\,\tfrac{\mu}{2}\right) \quad (5)$$

which can be written as

$$\begin{bmatrix} \sin(\mu/2) & \sin(\mu/\sqrt{6}) \\ \cos(\mu/2) & -\sqrt{6}\cos(\mu/\sqrt{6}) \end{bmatrix} \begin{bmatrix} A \\ C \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \qquad (6)$$

For there to be nontrivial solutions, the determinant of the matrix above must be zero,

$$f(\mu) := \sqrt{6}\sin\left(\frac{\mu}{2}\right)\cos\left(\frac{\mu}{\sqrt{6}}\right) + \sin\left(\frac{\mu}{\sqrt{6}}\right)\cos\left(\frac{\mu}{2}\right) = 0.$$

Can use rootfinding (Newton's method) to find the first few roots

$$\mu_1 \approx 3.6099, \quad \mu_2 \approx 6.6502$$

$$\mu_3 \approx 10.7716$$

Note, $\pi < \mu_1 < 3\pi/2$ so the slowest decay rate of the mixed bar is between the decay rates of bars made of the pure materials. What are the eigenfunctions $\varphi_n(x)$? Can take $A_n = 1$ in (6) and then

$$C_n = -\frac{\sin(\mu_n/2)}{\sin(\mu_n/\sqrt{6})}$$

giving
$$\varphi_n(x) = \begin{cases} \sin \mu_n x \\ -\dfrac{\sin(\mu_n/2)}{\sin(\mu_n/\sqrt{6})} \sin\left(\sqrt{\dfrac{2}{3}}\,\mu_n(x-1)\right). \end{cases}$$

Then the general solution to the example we are considering is

$$u(x,t) = \sum_{n=1}^{\infty} \underbrace{\alpha_n}_{\text{scalar coefficients.}} \varphi_n(x) e^{-\mu_n^2 t}$$

To match the initial data, need to choose coefficients α_n so that

$$\sum_{n=1}^{\infty} \alpha_n \varphi_n(x) = u_0(x).$$

Need to be a bit careful. S-L property [3] says that $\int_0^1 \rho(x)c(x)\,\varphi_n(x)\varphi_m(x)\,dx = 0$ if $n \neq m$, so

$$\alpha_n = \frac{\int_0^1 \rho(x)c(x)\,\varphi_n(x)\,u_0(x)\,dx}{\int_0^1 \rho(x)\,c(x)\,[\varphi_n(x)]^2\,dx}$$

$$= \frac{\int_0^{1/2} \varphi_n(x)\,u_0(x)\,dx + 2\int_{1/2}^{1} \varphi_n(x)\,u_0(x)\,dx}{\int_0^{1/2} [\varphi_n(x)]^2\,dx + 2\int_{1/2}^{1} [\varphi_n(x)]^2\,dx.}$$

<u>Example</u> Consider heat conduction in a metal bar with $K \equiv 1$, $\rho \equiv 1$ but $c(x)$ slowly varying,

$$c(x) = 1 + 0.01\,x.$$

so $u_t = \dfrac{1}{1+0.01\,x}\,u_{xx}$. Consider easy boundary conditions $u(0,t)=0$, $u(1,t)=0$. We are led naturally to the S-L problem

$$\left.\begin{aligned} &L\varphi := \frac{1}{1+0.01x}\,\varphi'' \qquad \varphi(0)=\varphi(1)=0 \\ &L\varphi = \lambda\varphi \;\Rightarrow\; \varphi'' = \lambda(1+0.01x)\,\varphi. \end{aligned}\right\} \quad (6)$$

Suppose we are interested in λ_1 and $\varphi_1(x)$. 6

We expect $\lambda_1 \approx -\pi^2$ and $\varphi_1(x) \approx \sin \pi x$.

<u>Option 1</u> (6) is solved by know functions known as Airy's functions. Could learn about these functions and proceed.

<u>Option 2</u> Consider (6) more generally as

$\varphi'' = \lambda(1+\varepsilon x)\varphi$, $\varphi(0) = \varphi(1) = 0$. We want $\varphi(x;\varepsilon)$ and $\lambda(\varepsilon)$ smoothly varying from $\varphi(x;0) = \sin \pi x$ and $\lambda(0) = -\pi^2$. actually, we'd be OK with linear approximation,

$$\varphi(x;\varepsilon) \approx \sin \pi x + \varepsilon \frac{\partial \varphi}{\partial \varepsilon}(x;0)$$

$$\lambda(\varepsilon) \approx -\pi^2 + \varepsilon \lambda'(0).$$

But we'd have to find $\frac{\partial \varphi}{\partial \varepsilon}(x;0)$ and $\lambda'(0)$. Start with

$$\varphi'' = \lambda(1+\varepsilon x)\varphi \quad (7) \quad \text{take the derivative wrt } \varepsilon.$$

$$\left(\frac{\partial \varphi}{\partial \varepsilon}\right)'' \leftarrow x \text{ derivatives}$$
$$= \lambda'(1+\varepsilon x)\varphi + \lambda x \varphi + \lambda(1+\varepsilon x)\frac{\partial \varphi}{\partial \varepsilon}.$$

Set $\varepsilon = 0$ in the above equation since we want $\frac{\partial \varphi}{\partial \varepsilon}(x;0)$ and $\lambda'(0)$: $\quad \varphi = \sin \pi x$ when $\varepsilon = 0$

$\lambda = -\pi^2$ when $\varepsilon = 0$

$$\left(\frac{\partial \varphi}{\partial \varepsilon}(x;0)\right)'' = \lambda'(0)\sin \pi x - \pi^2 x \sin \pi x - \pi^2 \frac{\partial \varphi}{\partial \varepsilon}(x;0).$$

$$\frac{\partial \varphi}{\partial \varepsilon}(0;0) = 0, \quad \frac{\partial \varphi}{\partial \varepsilon}(1;0) = 0.$$

Let's simplify notation a bit, letting
$V(x) = \frac{\partial \psi}{\partial \varepsilon}(x;0)$ and $\beta = \lambda'(0)$.

$$V'' + \pi^2 V = -\pi^2 x \sin \pi x + \beta \sin \pi x. \quad \left.\begin{matrix} V(0)=0 \\ V(1)=0 \end{matrix}\right\} (8)$$

Now it is straightforward since (8)
is a constant coefficient problem for $V(x)$.
Use the Method of Undetermined Coefficients.
It is an exceptional (resonant) case.

$$V(x) = A \sin \pi x + B \cos \pi x \quad \leftarrow \text{homogeneous solution}$$

$$\left.\begin{matrix} + \; c x \sin \pi x + d\, x \cos \pi x \\ + \; e x^2 \sin \pi x + f\, x^2 \cos \pi x. \end{matrix}\right\} \text{MUC particular solution.}$$

put form into ODE in (8) to find

$$c = -1/4, \quad d = -\frac{\beta}{2\pi}, \quad e = 0, \quad f = \pi/4.$$

To match the boundary conditions

$$V(0) = 0 \quad \Rightarrow \quad B = 0$$

$$V(1) = 0 \quad \Rightarrow \quad d + f = 0 \quad \Rightarrow \quad \frac{-\beta}{2\pi} + \frac{\pi}{4} = 0$$

so $\beta = \pi^2/2$

$$V(x) = A \sin \pi x - \frac{1}{4} x \sin \pi x - \frac{\pi}{4} x \cos \pi x + \frac{\pi}{4} x^2 \cos \pi x$$

Note: A is still undetermined. There is a bit of a subtlety here. Since $\sin \pi x$ is the original, unperturbed ($\varepsilon=0$) eigenfunction, we can take $A=0$ since this would just change the size of the original eigenfunction, which is arbitrary up to scaling.

So now

$$\lambda_1(\varepsilon) \approx -\pi^2 + \frac{\pi^2}{2}\varepsilon$$

$$\varphi_1(\varepsilon) \approx \sin\pi x + \varepsilon\left(\frac{\pi}{4}x^2\cos\pi x - \frac{\pi}{4}x\cos\pi x - \frac{1}{4}x\sin\pi x\right).$$

We could put in $\varepsilon = 0.01$ to get an approximate result for the original problem.

<u>Note:</u> The method above is called asymptotic analysis. How (8) is usually derived is to start with the expressions

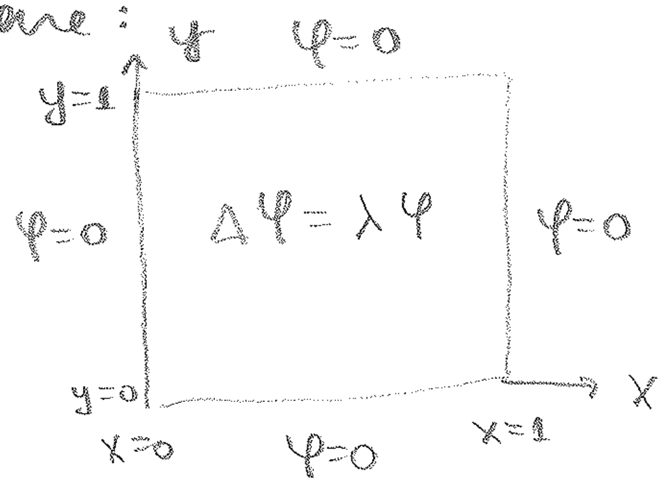
$$\varphi(x) \approx \sin\pi x + \varepsilon V(x)$$

$$\lambda(x) \approx -\pi^2 + \varepsilon\beta$$

These expressions are put into (7) and terms of order ε^2 and higher are neglected, leading to (8). In this sense, the process is a kind of linearization. For some asymptotic problems, higher order terms in ε must be considered.

Math 400, Notes V c
Fall, 2012

We can use the S-L theory to understand some problems in higher dimensions. Let's consider the 2D eigenvalue problem in the square:



$\varphi(x,y)$.

Note that $\Delta\varphi = \varphi_{xx}+\varphi_{yy} = \varphi_{yy} + \mathcal{L}\varphi = \lambda\varphi$ (1)
where $\mathcal{L}$ is our old friend

$$\mathcal{L}\phi = \phi'' \quad \text{with } \phi(x),\ \phi(0)=\phi(1)=0.$$

Look for $\varphi(x,y) = f(y)\sin n\pi x$
expression into (1) to obtain

$$f''(y)\sin n\pi x - n^2\pi^2 f(y)\sin n\pi x = \lambda f(y)\sin n\pi x$$

or $\quad f'' = (\lambda+n^2\pi^2)f \quad$ with $f(0)=0,\ f(1)=0$.

So $f(y)$ satisfies $\mathcal{L}f = (\lambda+n^2\pi^2)f \quad f(0)=f(1)=0$.
But then $f(y)=\sin(m\pi y)$ and

$$\lambda + n^2\pi^2 = -m^2\pi^2 \Rightarrow \lambda = -(n^2+m^2)\pi^2.$$

Thus the eigenvalues of the original problem satisfy $\lambda = -(n^2+m^2)\pi^2$ with eigenfunctions

$$\varphi_{nm}(x,y) = \sin n\pi x \sin m\pi y.$$

Note: This shows that eigenvalues for 2D problems don't have to be *simple*, since for $\lambda = -5\pi^2$ there are two eigenfunctions

$$\varphi_{12} = \sin \pi x \sin 2\pi y$$

and $\varphi_{21} = \sin 2\pi x \sin \pi y$.

However, many of the other properties of S-L problems extend to higher dimensional variants. Note that in this example we still have an orthogonality relationship

$$\int_0^1 \int_0^1 \varphi_{nm} \varphi_{n'm'} \, dx\, dy = \begin{cases} 1/4 & \text{if } n = n' \text{ and } m = m' \\ 0 & \text{otherwise} \end{cases}$$

inherited from the 1D sine series properties.

Applications of this result include

(i) If we considered the heat conduction problem for $u(x,y,t)$ with

$$u_t = \Delta u$$

$0 \leq x \leq 1$, $0 \leq y \leq 1$ (unit square)

with $u = 0$ on all boundaries, the general solution could be written as a series

$$u(x,y,t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} u_{nm} \sin n\pi x \sin m\pi y \, e^{-(n^2+m^2)\pi^2 t} \qquad (2)$$

If initial condions $u_0(x,y)$ were given then

$$u_{nm} = 4 \int_0^1 \int_0^1 u_0(x,y) \sin n\pi x \sin m\pi y \, dx\, dy.$$

of application interest is that the slowest decaying mode has decay rate $2\pi^2$.

(ii) If $u_0(x,y) \equiv 0$ but

$$u_t = \Delta u + f(x,y,t)$$

then $u(x,y,t)$ has a series solution in the same form as (2) but now with $u_{nm}(t)$ given by

$$u_{nm}(t) = \int_0^t e^{-(n^2+m^2)\pi^2 (t-s)} f_{nm}(s)\, ds \qquad (3)$$

where $f_{nm}(t) = 4\int_0^1\int_0^1 f(x,y,t) \sin n\pi x \sin m\pi y \, dx\, dy.$

(iii) If we considered the wave equation problem $u_{tt} = \Delta u$ in the same domain with the same boundary conditions, then u has a series solution

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \left(A_n \cos\left(\sqrt{n^2+m^2}\,\pi t\right) + B_n \sin\left(\sqrt{n^2+m^2}\,\pi t\right)\right) \sin n\pi x \sin m\pi y.$$

The three lowest frequencies of vibration are $\frac{\sqrt{2}}{2}$, $\frac{\sqrt{5}}{2}$ (two modes), $\sqrt{2}$

$n=m=1$ $n=1, m=2$; $n=2, m=1$ $n=2, m=2$

(iv) If we considered the Poisson problem $\Delta u = g(x,y)$ for $u(x,y)$ in the same domain, then u has a series solution

$$u(x,y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{-g_{nm}}{(n^2+m^2)\pi^2} \sin n\pi x \sin m\pi y$$

where $g_{nm} = 4 \int_0^1 \int_0^1 g(x,y) \sin n\pi x \sin m\pi y \, dx\, dy$

Example Consider now heat conduction in the unit sphere. It is natural to use spherical coordinates here $u(r, \theta, \varphi, t)$.

$$u_t = \Delta u \qquad u(r,\theta,\varphi,0) = u_0(r,\theta,\varphi) \text{ given}$$

$$u = 0 \text{ at } r = 1.$$

We will restrict ourselves to radially symmetric solutions, that is $u(r,t)$. The series solution ideas can be extended to the general case, but it gets pretty complicated (spherical harmonics). It is known that

$$\Delta u = \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right)$$

if $u(r,t)$. We consider naturally the S-L problem for $\varphi(r)$, $\varphi(1) = 0$ [and a "hidden" condition at $r=0$ discussed below]

$$L\varphi := \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \varphi}{\partial r}\right) = \lambda \varphi. \tag{4}$$

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial \varphi}{\partial r}\right) = \lambda r^2 \varphi.$$

$\varphi(1) = 0$ & condition at $r=0$ to be determined

To proceed, consider the new variable $y(r) = r\varphi(r)$. Now (4) becomes

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial (y/r)}{\partial r}\right) = \lambda r y.$$

$$\frac{\partial}{\partial r}\left(r^2 \left(\frac{\partial y/\partial r}{r} - y/r^2\right)\right) = \lambda r y.$$

$$\frac{\partial}{\partial r}\left(r \frac{\partial y}{\partial r} - y\right) = \lambda r y.$$

$$r \frac{\partial^2 y}{\partial r^2} + \cancel{\frac{\partial y}{\partial r}} - \cancel{\frac{\partial y}{\partial r}} = \lambda r y$$

$$\Rightarrow \quad \frac{d^2 y}{dr^2} = \lambda y \qquad \varphi(1) = 0 \Rightarrow y(1) = 0$$

Also, since $\varphi(r) = \frac{y(r)}{r}$ and $\varphi(0)$ is bounded, $y(0) = 0$ and the expression above must be interpreted as a limit. This is the "hidden" condition for φ at $r=0$ (that it is bounded). The problem for $y(r)$ is our old friend that leads to the sign series.

$$y_n(r) = \sin n\pi r, \quad \lambda_n = -n^2\pi^2.$$

and $\varphi_n(r) = \frac{1}{r}\sin n\pi r.$

Note that the $y_n(r)$ are orthogonal "as usual"

$$\int_0^1 y_n(r)\, y_m(r)\, dr = \begin{cases} 1/2 & \text{if } n = m \\ 0 & \text{otherwise}\end{cases}$$

while the $\varphi_n(r)$ are orthogonal in the following way:

$$\int_0^1 r^2 \varphi_n(r)\, \varphi_m(r) = 0 \quad \text{if } n \neq m.$$

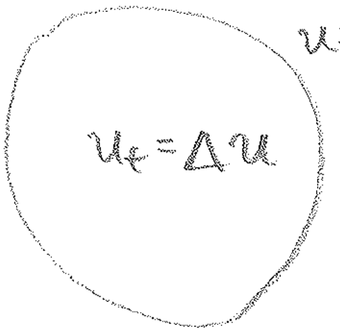
this can be seen in two ways: from the definition of φ_n as $\frac{1}{r}\sin n\pi r$ and also from matching (4) to the S-L theory [3].

<u>Example</u> Consider now heat conduction in a thin plate with the shape of a unit circle. It is natural to consider u in polar coordinates here $u(r,\theta,t)$:

$$u_t = \Delta u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

$$u(r,\theta,0) = u_0(r,\theta) \text{ given}$$

$$u(1,\theta,t) = 0$$



We want to investigate $\mathcal{L}\varphi(r,\theta) := \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \varphi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \varphi}{\partial \theta^2}$

In particular to find eigenfunctions

$$r\frac{\partial}{\partial r}\left(r\frac{\partial \varphi}{\partial r}\right) + \frac{\partial^2 \varphi}{\partial \theta^2} = \lambda r^2 \varphi.$$

Rewrite as

does not depend on r

$$r\frac{\partial}{\partial r}\left(r\frac{\partial \varphi}{\partial r}\right) - \lambda r^2 \varphi = -\frac{\partial^2 \varphi}{\partial \theta^2} = -M\varphi \qquad (5)$$

where M acts upon functions $\phi(\theta)$ with ϕ 2π-periodic and $M\phi = \phi''$. In (5) consider r to be playing a time-like role, so we would want to do the eigenanalysis of M.

$M\phi = \beta\phi \Rightarrow \phi'' = \beta\phi$ and ϕ 2π-periodic.
Note that this leads naturally to the Fourier series, $\phi = \cos(n\theta) \Rightarrow \beta = -n^2$ and $\phi = \sin(n\theta) \Rightarrow \beta = -n^2$. Note that we can allow the $n=0$ term in the cosine ($\phi = 1, \beta = 0$). Note also that there is a two-dimensional eigenspace associated with $\beta = -n^2$ for $n > 0$ (the eigenvalue is not simple). Note that this does not violate the S-L theory, since the boundary conditions associated with M are

$$\phi(0) = \phi(2\pi)$$
$$\phi'(0) = \phi'(2\pi)$$

which are not of the same form as the S-L theory (they mix data at different boundary points).

<u>Aside</u> It is convenient to introduce the complex fourier series at this point. Consider $\phi_n(\theta) = e^{in\theta}$ and $\phi_{-n} = e^{-in\theta}$ for $n > 0$. We know that

$$\left.\begin{aligned} \phi_n(\theta) &= \cos n\theta + i\sin n\theta \\ \phi_{-n}(\theta) &= \cos n\theta - i\sin n\theta. \end{aligned}\right\} (6)$$

Note that $\frac{1}{2}(\phi_n + \phi_{-n}) = \cos n\theta$ and

$$\frac{1}{2i}(\phi_n - \phi_{-n}) = \sin n\theta$$

so we can consider $\{\varphi_n; \varphi_{-n}\}$ as another basis for the eigenspace associated to $\beta = -n^2$. These functions are orthogonal in the sense that

conjugate

$$\int_0^{2\pi} \phi_n(x)\overline{\phi_m(x)}\,dx =$$

$$\int_0^{2\pi} e^{inx} e^{-imx}\,dx = \int_0^{2\pi} e^{i(n-m)x}\,dx$$

$$= \begin{cases} \int_0^{2\pi} 1\,dx = 2\pi & \text{if } n = m \\ \frac{1}{i(n-m)} e^{i(n-m)x}\Big|_0^{2\pi} = \frac{1}{i(n-m)}\left[\cos(n-m)x + i\sin(n-m)x\right]\Big|_0^{2\pi} = 0. & \text{if } n \neq m. \end{cases}$$

Thus $f(x) = \sum_{n=-\infty}^{\infty} f_n e^{inx}$

with $f_n = \frac{1}{2\pi}\int_0^{2\pi} f(x) e^{-inx}\,dx$

complex form of Fourier series.

is an alternate form of the Fourier series.

Now return to (5) using the complex fourier series that comes from the eigenanalysis of M. Take

$$\varphi(r,\theta) = \psi(r)\, e^{in\theta}$$

so (5) becomes

$$r(r\psi')' - \lambda r^2 \psi = n^2 \psi.$$

$\psi(1) = 0$

$\psi(0)$ bounded

It can be shown that $\lambda < 0$ is needed for nontrivial solutions, write $\lambda = -\mu^2$

$$r(r\psi')' + (\mu^2 r^2 - n^2)\psi = 0. \qquad (7)$$

Let $x = r\mu$, $\frac{d\psi}{dr} = \frac{d\psi}{dx} \cdot \frac{dx}{dr} = \mu \frac{d\psi}{dx}$

$$(r\psi')' = \mu \frac{d}{dx}\left(x \frac{d\psi}{dx}\right)$$

$$r(r\psi')' = x \frac{d}{dx}\left(x \frac{d\psi}{dx}\right)$$ so (7) becomes

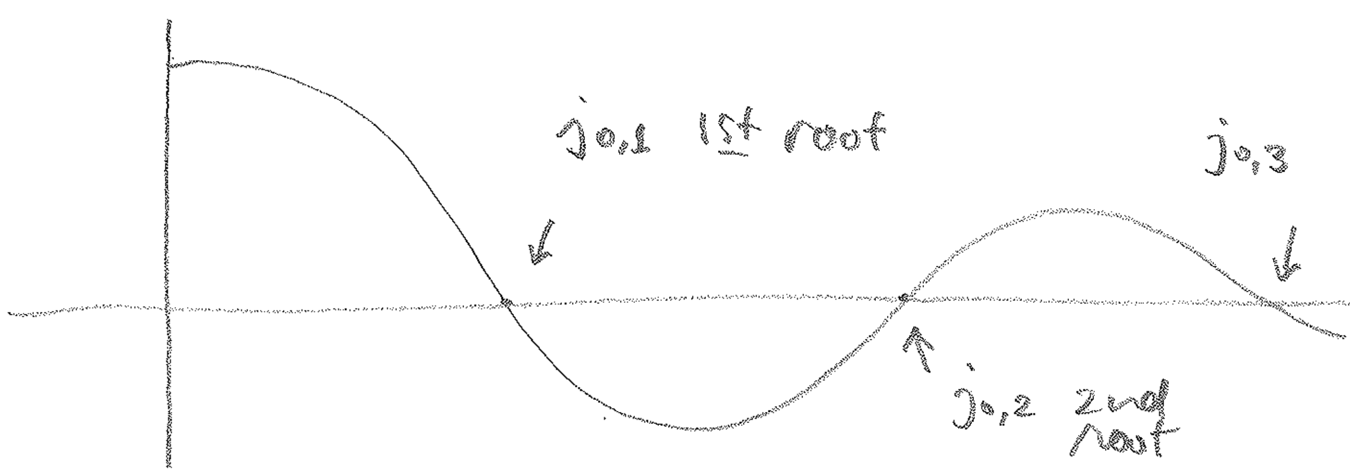
$$x \frac{d}{dx}\left(x \frac{d\psi}{dx}\right) + (x^2 - n^2)\psi = 0. \qquad \begin{array}{l}\psi(x=\mu)=0\\ \psi(0) \text{ bounded}\end{array} \quad (8).$$

Now (8) has well-studied solutions known as Bessel functions. To satisfy (8) and $\psi(0)$ bounded,

$$\psi(x) = A \, J_n(x)$$

↑ Bessel function of order n.

Graph of $J_0(x)$:



$j_{0,1} \approx 2.4048$, $j_{0,2} \approx 5.5201$,

$j_{0,3} \approx 8.6537$.

Roots of $J_n(x)$ [labelled $j_{n,s}$ for the s'th root] are listed in a table at the end of these notes. They are also available online at

mathworld.wolfram.com/BesselFunctionZeros.html

Now for $\psi_n(\overset{x=}{\mu}) = 0$, μ must be one of the zeros $j_{n,s}$. Then

$$\lambda = -j_{n,s}^2$$

and $\psi_{n,s}(r) = J_n(r\mu)$. Now we can go back to the original problem on p. 6 and write

$$\varphi_{n,s}(r,\theta) = e^{in\theta} J_n(r\mu)$$

$$\lambda = -j_{n,s}^2$$

as the eigenfunctions and eigenvalues of L. Then

$$u(r,\theta,t) = \sum_{n=-\infty}^{\infty} \sum_{s=1}^{\infty} \overset{u_{n,s}\text{ coefficient}}{\downarrow}\, e^{-j_{n,s}^2 t} e^{in\theta} J_n(r\mu)$$

is the series solution to the original heat conduction problem and

$$u_{n,s} = \frac{1}{A_{n,s}} \int_0^1 \int_0^{2\pi} r J_n(j_{n,s} r) \underbrace{u_0(r,\theta) e^{-in\theta} \, d\theta}_{\text{complex Fourier coefficient for each } r.} \, dr.$$

where the normalizing term is

$$A_{n,s} = 2\pi \int_0^1 r \left[J_n (j_{n,s}\, r) \right]^2 dr.$$

The orthogonality weight function r can be seen from the form of the eigenvalue problem on the bottom of p. 8.

$$r(r\psi')' - \lambda r^2 \psi = n^2 \psi$$

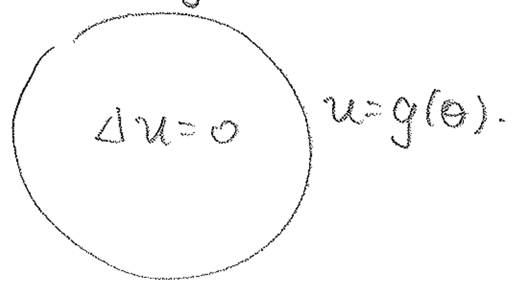
$$\Rightarrow \quad (r\psi')' - \frac{n^2 \psi}{r} = \lambda r \psi$$

weight r in S-L theory [3].

<u>Final example.</u> Consider the Laplace problem in the unit disk. As above, it is natural to consider u in polar coordinates $u(r,\theta)$.

$$\Delta u = 0 \qquad r < 1$$

$$u(1,\theta) = g(\theta) \quad \text{given}.$$



$$\Delta u = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0. \qquad (9).$$

From the previous problem, we know that we should consider $u(r,\theta)$ as its Fourier series in θ at each r:

$$u(r,\theta) = \sum_{n=-\infty}^{\infty} u_n(r)\, e^{in\theta}$$

We can insert this into (9) and use the orthogonality of the Fourier series to get for each n,

$$\frac{1}{r}(r u_n')' - \frac{n^2}{r^2} u_n = 0.$$

$$(r u_n')' - \frac{n^2 u_n}{r} = 0. \qquad u_n(1) = g_n,\ u_n(0) \text{ bounded} \tag{10}$$

Like many times before in the course, we have reduced the solution of a PDE to a family of ODEs and have been able to make progress.

n'th component of the Fourier Series of $g(\theta)$,

$$g_n = \frac{1}{2\pi}\int_0^{2\pi} g(\theta) e^{-in\theta}\, d\theta$$

To proceed, let $x = \ln r$, $r = e^x$

$$u_n' = \frac{du_n}{dr} = \frac{du_n}{dx}\cdot\frac{dx}{dr} = \frac{du_n}{dx}\frac{1}{r}.$$

So $(r u_n')' = \frac{1}{r}\frac{d^2 u_n}{dx^2}$ and (10) reads.

$$\frac{d^2 u_n}{dx^2} - n^2 u_n = 0. \qquad n \neq 0$$

This gives $u_n(x) = A e^{|n|x} + B e^{-|n|x}$ $\quad x = \ln r.$

$$\Rightarrow u_n(r) = A r^{|n|} + B r^{-|n|}$$

Using the Boundary conditions for (10), $B = 0$, and $A = g_n$. When $n = 0$,

$$u_n = A + Bx = A + B\ln r$$

and then $B = 0$ and $A = g_0$.

Thus the solution to the Laplace problem in a disk is

$$u(r,\theta) = \sum_{n=-\infty}^{\infty} g_n\, r^{|n|} e^{in\theta} \qquad (11).$$

Note: This shows that high frequency boundary data ($|n|$ large) does not penetrate very far into the disk in this problem.

The result (11) can be written in a different form. Write

$$u(r,\theta) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \left(\int_0^{2\pi} g(\theta') e^{-in\theta'} d\theta' \right) r^{|n|} e^{in\theta} \qquad (12)$$

Can justify the exchange of summation and integration in cases of interest to obtain

$$u(r,\theta) = \frac{1}{2\pi} \int_0^{2\pi} \left(\sum_{n=-\infty}^{\infty} e^{in(\theta-\theta')} r^{|n|} \right) g(\theta') d\theta'. \qquad (12)$$

Consider the sum broken into 3 pieces:

$$1 + \sum_{n=1}^{\infty} e^{in(\theta-\theta')} r^{|n|} + \sum_{n=1}^{\infty} e^{-in(\theta-\theta')} r^{|n|}$$

$\sum_{n=1}^{\infty} \alpha^n$ with $\alpha = e^{i(\theta-\theta')} r$ — similar geometric series

geometric series $\frac{\alpha}{1-\alpha}$

$$1 + \frac{re^{i(\theta-\theta')}}{1 - re^{i(\theta-\theta')}} + \frac{re^{-i(\theta-\theta')}}{1 - re^{-i(\theta-\theta')}}$$

$$= \text{some algebra} = \frac{1-r^2}{1 - 2r\cos(\theta-\theta') + r^2}$$

Thus, (12) reads

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \overbrace{\frac{(1-r^2)}{1-2r\cos(\theta-\theta')+r^2}}^{\text{Green's function}} g(\theta')\,d\theta'.$$

This is known as Poisson's formula and can be derived from complex variable theory in a different way. This result is known as the Green's function representation of the solution to this problem. It is a map from the data g to the solution u. Note that all the data affects all the solution values, there is no collapse of information movement along characteristics here.

Math 400, Notes $\underline{VI}$

Fall, 2012.

Consider $f(x)$, defined for all $x \in \mathbb{R}$. Suppose f is continuous and that

$$\int_{-\infty}^{\infty} |f(x)|\, dx$$

is finite. Then $\hat{f}(\alpha)$ (the Fourier Transform of f) is defined as

$$\hat{f}(\alpha) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x)\, e^{-i\alpha x}\, dx \qquad [\hat{f} = F(f)]$$

and

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\alpha)\, e^{i\alpha x}\, d\alpha \qquad [f = F^{-1}(\hat{f})]$$

at every x. The fourier transform represents a decomposition of the information in f into frequency (wavenumber) representation.

The Fourier transform is naturally extended to square integrable functions and here,

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = 2\pi \int_{-\infty}^{\infty} |\hat{f}(\alpha)|^2 d\alpha$$

Some selected properties of the Fourier Transform, with notation $F(f) = \hat{f}$.

(i) $F(f') = i\alpha \hat{f}$

"The Fourier transform diagonalizes derivatives"

(ii) Convolution results,

$$f * g := \int_{-\infty}^{\infty} f(x-s)\, g(s)\, ds.$$

$$F(f*g) = 2\pi \hat{f}(\alpha)\, \hat{g}(\alpha)$$

$$F(f(x)\, g(x)) = 2\pi \hat{f}(\alpha) * \hat{g}(\alpha)$$

$$F^{-1}(\hat{f}(\alpha)\hat{g}(\alpha)) = \frac{1}{2\pi} f * g.$$

just read backwards

(iii) $F(f(x-c)) = e^{-i\alpha c} \hat{f}(\alpha).$

(shift).

(iv) Gaussians transform into Gaussians.

If $f(x) = e^{-x^2/(2a^2)}$ then

$$\hat{f}(\alpha) = \frac{a}{\sqrt{2\pi}} e^{-a^2\alpha^2/2}$$

Fourier transforms "diagonalise" every constant coefficient, linear Cauchy problem

<u>Example</u> Consider $u(x,t)$ that solves

$$u_t + u_x = 0, \qquad (1)$$

$t > 0$, all x, with $u(x,0) = u_0(x)$ given.

[This is our old friend W1, with solution $u(x,t) = u_0(x-t)$, let's get this with the Fourier transform method].

Using property (ii) we see that $u(x,t)$ can be written as a convolution,

$$u(x,t) = \frac{1}{2\pi} u_0 * g \qquad \downarrow \; a^2 = 2t$$

where $\hat{g}(\alpha) = e^{-\alpha^2 t}$. From property (iv) this gives

$$g(x) = \frac{\sqrt{\pi}}{\sqrt{t}} e^{-x^2/(4t)}$$

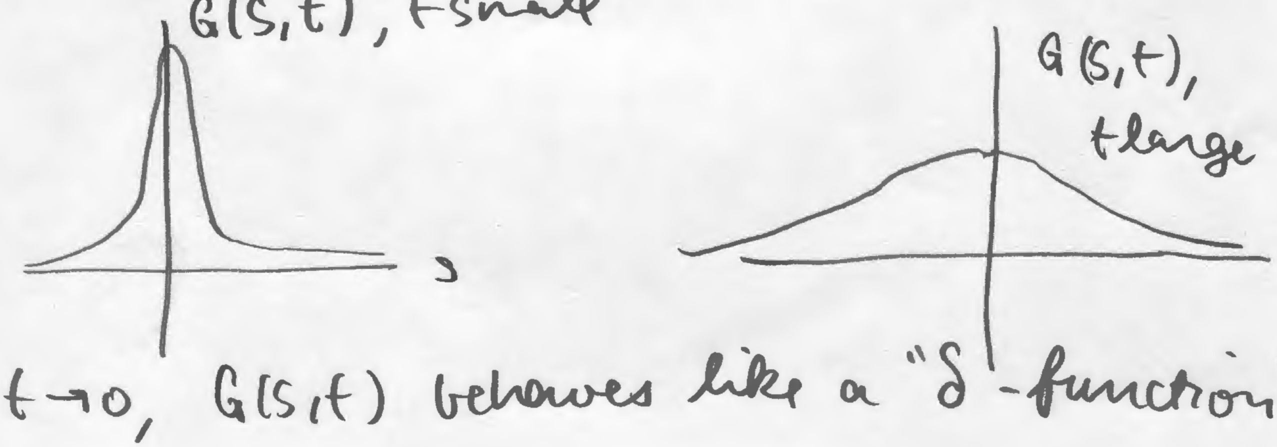
and so $u(x,t)$ can be written as

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} u_0(x-s) e^{-s^2/4t} \, ds. \qquad (3)$$

This is another Green's function representation of the solution. Note that $u(x,t)$ for $t>0$ depends on <u>all</u> the initial data $u_0(x)$.

$$\underbrace{\frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-s^2/4t}}_{G(s,t)} \, ds = 1 \quad \text{for all } t,$$

so (3) is a weighted average of $u_0(s)$ values, with s "near" x. As $t \to 0$ the average is over a narrower and narrower region.



as $t \to 0$, $G(s,t)$ behaves like a "δ-function"

At each time t, take the Fourier Transform of $u(x,t)$ in x:

$$F(u(x,t)) = \hat{u}(\alpha,t).$$

Take the transform of equation (1), using property (i),

$$\frac{\partial \hat{u}}{\partial t} + i\alpha\,\hat{u} = 0 \qquad \hat{u}(\alpha,0) = \hat{u}_0(\alpha).$$

Now this is an ODE for every α (no α derivatives) with solution

$$\hat{u}(\alpha,t) = e^{-i\alpha t}\,\hat{u}_0(\alpha).$$

Reading property (iii) backwards gives

$$u(x,t) = u_0(x-t)$$

as expected.

<u>Example</u> Consider $u(x,t)$ that solves

$$u_t = u_{xx} \tag{2}$$

$t \geq 0$, all x, with $u(x,0) = u_0(x)$ given. As above take the Fourier transform of (2) to obtain

$$\frac{d\hat{u}}{dt} = \underbrace{-\alpha^2}_{(i\alpha)^2}\,\hat{u} \qquad \hat{u}(\alpha,0) = \hat{u}_0(\alpha)$$

so $\hat{u}(\alpha,t) = e^{-\alpha^2 t}\,\hat{u}_0(\alpha).$

<u>Example</u> Consider the Laplace problem for $u(x,y)$ in the domain $y \geq 0$, with data $u(x,0) = u_0(x)$ given. Look for solutions that are bounded as $y \to \infty$.

<u>Note</u>: This is not a Cauchy problem. The boundedness condition as $y \to \infty$ is really a boundary condition.

$$u_{xx} + u_{yy} = 0 \qquad y \geq 0.$$

Take the Fourier Transform in x,

$$-\alpha^2 \hat{u} + \hat{u}_{yy} = 0 \qquad \hat{u}(\alpha, 0) = \hat{u}_0(\alpha).$$

$$\hat{u}(\alpha,t) = \cancel{A e^{|\alpha| y}} + B e^{-|\alpha| y}$$

by boundedness.

$$\hat{u}(\alpha,t) = u_0(\alpha)\, e^{-|\alpha| y}.$$

This is again gives $u(x,t)$ as a convolution, $u(x,t) = \frac{1}{2\pi} u_0 * g$, with

$$\hat{g}(\alpha,y) = e^{-|\alpha| y}$$

y is just a parameter here

Integrating $g(x,y) = \int_{-\infty}^{\infty} e^{-|\alpha| y} e^{i\alpha x}\, d\alpha$ or looking in tables gives

$$g(x) = \frac{2y}{x^2+y^2}, \quad \text{leading to}$$

$$u(x,y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{u_0(x)\, y}{x^2 + y^2}.$$

Again, this shows that $u(x,y)$ for $y > 0$ is a weighted average of $u_0(x)$ values. In both the previous cases, the resulting formula can be used to show a maximum principle.

Math 400, Notes VI b
Fall, 2012.

Introduce the Laplace transform of a function $f(t)$, defined for $t \geq 0$:

$$\hat{f}(s) = \int_0^\infty e^{-st} f(t)\, dt. \qquad \hat{f} = \mathcal{L} f$$

Think of s as a complex variable. If $|f(t)| < K e^{at}$ for all t with constants $K > 0$, $a > 0$, then $\hat{f}(s)$ is defined by the formula above for all s with $\mathrm{Re}(s) > a$.

Also, $\hat{f}(s)$ is a complex analytic function, can extend the function from where it is defined by analytic continuation (with possible singularities and branch lines).

Some examples that can be worked out by explicit integration of the formula:

1. $f(t) = e^{at} \Rightarrow \hat{f} = \dfrac{1}{s-a}$ (analytic function with simple pole at a).

2. $f(t) = t^n \Rightarrow \hat{f} = \dfrac{n!}{s^{n+1}}$ (pole of order $n+1$ at $s=0$)

3. $f(t) = \sin t \Rightarrow \dfrac{1}{1+s^2}$

   $f(t) = \cos t \Rightarrow \dfrac{s}{1+s^2}$

4. $f(t) = \mathrm{erfc}\left(\dfrac{1}{2\sqrt{t}}\right) \Rightarrow \hat{f}(s) = \dfrac{1}{s} e^{-\sqrt{s}}$

where $\mathrm{erfc}(x) = \dfrac{2}{\sqrt{\pi}} \displaystyle\int_x^\infty e^{-\tau^2} d\tau$

Properties of the Laplace transform:

(i) $\mathcal{L}(f') = s\tilde{f} - f(0)$. [integration by parts].

(ii) $\mathcal{L}$ is linear and invertible on an appropriate set of functions, i.e. there is a unique $f(t)$ associated with a suitable $\hat{f}(s)$.

(iii) $\mathcal{L}(f * g) = f(s)\, g(s)$

$\mathcal{L}^{-1}(f(s) g(s)) = f * g$

reading first result backwards.

So far, I haven't shown you how to compute $\mathcal{L}^{-1}$ in general, but we can still make some progress using our small table of known transforms.

Example Use the Laplace transform to solve for $u(t)$ that solves

$$\ddot{u} + u = 0 \qquad u(0) = 1, \quad \dot{u}(0) = 0.$$

[solution is $u = \cos t$ ✓]

Take Laplace transform, using (i)

$$s\,\mathcal{L}(\dot{u}) - \cancel{\dot{u}(0)} + \mathcal{L}(u) = 0.$$

use (i) again.

$$s^2 \mathcal{L}(u) - s\, u(0) + \mathcal{L}(u) = 0$$

$\hat{u} = \mathcal{L}(u)$

$$\hat{u}(s) = \frac{s}{1+s^2}$$

$$\Rightarrow u(t) = \cos t \qquad ✓.$$

Example Use the Laplace transform to solve for $u(t)$ that solves

$$\dot{u} = au + f(t), \; u(0) = 0$$

with a constant. $\left[u(t) = \int_0^t e^{a(t-\tau)} f(\tau)\, d\tau \right]$

$$s\tilde{u} - \cancel{u(0)} = a\tilde{u} + \tilde{f}$$

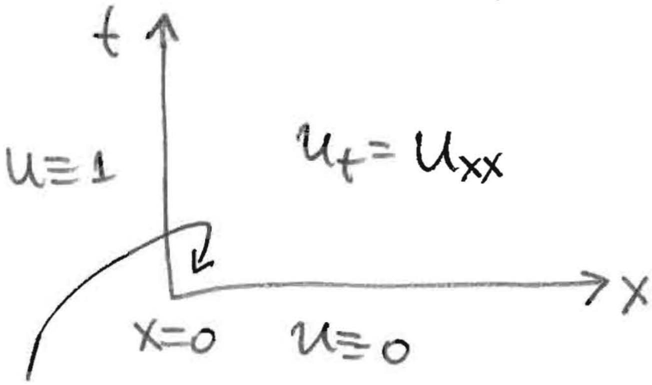
$$\hat{u}(s) = \frac{\hat{f}(s)}{s-a} = \frac{1}{s-a}\hat{f}(s)$$

so u is a convolution of e^{at} and $f(t)$,

$$u(t) = \int_0^t e^{a(t-\tau)} f(\tau)\, d\tau.$$

Example consider the following heat conduction problem for $u(x,t)$, $x \geq 0$, $t \geq 0$.

$u(x,0) = u_0(x) \equiv 0$, $u(0,t) \equiv 1$.



$\rightarrow$ "hidden" condition, $u(x,t) \to 0$ as $x \to \infty$ for each t.

discontinuity here at $x=0$, $t=0$, the "space-time corner). Take the Laplace transform w.r.t. t at each x,

$$s\tilde{u} = \hat{u}_{xx}$$

$\uparrow$

$-u(x,0) = 0$ for all x

Note that for each s, this is an ODE in x with s as a parameter.

$$\tilde{u}(x,s) = A e^{-\sqrt{s}\,x} + B e^{+\sqrt{s}\,x}$$

Since s is complex, we have to be a bit careful defining the $\sqrt{s}$. Here, recall that the s with $\mathrm{Re}(s)$ sufficiently large are the ones that define $\tilde{u}$, so we can take $+\sqrt{s}$ to have positive real part. Thus, $B=0$ to match the boundedness condition as $x \to \infty$.

Note that since $\overset{x=0}{u(0,t)} = 1$ for all t,

$\tilde{u}(0,s) = \frac{1}{s}$. Thus,

$$\tilde{u}(x,s) = \frac{1}{s} e^{-\sqrt{s}\,x}$$

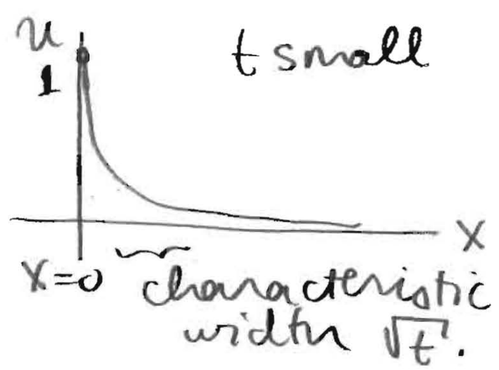
To proceed, we need another property of Laplace transforms, that

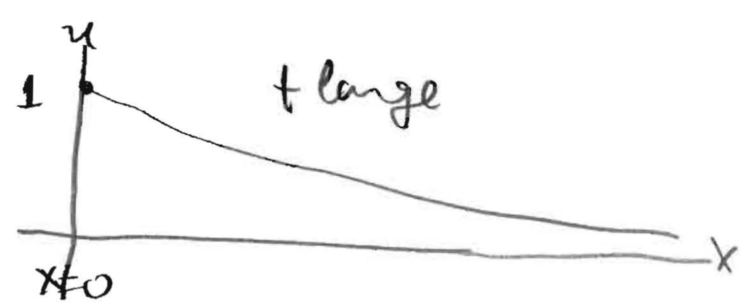
(iv) $\tilde{F}(cs) \Rightarrow \frac{1}{c} f\left(\frac{t}{c}\right)$ for constant $c > 0$.

rewrite $\tilde{u}(x,s)$ above as $x^2 \frac{1}{x^2 s} e^{-\sqrt{s x^2}}$, so in form (iv) with $c = x^2$, leading to

$$u(x,t) = \mathrm{erfc}\left(\frac{x}{2\sqrt{t}}\right)$$

This solution is reasonable,





So far, using the Laplace transform has been a pattern matching game when it comes to inversion. To develop an inversion formula, we do the following steps to identify the Laplace transform to the Fourier transform.

Consider $f(t)$ as on page 1, defined for $t \geq 0$ and satisfying $|f(t)| < Ke^{at}$ for all $t \geq 0$. Define for any constant $\eta > a$,

$$g(t) = \begin{cases} f(t)\, e^{-\eta t} & t \geq 0 \\ 0 & t < 0. \end{cases}$$

g is such that its Fourier tranform is well defined,

$$\hat{g}(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(t)\, e^{-i\alpha t}\, dt$$

$$= \frac{1}{2\pi}\int_0^{\infty} f(t)\, e^{-(\eta + i\alpha)t}\, dt$$

$$= \frac{1}{2\pi}\hat{f}(s) \quad \text{with } s = \eta + i\alpha.$$

But we have the inverse Fourier transform, so

$$g(t) = f(t)e^{-\eta t} = \int_{-\infty}^{\infty} \hat{g}(\alpha)\, e^{i\alpha t}\, d\alpha$$
for $t \geq 0$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(\eta + i\alpha)\, e^{(\eta + i\alpha)t}\, d\alpha$$

same complex argument.

which can be interpreted as a contour integral over the vertical line $Re(z) = \eta$ in the complex plane

$$f(t) = \frac{1}{2\pi i} \int_{C} \hat{f}(z) e^{zt} \underbrace{dz}_{dz = i\, d\alpha}$$

Can use complex contour integration techniques to evaluate such integrals.

Math 400, Notes VII

Fall, 2012

Consider a general time-harmonic (sinusoidal, pure frequency) signal with angular frequency ω:

$$f(t) = c\cos\omega t + d\sin\omega t \tag{1}$$

This kind of signal can be put into amplitude-phase form

$$f(t) = a\cos(\omega t - \delta) \tag{2}$$

where a is the signal amplitude. Using the trig identity

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta.$$

We see that the two forms match when

$$c = a\cos\delta \quad \text{and} \quad d = a\sin\delta$$

That is, $a = \sqrt{c^2+d^2}$ and δ is the angle from the positive x-axis to the vector (c,d). We can take

$$\delta = \operatorname{Arg}(c+id)$$

giving $\delta \in (-\pi, \pi]$. Note that different fields can use different forms of (2), sine instead of cosine, $+\delta$ instead of $-\delta$, and the range of $[0, 2\pi)$ for δ.

Another form for (1) is the following:

$$f(t) = \mathrm{Re}\left\{ A e^{i\omega t} \right\} \qquad (3)$$

with complex $A = c - id$. This matches (2) with $a = |A|$ and $\delta = -\operatorname{Arg} A$. This is convenient for some calculations.

<u>Example</u> Find a particular solution $u(t)$ to

$$\ddot{u} + \dot{u} - u = \sin(2t) = -\mathrm{Re}\left(i e^{i2t}\right)$$

using the method of undetermined coefficients. Note that this is not a resonant case, so we would look for $u(t)$ of the form

$$u(t) = c \sin 2t + d \sin 2t \qquad \text{or equivalently}$$
$$= \mathrm{Re}\left\{ A e^{i2t} \right\}.$$

Actually, we could solve

$$\ddot{u} + \dot{u} - u = -i e^{i2t} \qquad (4)$$

with the form $u(t) = A e^{i2t}$ and take the real part afterwards since real and imaginary parts of the problem don't mix. Plugging that form into (4) gives.

$$A(-4)e^{i2t} + 2iAe^{i2t} - Ae^{i2t} = -ie^{i2t}$$

$$A = \frac{-i}{-5+2i} = \frac{-2+5i}{29}$$

So $$u(t) = \frac{-2+5i}{29} e^{i2t}$$

The solution to the original problem is the real part of this

$$\mathrm{Re}\{u(t)\} = \frac{-2}{29}\cos 2t - \frac{5}{29}\sin 2t.$$

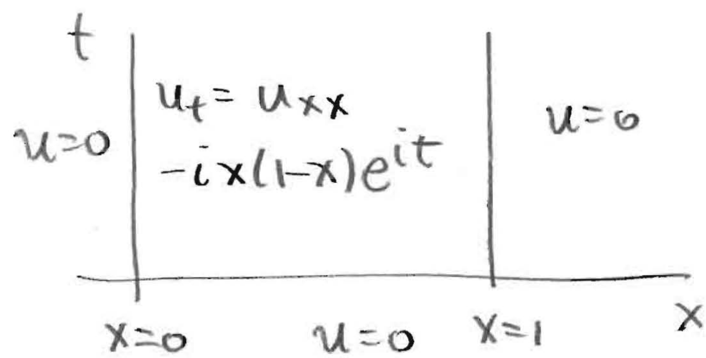
Example Let's look at a heat conduction problem with time harmonic forcing.

Find a series solution for $u(x,t)$, $0 \le x \le 1$, $t \ge 0$ that solves

$$u_t = u_{xx} + \overbrace{x(1-x)\sin t}^{\mathrm{Re}(-ix(1-x)e^{it})}$$

$$u(0,t) = 0, \quad u(1,t) = 0.$$

$$u(x,0) = 0, \qquad 0 \le x \le 1.$$



Following the work in assignment #2, question #2, we write $u(x,t)$ as a sine series at every t,

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t) \sin n\pi x$$

with $$\dot{u}_n = -n^2\pi^2 u_n + f_n(t) \qquad (5)$$

where $$f_n(t) = 2\int_0^1 \sin n\pi x \left[-ix(1-x)e^{it}\right] dx$$

But the time dependent terms can be pulled out of the integral

$$f_n(t) = -ie^{it} \cdot \underbrace{2\int_0^1 x(1-x)\sin n\pi x\, dx}_{\text{Sine series worked out in assignment 1}}$$

$$= \underbrace{-ie^{it}\, 4(1-(-1)^n)/\pi^3 n^3}_{c_n \text{ for short, not time dependent.}}$$

Returning to (5) gives

$$\dot{u}_n = -n^2\pi^2 u_n - i c_n e^{it} \qquad u_n(0) = 0 \qquad (6)$$

This has a solution of the form

$$u_n(t) = \underbrace{A_n e^{-n^2\pi^2 t}}_{\text{homogeneous}} + \underbrace{b_n e^{it}}_{\text{particular, MUC.}}$$

plugging in to (6) gives

$$i b_n = -n^2\pi^2 b_n - i c_n,$$

$$b_n = \frac{-i c_n}{n^2\pi^2 + i} = \frac{c_n(-1 - n^2\pi^2 i)}{(n^4\pi^4 + 1)}$$

Matching the initial conditions gives $A_n = -b_n$, so $u_n(t) = b_n\left(e^{it} - e^{-n^2\pi^2 t}\right)$

and the series solution we want is

$$\mathrm{Re}\{u(x,t)\} = \sum_{n=1}^{\infty} \left\{ \frac{-c_n}{n^4\pi^4+1}\left(\cos t - e^{-n^2\pi^2 t}\right) + \frac{n^2\pi^2 c_n}{n^4\pi^4+1}\sin t \right\} \sin n\pi x \qquad (7)$$

Consider (7) carefully. As $t \to \infty$, the exponential terms decay to zero, leaving only a time harmonic solution.

$$\text{as } t \to \infty, \quad u \to \cos t \left\{ \sum_{n=1}^{\infty} \frac{-C_n}{n^4\pi^4+1} \sin n\pi x \right\}$$

$$+ \sin t \left\{ \sum_{n=1}^{\infty} \frac{+n^2\pi^2 C_n}{n^4\pi^4+1} \sin n\pi x \right\}.$$

We'd probably be most interested in this limiting solution (not a steady state since there is still time dependence). We could try to identify the series sum, but can more easily compute it directly

$$u(x,t) \to a(x)\cos t + b(x)\sin t$$

$$= \mathrm{Re}\left(A(x)\, e^{it}\right)$$

Put this form back in the original problem (forgetting the initial conditions since they just lead to a transient) giving

$$i A e^{it} = A'' e^{it} - i\,x(1-x)\,e^{it}$$

or $\quad A'' - iA = +ix - ix^2 \qquad\qquad A(0)=0,\ A(1)=0$

This is a constant coefficient problem for $A(x)$ with a RHS that can be solved using standard methods, starting with the auxilliary equation $r^2 - i = 0$, $r = \pm\sqrt{i} \Rightarrow$

$$r = \pm\left(\frac{1+i}{\sqrt{2}}\right)$$

$$A(x) = Ce^{\left(\frac{1+i}{\sqrt{2}}\right)x} + De^{-\left(\frac{1+i}{\sqrt{2}}\right)x} + \quad \leftarrow \text{homogeneous}$$

$$a + bx + cx^2 \quad \leftarrow \text{particular.}$$

Now plug in the particular part into the equation to get

$$2c - i(a+bx+cx^2) = +ix - ix^2$$

Now equate coefficients:

x^2 : $-ic = -i \Rightarrow c = +1.$

x : $-ib = +i \Rightarrow b = -1$

1 : $2c - ia = 0 \Rightarrow ia = +2 \Rightarrow a = -2i$

apply the boundary conditions to determine C and D.

$$A(0) = 0 \Rightarrow C + D + a = 0 \Rightarrow C + D = +2i \qquad (7)$$

$$A(1) = 0 \Rightarrow Ce^{\frac{1}{\sqrt{2}}}e^{\frac{i}{\sqrt{2}}} + De^{-\frac{1}{\sqrt{2}}}e^{-i/\sqrt{2}}$$

$$+ a + b + c = 0.$$

$$\Rightarrow Ce^{\frac{1}{\sqrt{2}}}e^{\frac{i}{\sqrt{2}}} + De^{-1/\sqrt{2}}e^{-i/\sqrt{2}} = +2i \qquad (8)$$

(7) & (8) are a complex linear system with solution

$$C = \frac{+2ie^{-z}}{e^{-z}+1}, \quad D = \frac{2i}{e^{-z}+1} \quad \text{with } z = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$$

These can be further simplified to identify their real and imaginary parts:

$$C = \underbrace{\frac{1}{N}\left(2e^{-\frac{1}{\sqrt{2}}}\sin\frac{1}{\sqrt{2}} + 2i\left(e^{-1/2} + e^{-\frac{1}{\sqrt{2}}}\cos\frac{1}{\sqrt{2}}\right)\right)}_{C_r + iC_i}$$

with $N = (e^{-\frac{1}{2}}+1) + 2e^{-\frac{1}{\sqrt{2}}}\cos\frac{1}{\sqrt{2}}$

$$D = \underbrace{\frac{1}{N}\left(-2e^{-\frac{1}{\sqrt{2}}}\sin\frac{1}{\sqrt{2}} + 2i\left(1 + e^{-\frac{1}{\sqrt{2}}}\cos\frac{1}{\sqrt{2}}\right)\right)}_{D_r + iD_i}$$

and then $u(x,t) = \left\{Ce^{\frac{1}{\sqrt{2}}x}e^{\frac{i}{\sqrt{2}}x} + De^{-\frac{1}{\sqrt{2}}x}e^{-\frac{i}{\sqrt{2}}x} - 2i - x + x^2\right\}e^{it}$

Actually, we want the real part of this as our solution,

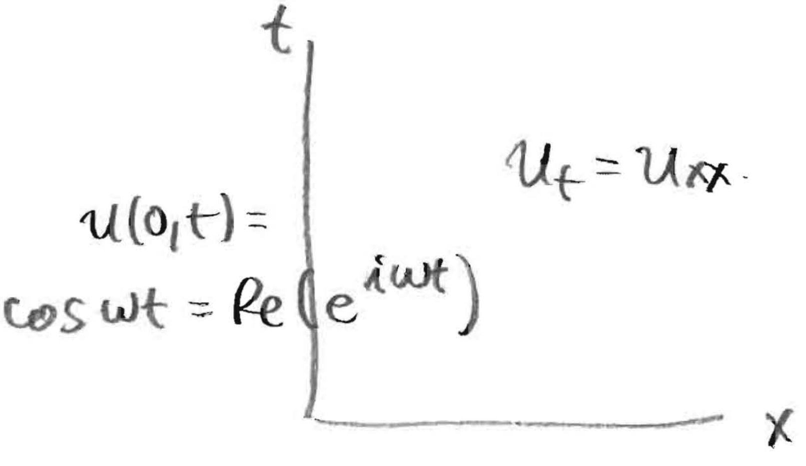
$$u(x,t) = C_r e^{\frac{x}{\sqrt{2}}}\cos\frac{x}{\sqrt{2}} - C_i e^{\frac{x}{\sqrt{2}}}\sin\frac{x}{\sqrt{2}}$$

$$+ D_r e^{-\frac{x}{\sqrt{2}}}\cos\frac{x}{\sqrt{2}} + D_i e^{\frac{-x}{\sqrt{2}}}\sin\frac{x}{\sqrt{2}}$$

$$+ 2i\sin t - x\cos t + x^2\cos t$$

<u>Note</u>: I can't guarantee the algebra above is error free. If you find errors, please let me know.

Example Find the time harmonic solution $u(t,x)$ to the problem for $x \geq 0$



↑
initial conditions not relevant, they give a transient we're not interested in.

Look for $u(x,t)$ in the form $u(x,t) = A(x)e^{it}$
this must solve

$$A(0) = 1, \qquad A(x) \text{ bounded as } x \to \infty.$$

$$i\omega A = A''$$

$$A(x) = C e^{-\frac{1}{\sqrt{2}}(1+i)\sqrt{\omega}\,x} + D e^{+\frac{1}{\sqrt{2}}(1+i)\sqrt{\omega}\,x}$$

as before. Here, the algebra is easier since $D = 0$ for solutions bounded as $x \to \infty$,

So $A(x) = e^{-\frac{1}{\sqrt{2}}(1+i)\sqrt{\omega}\,x}$,

$$u(x,t) = e^{-\sqrt{\frac{\omega}{2}}\,x} \; e^{i\left(\omega t - \sqrt{\frac{\omega}{2}}\,x\right)}.$$

↑
exponential decay

It's easy to find the real part here,

$$u(x,t) = e^{-\sqrt{\frac{\omega}{2}}\,x} \cos\left(\omega t - \sqrt{\tfrac{\omega}{2}}\,x\right).$$

Note that when $x = \pi\sqrt{\frac{2}{\omega}}$, the temperature is 180° out of phase with the applied temperature at the boundary, so the temperature will be coldest when the boundary is hottest and vice-versa.

Example Time harmonic solutions to the wave equation have the form

$$u_{tt} = u_{xx}$$

$$u(x,t) = \underbrace{A e^{-i\omega(x-t)}}_{\text{rightgoing}} + \underbrace{B e^{i\omega(x+t)}}_{\text{leftgoing.}}$$

real part of

Example consider the problem of

$$u_{tt} = (1 + \varepsilon V(x))\, u_{xx} \qquad (9)$$

with $V(x)$ given, nonzero in $x \in [0,1]$ only. Look for a time harmonic solution that is a perturbation of $u(x,t) = e^{-i\omega(x-t)}$. Consider the problem to be finding the (small) reflection of this wave at the anomaly and its transmission through it.

for $x<0$, take

$$u(x,t) \approx e^{-i\omega(x-t)} + \varepsilon R e^{i\omega(x+t)} \qquad (10)$$

For $x>1$, take

$$u(x,t) \approx e^{-i\omega(x-t)} + \varepsilon T e^{-i\omega(x-t-1)} \qquad (11)$$

no wave coming in from the left in this problem.

For $0<x<1$ take

$$u(x,t) \approx e^{-i\omega(x-t)} + \varepsilon A(x) e^{i\omega t} \qquad (12)$$

At $x=0$ and $x=1$ the solution and its derivatives should be continuous. We want to solve for R and T, have to find $A(x)$ along the way.

Plug (10)-(12) into (9). Actually, this gives no new information from (10) & (11). For (12) we have

$$-\cancel{\omega^2 e^{i\omega(x-t)}} - \varepsilon\omega^2 A(x) e^{i\omega t}$$
$$= -(\cancel{1} + \varepsilon V(x))\,\omega^2 e^{-i\omega(x-t)} + \varepsilon A''(x) e^{i\omega t}$$

the terms crossed out above indicate that we had the base solution correct. Now the following equation results:

$$A'' + \omega^2 A = \omega^2 V(x)\, e^{i\omega x}$$

The general solution is

$$A(x) = C e^{i\omega x} + D e^{-i\omega x} \qquad \text{homogeneous}$$

$$+ A_p(x) \qquad \text{particular solution}$$

Example: if $V(x) \equiv 1$ then $A_p(x) = -\frac{\omega i}{2} x e^{i\omega x}$

For continuity at $x=0$

$$C + D + A_p(0) = R.$$

at $x = 1$

$$C e^{i\omega} + D e^{-i\omega} + A_p(1) = T$$

Derivative continuity at $x=0$.

$$i\omega C - i\omega D + A_p'(0) = i\omega R$$

at $x=1$

$$i\omega C e^{i\omega} - i\omega D e^{-i\omega} + A_p'(1) = -i\omega T.$$

These are four (complex) linear equations for C, D, R and T. The matrix form is

$$\begin{bmatrix} 1 & 1 & -1 & 0 \\ e^{i\omega} & e^{-i\omega} & 0 & -1 \\ i\omega & -i\omega & -i\omega & 0 \\ i\omega e^{i\omega} & -i\omega e^{-i\omega} & 0 & +i\omega \end{bmatrix} \begin{bmatrix} C \\ D \\ R \\ T \end{bmatrix} = \begin{bmatrix} -A_p(0) \\ -A_p(1) \\ -A_p'(0) \\ -A_p'(1) \end{bmatrix}$$

influence of $V(x)$.

determinant $-4\omega^2 e^{i\omega}$ so never zero.

Note that in the case where the disturbance to the speed ($\varepsilon V(x)$ in this case) is not small, resonance can occur, signalled by the lack of solution to the $A(x)$ problem.

Table 9.5

ZEROS AND ASSOCIATED VALUES OF BESSEL FUNCTIONS AND THEIR DERIVATIVES

s	$j_{0,s}$	$J'_0(j_{0,s})$	$j_{1,s}$	$J'_1(j_{1,s})$	$j_{2,s}$	$J'_2(j_{2,s})$
1	2.40482 55577	−0.51914 74973	3.83171	−0.40276	5.13562	−0.33967
2	5.52007 81103	+0.34026 48065	7.01559	+0.30012	8.41724	+0.27138
3	8.65372 79129	−0.27145 22999	10.17347	−0.24970	11.61984	−0.23244
4	11.79153 44391	+0.23245 98314	13.32369	+0.21836	14.79595	+0.20654
5	14.93091 77086	−0.20654 64331	16.47063	−0.19647	17.95982	−0.18773
6	18.07106 39679	+0.18772 88030	19.61586	+0.18006	21.11700	+0.17326
7	21.21163 66299	−0.17326 58942	22.76008	−0.16718	24.27011	−0.16170
8	24.35247 15308	+0.16170 15507	25.90367	+0.15672	27.42057	+0.15218
9	27.49347 91320	−0.15218 12138	29.04683	−0.14801	30.56920	−0.14417
10	30.63460 64684	+0.14416 59777	32.18968	+0.14061	33.71652	+0.13730
11	33.77582 02136	−0.13729 69434	35.33231	−0.13421	36.86286	−0.13132
12	36.91709 83537	+0.13132 46267	38.47477	+0.12862	40.00845	+0.12607
13	40.05842 57646	−0.12606 94971	41.61709	−0.12367	43.15345	−0.12140
14	43.19979 17132	+0.12139 86248	44.75932	+0.11925	46.29800	+0.11721
15	46.34118 83717	−0.11721 11989	47.90146	−0.11527	49.44216	−0.11343
16	49.48260 98974	+0.11342 91926	51.04354	+0.11167	52.58602	+0.10999
17	52.62405 18411	−0.10999 11430	54.18555	−0.10839	55.72963	−0.10685
18	55.76551 07550	+0.10684 78883	57.32753	+0.10537	58.87302	+0.10396
19	58.90698 39261	−0.10395 95729	60.46946	−0.10260	62.01622	−0.10129
20	62.04846 91902	+0.10129 34989	63.61136	+0.10004	65.15927	+0.09882

s	$j_{3,s}$	$J'_3(j_{3,s})$	$j_{4,s}$	$J'_4(j_{4,s})$	$j_{5,s}$	$J'_5(j_{5,s})$
1	6.38016	−0.29827	7.58834	−0.26836	8.77148	−0.24543
2	9.76102	+0.24942	11.06471	+0.23188	12.33860	+0.21743
3	13.01520	−0.21828	14.37254	−0.20636	15.70017	−0.19615
4	16.22347	+0.19644	17.61597	+0.18766	18.98013	+0.17993
5	19.40942	−0.18005	20.82693	−0.17323	22.21780	−0.16712
6	22.58273	+0.16718	24.01902	+0.16168	25.43034	+0.15669
7	25.74817	−0.15672	27.19909	−0.15217	28.62662	−0.14799
8	28.90835	+0.14801	30.37101	+0.14416	31.81172	+0.14059
9	32.06485	−0.14060	33.53714	−0.13729	34.98878	−0.13420
10	35.21867	+0.13421	36.69900	+0.13132	38.15987	+0.12861
11	38.37047	−0.12862	39.85763	−0.12607	41.32638	−0.12366
12	41.52072	+0.12367	43.01374	+0.12140	44.48932	+0.11925
13	44.66974	−0.11925	46.16785	−0.11721	47.64940	−0.11527
14	47.81779	+0.11527	49.32036	+0.11343	50.80717	+0.11167
15	50.96503	−0.11167	52.47155	−0.10999	53.96303	−0.10838
16	54.11162	+0.10839	55.62165	+0.10685	57.11730	+0.10537
17	57.25765	−0.10537	58.77084	−0.10396	60.27025	−0.10260
18	60.40322	+0.10260	61.91925	+0.10129	63.42205	+0.10003
19	63.54840	−0.10004	65.06700	−0.09882	66.57289	−0.09765
20	66.69324	+0.09765	68.21417	+0.09652	69.72289	+0.09543

s	$j_{6,s}$	$J'_6(j_{6,s})$	$j_{7,s}$	$J'_7(j_{7,s})$	$j_{8,s}$	$J'_8(j_{8,s})$
1	9.93611	−0.22713	11.08637	−0.21209	12.22509	−0.19944
2	13.58929	+0.20525	14.82127	+0.19479	16.03777	+0.18569
3	17.00382	−0.18726	18.28758	−0.17942	19.55454	−0.17244
4	20.32079	+0.17305	21.64154	+0.16688	22.94517	+0.16130
5	23.58608	−0.16159	24.93493	−0.15657	26.26681	−0.15196
6	26.82015	+0.15212	28.19119	+0.14792	29.54566	+0.14404
7	30.03372	−0.14413	31.42279	−0.14055	32.79580	−0.13722
8	33.23304	+0.13727	34.63709	+0.13418	36.02562	+0.13127
9	36.42202	−0.13131	37.83872	−0.12859	39.24045	−0.12603
10	39.60324	+0.12606	41.03077	+0.12365	42.44389	+0.12137
11	42.77848	−0.12139	44.21541	−0.11924	45.63844	−0.11719
12	45.94902	+0.11721	47.39417	+0.11526	48.82593	+0.11342
13	49.11577	−0.11343	50.56818	−0.11166	52.00769	−0.10998
14	52.27945	+0.10999	53.73833	+0.10838	55.18475	+0.10684
15	55.44059	−0.10685	56.90525	−0.10537	58.35789	−0.10395
16	58.59961	+0.10396	60.06948	+0.10260	61.52774	+0.10129
17	61.75682	−0.10129	63.23142	−0.10003	64.69478	−0.09882
18	64.91251	+0.09882	66.39141	+0.09765	67.85943	+0.09652
19	68.06689	−0.09652	69.54971	−0.09543	71.02200	−0.09438
20	71.22013	+0.09438	72.70655	+0.09336	74.18277	+0.09237

MATH 400, Fall 2012, Wetton
Assignment #1 - due Monday, September 17

Part A Routine questions

1. *Review of linear constant coefficient inhomogeneous problems.* The solution $u(t)$ to the inhomogeneous ODE

$$\dot{u} + 2u = f(t)$$

with initial conditions $u(0) = 0$ is given by the integral formula

$$u(t) = \int_0^t e^{-2(t-s)} f(s) ds$$

(an example of Duhamel's Principle). Derive this formula.

2. *Review of the Method of Undetermined Coefficients.* Find the analytic solution (not in series form) for the boundary value problem for $u(x)$:

$$u'' - u = xe^x$$

with boundary values $u(0) = 1$ and $u(1) = 0$.

3. *Review of Newton's Method.* Find the smallest positive root of

$$f(x) = \tan x - 2x$$

using Newton's method. Describe how you chose your initial estimate of the root. Show the iteration formula you obtain and the list of iterates starting from your initial estimate.

4. *Review of Linear Systems.* Find the inverse of the matrix

$$A = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

Use Gaussian Elimination by hand, although you can check your answer with MATLAB afterward.

5. *Review of Eigenanalysis.* Find an orthonormal basis of eigenvectors of the symmetric matrix

$$B = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

6. *Linear, constant coefficient, homogeneous ODE solutions.* Let $\mathbf{u}(t)$ be a three component column vector at every t. Write the general solution to the ODE system

$$\dot{\mathbf{u}} = B\mathbf{u}$$

where B is the matrix from the previous question. Additionally, what is the solution to this problem with initial conditions $\mathbf{u}(0) = [1\ 0\ 0]^T$?

7. *Review of Fourier Series.* Consider the function $f(x) = x(1-x)$ on the interval $x \in [0,1]$. Find the sine series coefficients of this function. Plot (using for example MATLAB, excel or octave which is freely available online) the graph of the original function with superimposed the graph of the sum of the first three nonzero terms of the sine series.

8. Consider the general, quasi-linear third order ODE for $u(t)$:

$$\frac{d^3u}{dt^3} = F(t, u, \dot{u}, \ddot{u}).$$

Reformulate this ODE as a first order system.

Part B. More difficult questions.

9. Consider the ODE system for $\mathbf{u}$ with two components

$$\dot{\mathbf{u}} = A\mathbf{u} + \mathbf{f}(t)$$

where

$$A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$$

and $\mathbf{u}$ has initial conditions $\mathbf{u} = \mathbf{0}$. Analogously to the formula in question # 1, an integral formula for the solution can be derived of the form

$$\mathbf{u}(t) = \int_0^t B(t-s)\mathbf{f}(s)ds$$

where $B(t)$ is a 2×2 matrix. Find the entries of this matrix. (Another example of Duhamel's Principle)

10. Consider the boundary value eigen-problem for $u(x)$ given below

$$\big((1+x^2)u'\big)' = \lambda u$$

with homogeneous boundary conditions $u(0) = 0$ and $u'(1) = 0$. Show that all eigenvalues λ must be real. *Do not try to find the eigenvalues of this problem.* **Hint:** It is the same underlying argument as is used to show that eigenvalues of symmetric matrices must be real.

Math 400, Fall 2012

Assignment #1 Solutions

1. $\dot{u} + 2u = f(t)$

multiply the equation by the integrating factor e^{2t}.

$$\frac{d}{dt}\left(u e^{2t}\right) = e^{2t} f(t).$$

Now integrate from 0 to t.

$$u(t)\, e^{2t} - \underbrace{u(0)}_{\text{zero}} = \int_0^t e^{2s} f(s)\, ds.$$

Multiply by e^{-2t}

$$u(t) = e^{-2t} \int_0^t e^{2s} f(s)\, ds.$$

bring into integrand

$$u(t) = \int_0^t e^{-2(t-s)} f(s)\, ds.$$

2. $u'' - u = x e^x$

homogeneous solution $u_h = Ae^x + Be^{-x}$

Note: A & B to be determined later to match boundary conditions

particular solution

$u_p = x(ax+b)e^x$

special case since e^x is in the homogeneous solution.

<u>Note</u>: a & b determined to satisfy the ODE:

$$u_p' = e^x\left[ax^2+bx+2ax+b\right].$$
$$= e^x\left[ax^2+(b+2a)x+b\right].$$

$$u_p'' = e^x\left[ax^2+(b+2a)x+b+2ax+b+2a\right]$$
$$= e^x\left[ax^2+(b+4a)x+2b+2a\right].$$

$u_p'' - u_p = xe^x$, equate terms below:

e^x: $2b+2a=0$ $\quad$ $b=-1/4$

xe^x: $b+4a-b = 4a = 1 \Rightarrow a = 1/4$

x^2e^x: $a-a=0$ ✓ (always happens for resonant cases).

Now $u_p(x) = \frac{1}{4}(x^2-x)e^x$

Solution $u(x) = u_h + u_p$

$$= Ae^x + Be^{-x} + \frac{1}{4}(x^2-x)e^x \qquad (\bigstar)$$

$$\left.\begin{array}{ll} u(0)=1 \Rightarrow & A+B=1. \\ u(1)=0 \Rightarrow & Ae+B/e=0 \end{array}\right\} \Rightarrow \begin{array}{l} B=\left(1-\frac{1}{e^2}\right)^{-1} \\ A=-B/e^2 \end{array}$$

Solution is ($\bigstar$) with these values of A & B.

Alternate form of the solution

$$u(x) = \frac{\sinh(1-x)}{\sinh 1} + \frac{1}{4}(x^2-x)e^x$$

3. $f(x) = \tan x - 2x$

$f'(x) = \sec^2 x - 2.$

Newton iterations $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$.

From the notes, the desired root lies between $x=0$ and $x=\pi/2$ so I tried $x_0 = \pi/4$ but that did not converge $\left(f'(\pi/4) = 0\right)$. I then tried $x_0 = 3\pi/8$ which did converge

n	x_n
0	1.1781
1	1.1661
2	1.1656
3	1.1656

converged.

4. $A = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} -2 & 1 & 0 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & -1/2 & 0 & -1/2 & 0 & 0 \\ 0 & -3/2 & 1 & 1/2 & 1 & 0 \\ 0 & 1 & -2 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & -1/2 & 0 & -1/2 & 0 & 0 \\ 0 & 1 & -2/3 & -1/3 & -2/3 & 0 \\ 0 & 0 & -4/3 & 1/3 & 2/3 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -3/4 & -1/2 & -1/4 \\ 0 & 1 & 0 & -1/2 & -1 & -1/2 \\ 0 & 0 & 1 & -1/4 & -1/2 & -3/4 \end{array}\right]$$

So $\mathbb{A}^{-1} = \begin{bmatrix} -3/4 & -1/2 & -1/4 \\ -1/2 & -1 & -1/2 \\ -1/4 & -1/2 & -3/4 \end{bmatrix}$

I checked the result with MATLAB.

5. $\mathbb{B} = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$

Find the eigenvalues.

$$\det(\mathbb{B} - \lambda \mathbb{I}) = \det \begin{bmatrix} -2-\lambda & 1 & 1 \\ 1 & -2-\lambda & 1 \\ 1 & 1 & -2-\lambda \end{bmatrix}$$

$$= \cdots = -\lambda(\lambda^2 + 6\lambda + 9)$$

So eigenvalues are $\lambda = 0, -3, -3$.

$\lambda_1 = 0$ eigenvector $\underline{v}_1$ solves the homogeneous problem

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

So $\underline{v}_1 = c \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, choose $c = \frac{1}{\sqrt{3}}$ to normalize the vector.

$\lambda_{2,3} = -3$ eigenvectors $\underline{v}_{2,3}$ solve the homogeneous problem

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ this has a 2 dimensional space of homogeneous solutions,

we must pick $\underline{V}_2$ and $\underline{V}_3$ to be orthogonal. I chose.

$$\underline{V}_2 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix}, \quad \underline{V}_3 = \begin{bmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ -2/\sqrt{6} \end{bmatrix}$$

Note: Could have picked $\underline{V}_2$ and computed $\underline{V}_3 = \underline{V}_1 \times \underline{V}_2$. Could also have chosen $\underline{V}_2$ & $\underline{V}_3$ arbitrarily (different) and used the Gramm-Schmidt process.

6. General solution

$$\underline{u}(t) = c_1 \underline{V}_1 + e^{-3t}\left(c_2 \underline{V}_2 + c_3 \underline{V}_3\right).$$

To match $\underline{u}(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ use the fact that $\{\underline{V}_i\}$ are orthonormal to compute

$$c_i = \underline{V}_i \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \text{ so } \quad c_1 = 1/\sqrt{3}, \; c_2 = \frac{1}{\sqrt{2}}, \; c_3 = \frac{1}{\sqrt{6}}$$

$$\underline{u}(t) = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix} + e^{-3t} \begin{bmatrix} 2/3 \\ -1/3 \\ -1/3 \end{bmatrix}$$

7. $f(x) = x(1-x)$, $x \in [0,1]$

Fourier sine series coefficients.

$$f_n = 2\int_0^1 (x-x^2)\sin n\pi x\,dx$$

$$= \frac{4(1-(-1)^n)}{\pi^3 n^3}.$$

integrate by parts, MAPLE, tables, Wolfram Alpha

First three non-zero terms:

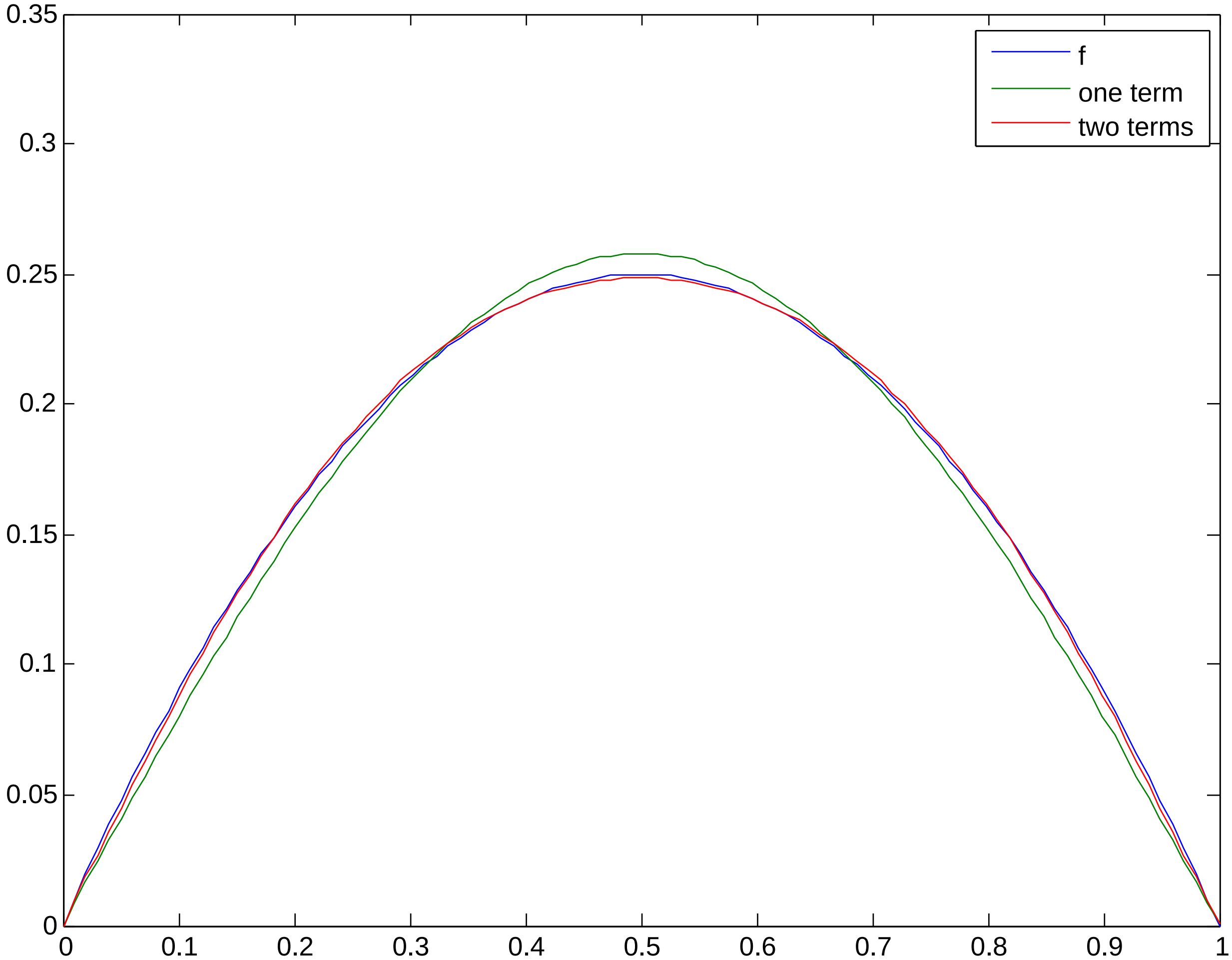
$$f(x) \approx S_3(x) = \underset{n=1}{\frac{8}{\pi^3}\sin\pi x} + \underset{n=3}{\frac{8}{27\pi^3}\sin 3\pi x} + \underset{n=5}{\frac{8}{125\pi^3}\sin 5\pi x}$$

Plot next page. Can't detect a graphical difference between f and S_3. I added the first and first two term approximations.

8. $\dfrac{d^3u}{dt^3} = F(t, u, \dot{u}, \ddot{u})$

Let $u_0 = u$, $u_1 = \dot{u}$, $u_2 = \ddot{u}$. The system is then

$$\dot{u}_0 = u_1$$
$$\dot{u}_1 = u_2$$
$$\dot{u}_2 = F(t, u_0, u_1, u_2).$$



0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
f
one term
two terms

9. $\dot{\underline{u}} = \mathbb{A}\underline{u} + \underline{f}(t)$, $\quad \dot{\underline{u}} - \mathbb{A}\underline{u} = \underline{f}(t) \quad (1)$

We can proceed exactly like question #1 if we could find a matrix function $\underline{\Phi}(t)$ such that $\dot{\underline{\Phi}} = -\underline{\Phi}\mathbb{A}$, with $\underline{\Phi}(0) = \underline{\mathbb{I}}$ (identity).

Leave aside the question of finding $\underline{\Phi}(t)$ for now and multiply (1) by $\underline{\Phi}$ with these properties:

$$\underline{\Phi}\dot{\underline{u}} - \underline{\Phi}\mathbb{A}\underline{u} = \underline{\Phi}\underline{f}$$

property of $\underline{\Phi}$

$$\frac{d}{dt}(\underline{\Phi}\underline{u}) = \underline{\Phi}\underline{f}$$

Integrate from 0 to t as in question #1

$$\underline{\Phi}(t)\underline{u}(t) = \int_0^t \underline{\Phi}(s)\underline{f}(s)\,ds.$$

$$\underline{u}(t) = \int_0^t \underbrace{\underline{\Phi}^{-1}(t)\,\underline{\Phi}(s)}_{} f(s)\,ds.$$

must match this to $\mathbb{B}(t-s)$ in the desired formula.

To proceed, do the eigenanalysis of A:

$$\lambda_1 = -1, \quad \underline{v_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

$$\lambda_2 = -3, \quad \underline{v_2} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

$$\mathbb{T} = \left[\underline{v_1} \,\vdots\, \underline{v_2} \right] = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$\mathbb{T}^{-1} = \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix}$$

$$\mathbb{T} A \mathbb{T}^{-1} = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix} \quad \text{diagonalization of } A. \ (2)$$

Now to compute Φ you can realize that

$$\Phi = \exp(-\mathbb{A}t) \underset{\substack{\uparrow \\ \text{defined as}}}{:=} \sum_{n=0}^{\infty} \mathbb{A}^n (-t)^n$$

has the desired properties and that

$$\Phi(t) = \exp(-\mathbb{A}t) = \mathbb{T} \begin{bmatrix} e^{t} & 0 \\ 0 & e^{3t} \end{bmatrix} \mathbb{T}^{-1}$$

$$\Phi^{-1}(t) = \exp(\mathbb{A}t) = \mathbb{T} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-3t} \end{bmatrix} \mathbb{T}^{-1}$$

and so

$$\underbrace{\Phi^{-1}(t)\,\Phi(s)}_{\exp(A(t-s))} = \mathbb{T} \begin{bmatrix} e^{-(t-s)} & 0 \\ 0 & e^{-3(t-s)} \end{bmatrix} \mathbb{T}^{-1}$$

So $\mathbb{B}(t) = \mathbb{T}\begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-3t} \end{bmatrix}\mathbb{T}^{-1}$

$$= \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-3t} \end{bmatrix}\begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} e^{-t} & e^{-t} \\ -e^{-3t} & e^{-3t} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2}e^{-t} + \frac{1}{2}e^{-3t} & \frac{1}{2}e^{-t} - \frac{1}{2}e^{-3t} \\ \frac{1}{2}e^{-t} - \frac{1}{2}e^{-3t} & \frac{1}{2}e^{-t} + \frac{1}{2}e^{-3t} \end{bmatrix}.$$

<u>Alternate path:</u> Without knowing the machinery of matrix exponentials, you could proceed by considering $\Psi = \mathbb{T}\,\Phi\,\mathbb{T}^{-1}$

(Φ acting on $\mathbb{A}$ eigen coordinates).

$$\dot{\Phi} = -\Phi\mathbb{A} \underset{\text{some algebra}}{\Longrightarrow} \dot{\Psi} = \Psi\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}.$$

which leads to $\Psi = \begin{bmatrix} e^{t} & 0 \\ 0 & e^{3t} \end{bmatrix}$ and you can proceed as above.

10. Suppose $((1+x^2)u')' = \lambda u$ (3)

with $u(0)=0$ and $u'(1)=0$ and $u(x)$ is not identically zero.

Multiply (3) by u^* (complex conjugate of u) and integrate from 0 to 1.

Switching the sides of the equality we have

$$\text{(A)} \quad \lambda \underbrace{\int_0^1 |u|^2\,dx}_{\text{not zero}} = \int_0^1 u^*\left((1+x^2)u'\right)'dx$$

by parts

$$= -\int_0^1 (u^*)'(1+x^2)u'\,dx + \left(u^*(1+x^2)u'\right)\Big|_0^1$$

but $u(0)=0$ and $u'(1)=0$ So this boundary term is zero.

by parts again

$$= \int_0^1 \underbrace{\left[(1+x^2)(u^*)'\right]'}_{\lambda^* u^* \text{ from (3).}} u\,dx$$

$$\text{(B)} \quad = \lambda^* \int_0^1 |u|^2 dx.$$

Considering (A) = (B) we see that $\lambda = \lambda^*$ so λ must be real.

MATH 400, Fall 2012, Wetton
Assignment #2 - due Monday, September 24

Part A Routine questions

1. Consider the following heat equation problem for $u(x,t)$:

$$u_t = u_{xx} + f(x,t)$$

for $t \geq 0$ and $0 \leq x \leq 1$ with initial conditions $u(x,0) = u_0(x)$ and boundary conditions $u(0,t) = b_0(t)$ and $u(1,t) = b_1(t)$. Write the solution of this problem as a sum of solutions of four problems each of which involves only one of the four data functions $f(x,t)$, $u_0(x)$, $b_0(t)$ and $b_1(t)$. You *do not* need to solve the four problems you specify.

2. Consider the problem in question #1 with u_0, b_1 and b_2 identically zero, so only $f(x,t)$ nonzero. Write a series solution for this problem. Note that the series coefficients will be integrals of $f(x,t)$ multiplied by functions of x and t that are to be determined. *Hint:* Write $u(x,t)$ as a sine series in x with coefficients that depend on t.

3. For the problem in question #1, consider the data functions to be smooth (infinitely differentiable) for all x and t. What necessary conditions on the data must be satisfied for the solution to be continuous at (0,0) and (1,0)? What further conditions on the data must be satisfied for the solution to have continuous first partial derivatives at these two points? The necessary conditions you give can be shown to be sufficient, but you do not need to prove this.

4. Consider the two-way wave equation with constant speed $c > 0$

$$u_{tt} - c^2 u_{xx} = 0$$

Write the form of D'Alembert's solution in this case.

5. Consider the one-way wave equation for $u(x,t)$ for $t \geq 0$ and $x \geq 0$

$$u_t + u_x = 0$$

with initial data $u(x,0) = u_0(x)$ given for $x \geq 0$ and boundary data $u(0,t) = f(t)$ given for $t \geq 0$. Write down the solution $u(x,t)$ in terms of the data u_0 and f. Note that there will be an "if" statement in your solution.

6. Consider the problem in the previous question where the region of interest is reduced to $0 \leq x \leq 1$. Discuss why arbitrary boundary data $u(1,t) = g(t)$ cannot be added to the specification of this problem.

7. Consider the two-way wave equation

$$u_{tt} - u_{xx} = 0$$

for $t \geq 0$ and all x with initial data $u(x,0) = 0$ and $u_t(x,0) = u_1(x)$. Find the solution of this problem. Note that it will involve the anti-derivative of u_1.

8. Consider the two-way wave equation

$$u_{tt} - u_{xx} = 0$$

for $t \geq 0$ and $x \geq 0$ with initial data $u(x,0) = u_0(x)$ and $u_t(x,0) = 0$ for $x \geq 0$ and boundary data $u(0,t) = 0$ for $t > 0$. Write the solution of this problem in terms of $u_0(x)$.

Part B. More difficult questions.

9. Consider the one-way wave equation for $u(x,t)$

$$u_t + u_x = 0$$

Consider the alternative coordinates $\eta = x + t$ and $\xi = x - t$. Using the chain rule, write the equation above in terms of these coordinates. Repeat this exercise for the two-way wave equation

$$u_{tt} - u_{xx} = 0.$$

10. Consider the two-way wave equation for $u(x,t)$, $t \geq 0$ and all x with piecewise constant speed:

$$\begin{aligned} u_{tt} - u_{xx} &= 0 \quad \text{for } x \geq 0 \\ u_{tt} - 4u_{xx} &= 0 \quad \text{for } x < 0. \end{aligned}$$

At $x = 0$, u and u_x are continuous. Data $u_0(x)$ is given for $x \geq 0$. $u(x,0) = u_0(x)$ for $x \geq 0$ and $u(x,0) = 0$ for $x < 0$. $u_t(x,0) = 0$ for all x. Write the solution for all x for $t > 0$ in terms of $u_0(x)$. There will be several cases.

Math 400 Assignment #2 solutions

1. $u = u_1 + u_2 + u_3 + u_4$ where u_i solve the problems below.

u_1: $u_t = u_{xx} + f(x,t)$, $u(x,0) = 0$, $u(0,t) = 0$, $u(1,t) = 0$

u_2: $u_t = u_{xx}$, $u(x,0) = u_0(x)$, $u(0,t) = 0$, $u(1,t) = 0$

u_3: $u_t = u_{xx}$, $u(x,0) = 0$, $u(0,t) = b_0(t)$, $u(1,t) = 0$

u_4: $u_t = u_{xx}$, $u(x,0) = 0$, $u(0,t) = 0$, $u(1,t) = b_1(t)$

2. Write $f(x,t)$ as a sine series at each t.

$$f(x,t) = \sum_{n=1}^{\infty} f_n(t) \sin n\pi x \qquad (1)$$

where $f_n(t) = 2\int_0^1 f(x) \sin n\pi x \, dx$.

Write $u(x,t)$ as a sine series at each t

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t) \sin n\pi x \qquad (2)$$

where $u_n(t)$ are unknown. Put (1) and (2) into the PDE $u_t = u_{xx} + f(x,t)$ and at each time equate the sine coefficients (using the orthogonality of the sine functions)

$$\dot{u}_n = -n^2\pi^2 u_n + f_n(t).$$

Thus we have independent c.c. linear ODEs for each $u_n(t)$ that are solved like assignment #1, Q 1.

$$u_n(t) = \int_0^t e^{-n^2\pi^2(t-s)} f_n(s)\,ds.$$

This expression in the series (2) is the desired result.

3. To be continuous at $(0,0)$ the initial and boundary data must match:

$$u_0(0) = b_0(0)$$

and similarly at $(1,0)$ we must have

$$u_0(1) = b_1(0).$$

To be differentiable at $(0,0)$ the time derivative from the boundary data must match the time derivative from the initial data and forcing in the PDE, ie.

$$\dot{b}_0(0) = u_0''(0) + f(0,0)$$

and similarly at $(1,0)$ we must have

$$\dot{b}_1(0) = u_0''(1) + f(1,0)$$

4. $u(x,t) = f(x-ct) + g(x+ct)$

5. We know that the general solution of the equation is

$$u(x,t) = g(x-t) \qquad (3)$$

[Sorry, I used $f(t)$ for the BC's].

Consider (3) at $t=0$,

$$g(x) = u_0(x) \qquad \text{for} \quad x \geq 0$$

This determines $g(\cdot)$ for positive arguments but for values of $x<t$ we also need the values of $g(\cdot)$ for negative arguments.

Consider (3) along the boundary $x=0$,

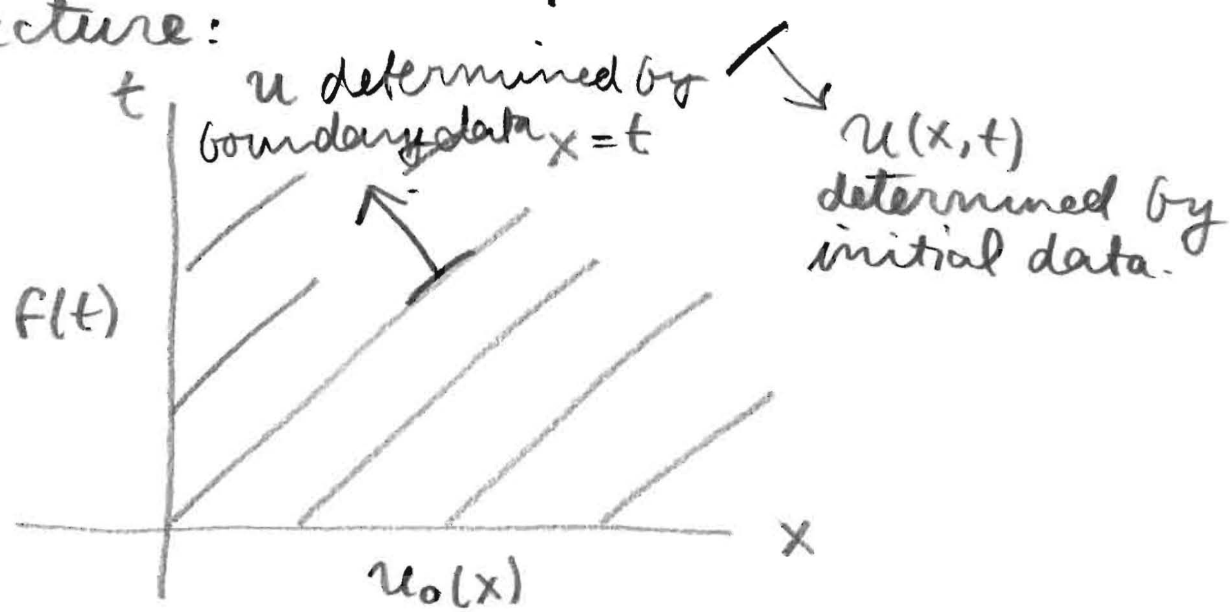
$$f(t) = g(-t) \qquad t \geq 0$$

$g(s) = f(-s)$ for $s \leq 0$.

So we can write

$$g(s) = \begin{cases} u_0(s) & s \geq 0 \\ f(-s) & s \leq 0 \end{cases}$$

Note, This matches the picture I drew in the lecture:



6. Along the line $x=1$, the solution is already determined by the initial data or the boundary data. Thus, the solution cannot be given independent data on this line.

7. Begin with the D'Alembert solution

$$u(x,t) = f(x-t) + g(x+t).$$

Proceeding as in the lecture notes consider this form at $t=0$

$$f(x) + g(x) = 0 \qquad (4)$$

$$-f'(x) + g'(x) = u_1(x).$$

integrate the last equation to get

$$-f(x) + g(x) = U_1(x) + c$$

where U_1 is the antiderivative (indefinite integral) of u_1. Combine with (4) to get

$$g(x) = U_1(x)/_2 + c/2.$$

$$f(x) = -U_1(x)/_2 - c/2.$$

so now

$$u(x,t) = f(x-t) + g(x+t) \qquad \text{c's cancel}$$

$$= \frac{1}{2}\left(U_1(x+t) - U_1(x-t)\right)$$

which can also be written as

$$u(x,t) = \frac{1}{2} \int_{x-t}^{x+t} u_1(s)\, ds$$

8. Begin with the D'Alembert solution

$$u(x,t) = f(x-t) + g(x+t) \qquad (5).$$

Matching the initial conditions as in the lecture notes gives

$$\left.\begin{aligned} f(s) &= \tfrac{1}{2} u_0(s) \\ g(s) &= \tfrac{1}{2} u_0(s) \end{aligned}\right\} \quad s \geq 0.$$

However, notice in (5) that we also need the negative arguments of $f(\cdot)$, to describe the solution in the region $x<t$. This is a similar idea to question 5.

Apply the boundary condition $u(0,t)=0$ with the form (5):

$$f(-t) + g(t) = 0 \qquad t \geq 0.$$

$$\text{or} \quad f(s) = -g(-s) = -\tfrac{1}{2}u_0(-s) \quad s \leq 0.$$

This gives the negative arguments of f as required. So the solution is (5) with the scalar functions

$$g(s) = \tfrac{1}{2} u_0(s) \qquad s \geq 0$$

$$f(s) = \begin{cases} \tfrac{1}{2} u_0(s) & s \geq 0 \\ -\tfrac{1}{2} u_0(-s) & s \leq 0. \end{cases} \qquad (6)$$

Additional: The physical interpretation of (6) is a reflection of the left moving wave from the $x=0$ boundary where the wire is held fixed. Note that the reflected wave is inverted. From the solution, it can be seen that the effect of the $u=0$ boundary condition is the same as extending u_0 as an odd function in the problem without the boundary.

9. $$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial t} + \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial t}$$ by the chain rule.

$$= \frac{\partial u}{\partial \eta} - \frac{\partial u}{\partial \xi}$$

similarly $$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \eta} + \frac{\partial u}{\partial \xi}.$$

so $$u_t + u_x = 2\frac{\partial u}{\partial \eta} = 0.$$

Additional This shows us that solutions of $u_t + u_x = 0$ only depend on ξ, not η so $u(x,t) = F(\xi) = F(x-t)$, the general solution I stated in class. This is the proof of that statement.

Continuing on from the expressions above we find that

$$u_{tt} = \frac{\partial^2 u}{\partial \eta^2} - 2\frac{\partial^2 u}{\partial \eta \partial \xi} + \frac{\partial^2 u}{\partial \xi^2}$$

and $$u_{xx} = \frac{\partial^2 u}{\partial \eta^2} + 2\frac{\partial^2 u}{\partial \eta \partial \xi} + \frac{\partial^2 u}{\partial \xi^2}$$

so $$u_{tt} - u_{xx} = -4\frac{\partial^2 u}{\partial \eta \partial \xi} = 0.$$

Additional: Thus we see the general solution of the wave equation is $f(\xi) + g(\eta)$, the D'Alembert solution.

10. Write the D'Alembert form of the solutions on both sides of the interface $x = 0$:

$$\left.\begin{aligned} u(x,t) &= f_1(x-t) + g_1(x+t) & x \geq 0 \\ u(x,t) &= f_0(x-2t) + g_0(x+2t) & x \leq 0 \end{aligned}\right\} \quad (7a)$$

Match initial data,

$$\left.\begin{aligned} f_1(s) &= g_1(s) = \tfrac{1}{2}u_0(s) & s \geq 0. \\ f_0(s) &= g_0(s) = 0 & s \leq 0. \end{aligned}\right\} \quad (7b)$$

However from the form (7a), negative arguments of f_1 (reflected wave) and positive arguments of g_0 (transmitted wave) are still required.

Consider the junction conditions, u and u_x continuous at $x=0$:

$$\left.\begin{aligned} f_1(-t) + g_1(t) &= f_0(-2t) + g_0(2t) \\ f_1'(-t) + g_1'(t) &= f_0'(-2t) + g_0'(2t). \end{aligned}\right\} t \geq 0.$$

which we can put the known data into

$$f_1(-t) + \tfrac{1}{2}u_0(t) = g_0(2t) \qquad (8)$$

$$f_1'(-t) + \tfrac{1}{2}u_0'(t) = g_0'(2t) \qquad (9).$$

Integrate (9) to obtain

$$-f_1(-t) + \tfrac{1}{2}u_0(t) = \tfrac{1}{2}g_0(2t) + C \quad (10).$$

Notice that (8) and (10) are a linear system for $f_1(-t)$ and $g_0(2t)$ for $t \geq 0$ so we will be able to determine the missing values of f_1 and g_0 from these relationships. First, notice that if we take u_0 to be continuous, $u_0(0)=0$, and since then $f_1(0) = 0$ and $g_0(0)=0$, we have $C=0$ in (10).
Solving (8) & (10) we have

$$g_0(2t) = \tfrac{2}{3}u_0(t) \Rightarrow g_0(s) = \tfrac{2}{3}u_0(s/2) \quad s \geq 0.$$

$$f_1(-t) = \tfrac{1}{6}u_0(t) \Rightarrow f_1(s) = \tfrac{1}{6}u_0(-s) \quad s \leq 0.$$

These two relationships, along with (7b) in (7a) give the solution.

<u>Additional</u>: This can be interpreted as a reflection coefficient of $1/6$ and a transmission coefficient of $2/3$.

MATH 400, Fall 2012, Wetton
Assignment #3 - due Monday, October 1

Part A Routine questions

1. Consider the following heat equation problem for $u(x,t)$, $0 \le x \le 1$ and $t \ge 0$:

$$u_t = (a(x)u_x)_x$$

$u(x,0) = u_0(x)$ and boundary conditions $u(0,t) = 0$ and $u(1,t) = 0$. The function $a(x)$ is discontinuous at $x = 1/2$ and the equation above is satisfied for $x \ne 1/2$. At $x = 1/2$, the values of u and au_x are continuous. Show that this problem has a maximum principle, that is that

$$u(x,t) \le \max(\max_{x\in[0,1]} u(x), 0)$$

for all x and t.

2. Consider the following problem for $u(x,t)$, $0 \le x \le L$ and $t \ge 0$:

$$u_t = -C_1 u - C_2 u^4 + C_3 [u_x]^2 + C_4 u_{xx} + Q$$

where C_1, C_2, C_3, C_4 and Q are positive constants with appropriate units. Non-dimensionalize the equation to make it as simple as possible. There will still be some dimensionless parameters in the resulting equation. *Note:* This is an equation I just made up for this assignment, it has no physical meaning that I know of.

3. Consider the problem above with boundary conditions $u_x = 0$ at $x = 0$ and $x = L$. Show that there is a unique, positive constant (for all x and t) solution to this problem. Linearize the equation around this constant state.

Part B. More difficult questions.

4. Consider the heat conduction problem with discontinuous parameters as discussed on pages 6 and 7 of part II of the posted course notes. Show that conditions (6) and (7) are physically correct in the sense that they can be derived from the limit of solutions of problems in which the coefficients vary smoothly over an interval of width δ and that width goes to zero. It is not necessary to be completely rigorous but carefully state any assumptions you make on the solutions and their behaviour in δ.

5. Consider the coupled, nonlinear wire wave equation derived on pages 7-9 of part II of the posed course notes. Consider the case where a weight is attached to the wire. Model this weight as a point mass m at $s = s^*$. Equation (8) of the notes will apply at all points $s \neq s^*$. Describe the conditions at $s = s^*$ that describe the motion of the attached weight. As in the notes, you may neglect gravity.

Math 400 Assignment #3 Solutions

Fall, 2012

1. As in the notes, consider a modification of $u(x,t)$:

$$V(x,t) := u(x,t) + \varepsilon f(x)$$

where $f(x) = \int_{1/2}^{x} \frac{s-1/2}{a(s)}\, ds \qquad (1).$

Note: This is where we require a to be bounded away from zero.

f is the function defined by the properties:

$$f(\tfrac{1}{2}) = 0$$

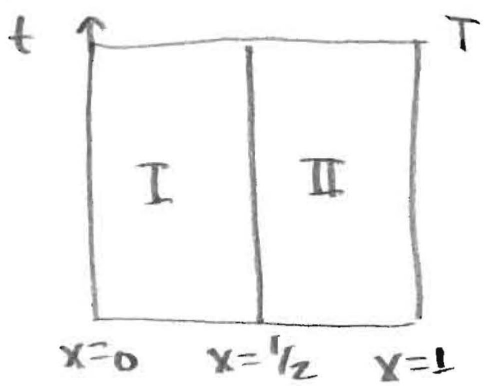
$$f'(\tfrac{1}{2}\pm) = 0 \qquad (2)$$

$$(af')' = \left(a(x)\,\frac{x-\frac{1}{2}}{a(x)}\right)' = (x-\tfrac{1}{2})' = 1. \qquad (3)$$

Now consider $V_t - (a(x)V_x)_x = -\varepsilon < 0 \qquad (4)$

Using $u_t - (a(x)u_x)_x = 0$ and (3).

Consider V in the domains I & II described below:



I: $0<x<1/2$, $0<t<T$

II: $1/2<x<1$, $0<t<T$

Suppose $V(x,t)$ has a maximum in regions I or II. At such a point, $V_t = 0$, $V_x = 0$ and $V_{xx} \le 0$. Thus,

Here, positivity of a is needed.

$$V_t - (a(x)V_x)_x = V_t - a' V_x - a(x) V_{xx} \geq 0$$

which contradicts (4). Thus the maximum cannot be attained in regions I or II.

At the upper boundary $t = T$, excluding $x = 1/2$, a maximum point must satisfy $V_t \geq 0$, $V_x = 0$, $V_{xx} \le 0$ which again leads to a contradiction.

Consider carefully the line $x = 1/2$ at which u and f and so V is continuous. Suppose V attains its maximum on this line, then

$$V_t = 0, \quad V_x\left(\tfrac{1}{2}_-, t\right) \geq 0, \quad V_x\left(\tfrac{1}{2}_+, t\right) \le 0 \qquad (5)$$

$$\left.\begin{aligned} \text{However, } V_x\left(\tfrac{1}{2}_-, t\right) &= u_x\left(\tfrac{1}{2}_-, t\right) \\ \text{and } V_x\left(\tfrac{1}{2}_+, t\right) &= u_x\left(\tfrac{1}{2}_+, t\right) \end{aligned}\right\} \quad (6)$$

because of (2). Now using $a(\tfrac{1}{2}_-)\, u_x(\tfrac{1}{2}_-, t) = a(\tfrac{1}{2}_+)\, u_x(\tfrac{1}{2}_+, t)$ [given junction condition] and the positivity of a we see that $u_x(\tfrac{1}{2}_\pm, t)$ must have the same sign. Now using (5) & (6) we see that

$$V_x\left(\tfrac{1}{2}_\pm, t\right) = 0.$$

With this condition, $V_{xx}(\frac{1}{2}\pm, t) \le 0$ for a maximum which leads to a contradiction as above.

We see now that the maximum of V is attained on the initial line segment or on the boundaries $x=0$ or $x=1$. Since

$$|f(x)| \le \frac{1}{8A}$$

where $A = \min_x a(x)$, we prove the maximum principle as in the notes as $\varepsilon \to 0$.

<u>Note</u>: You can also use

$$V(x,t) = u(x,t) - \varepsilon t$$

in the argument. In this case, some of the steps are simpler.

2. A natural length scale is the interval length L. Let T and U be the scales for t and u, to be determined.

$x = Ly$ y dimensionless distance, $y \in [0,1]$.

$t = T\tau$ τ dimensionless time

$u = Uv$ v dimensionless u

$$\frac{U}{T}\frac{\partial v}{\partial \tau} = -C_1 U v - C_2 U^4 v^4 + \frac{C_3 U^2}{L^2}\left(\frac{\partial v}{\partial y}\right)^2 + C_4 U/L^2\, v_{yy} + Q$$

$$\frac{\partial V}{\partial \tau} = -C_1 T V - C_2 T U^3 V^4 + \frac{C_3 U T}{L^2}(V_y)^2 + \frac{C_4 T}{L^2} V_{yy} + \frac{TQ}{U}.$$

Choose T & U to make the coefficients in two of the five terms equal to 1. That will leave 3 dimensionless parameters. It is arbitrary which three we leave in this abstract example. In an application, you might know which two terms are dominant. Then, the remaining three parameters tell you the importance of the other three terms.

I chose the last two terms to have unit coefficients:

$$\frac{C_4 T}{L^2} = 1 \Rightarrow T = L^2/C_4$$

$$\frac{TQ}{U} = 1 \Rightarrow U = TQ = \frac{L^2 Q}{C_4}.$$

Now $\frac{\partial V}{\partial \tau} = -\beta_1 V - \beta_2 V^4 + \beta_3 (V_y)^2 + V_{yy} + 1.$

where β_i are dimensionless

$$\beta_1 = C_1 T = C_1 L^2/C_4$$

$$\beta_2 = C_2 T U^3 = \frac{C_2 L^2}{C_4} \cdot \left(\frac{L^2 Q}{C_4}\right)^3 = \frac{C_2 L^8 Q^3}{C_4^4}$$

$$\beta_3 = \frac{C_3 U T}{L^2} = \frac{C_3}{L^2}\,\frac{L^2 Q}{C_4}\cdot\frac{L^2}{C_4} = \frac{C_3 Q}{C_4^2}\,L^2.$$

3. A constant solution K (independent of x and t) satisfies

$$-C_1 K - C_2 K^4 + Q = 0.$$

or
$$C_1 K + C_2 K^4 = Q. \qquad (7)$$

Since $C_1 > 0$, $C_2 > 0$ the left hand side is increasing with K from value 0 when $K=0$ to ∞ as $K \to \infty$. Thus, (7) has a single positive solution K for every $Q > 0$.

Now take $u(x,t) = K + \varepsilon V(x,t)$ and find the linear equation for V. Put this expression into the PDE, dropping terms of $\varepsilon^2, \varepsilon^3, \ldots$

$$\varepsilon V_t = -C_1(\cancel{K} + \varepsilon V) - C_2(\cancel{K^4} + 4\varepsilon K^3 V)$$

$$+ \underset{\substack{\uparrow \\ \text{only } \varepsilon^2 \text{ terms in } (u_x)^2}}{C_3(0)} + \varepsilon C_4 V_{xx} + \cancel{Q}.$$

where the cancelled terms above satisfy (7). Thus,

$$V_t = -C_1 V - 4C_2 K^3 V + C_4 V_{xx}$$

4. Let the solutions with smoothed coefficients be $u_\delta(x,t)$. Assume that

$u_\delta, \frac{\partial u_\delta}{\partial x}, \frac{\partial u_\delta}{\partial t}$ are bounded in absolute value (by M) as $\delta \to 0$, converge pointwise to u u_x and u_t except possibly at $x = x^*$.

Consider
$$\frac{u_\delta(x^*+\varepsilon,t) - u_\delta(x^*-\varepsilon,t)}{2\varepsilon} \underset{\substack{\uparrow \\ \text{Mean Value} \\ \text{Theorem}}}{=} u_{\delta,x}(\theta,t)$$

where $\theta \in [x^*-\varepsilon, x^*+\varepsilon]$. Now $|u_{\delta,x}(\theta,t)| < M$ by assumption so

$$|u_\delta(x^*+\varepsilon,t) - u_\delta(x^*-\varepsilon,t)| < \varepsilon M.$$

Now let $\delta \to 0$ using the pointwise convergence assumption

$$|u(x^*+\varepsilon,t) - u(x^*-\varepsilon,t)| < \varepsilon M.$$

Now let $\varepsilon \to 0$, obtaining

$$u(x^*_+,t) = u(x^*_-,t) \qquad \checkmark.$$

Now consider

$$|K_\delta(x^*+\varepsilon)\, u_{\delta,x}(x^*+\varepsilon,t) - K_\delta(x^*-\varepsilon)\, u_{\delta,x}(x^*-\varepsilon,t)$$

$$\underset{\substack{\uparrow \\ \text{MVT}}}{=} 2\varepsilon \left| \left(K_\delta(\theta)\, u_{\delta,x}(\theta,t)\right)_x \right| = (\text{use PDE}) = \cdots$$

$$2\varepsilon \left| c(\theta)\rho(\theta)\, u_{\delta,t}(\theta,t) \right|$$
$$\leq 2\varepsilon M C R$$

where $C = \max\limits_{x} c(x)$ and $R = \max\limits_{x} \rho(x)$.

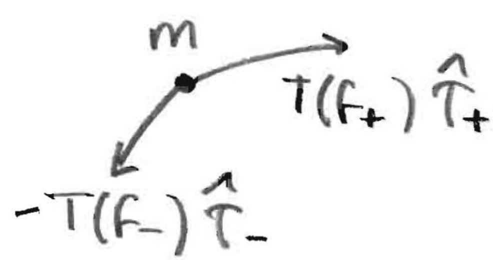
Now letting $\delta \to 0$ and then $\varepsilon \to 0$ as above, we obtain

$$K(x^*_-)\frac{\partial u}{\partial x}(x^*_-, t) = K(x^*_+)\frac{\partial u}{\partial x}(x^*_+, t) \checkmark$$

5. Four conditions are needed at $s = s^*$. Two, you may not have written explicitly but they are needed to specify the problem mathematically. These are merely that $x(s,t)$ and $y(s,t)$ are continuous at $s = s^*$:

$$\left.\begin{aligned} x(s^*_-, t) &= x(s^*_+, t) \\ y(s^*_-, t) &= y(s^*_+, t). \end{aligned}\right\} \quad (8)$$

However, the first derivatives of x and y with respect to s are not continuous. Consider the force balances on the mass m at $s = s^*$:



where $f_\pm = \sqrt{x_s^2(s^*_\pm, t) + y_s^2(s^*_\pm, t)}$ defines f_+ with all + signs, f_- with all − signs.

and $\tau_\pm = \left(x_s(s^*_\pm, t),\, y_s(s^*_\pm, t)\right) / f_\pm$

Net force on the mass is

$$T(f_+)\hat{T}_+ - T(f_-)\hat{T}_-$$

must equal m times the acceleration of the mass, i.e.

$$m\left(x_{tt}(s^*,t),\, y_{tt}(s^*,t)\right) = T(f_+)\hat{T}_+ - T(f_-)\hat{T}_- \quad (9)$$

x continuous at s^* for all t, so also is x_{tt} ✓.

(8) and (9) are the four desired conditions.

MATH 400, Fall 2012, Wetton
Assignment #4 - due Wednesday, October 10

Part A Routine questions

1. Consider the following PDE for $u(x, y)$:

$$4u_{xx} - 4u_{xy} + u_{yy} + 4u_x = 0.$$

Determine whether the problem is elliptic, hyperbolic or parabolic.

2. Consider the following PDE for $u(x, y)$:

$$2u_{xx} - 3u_{xy} + u_{yy} = 0.$$

Determine whether the problem is elliptic, hyperbolic or parabolic. What change of variables would take the equation to its standard form?

3. Consider the following problem for $u(x, t)$, $t \geq 0$ and all x:

$$u_t + x^2 u_x = 0$$

with initial conditions $u(x, 0) = 1/(1 + x^2)$. Find the solution $u(x, t)$.

Part B. More difficult questions.

4. Consider the following equation for $\Phi(x, y)$:

$$(1 - M_x^2)\Phi_{xx} - 2M_x M_y \Phi_{xy} + (1 - M_y^2)\Phi_{yy} = 0$$

where $M_x = \Phi_x/a$ and $M_y = \Phi_y/a$ and a is a given positive constant. Show that $\Phi = Vx$ with V constant satisfies this equation. Linearize the equation about this solution. Determine whether the resulting linear equation is elliptic, hyperbolic, or parabolic. *Note:* your answer will depend on $|V|$.

5. Consider the characteristics for the problem

$$u_t + a(x, t)u_x = 0.$$

These are curves $x(t)$ that satisfy

$$\dot{x} = a(x, t) \text{ with } x(0) = s.$$

We can consider characteristics over a range of s, $x(s,t)$. Assume that the given function $a(x,t)$ and a_x and a_t are continuous and bounded for all x and all $t \geq 0$. It can be shown that $x(s,t)$ exists for all s and $t \geq 0$ and is differentiable (you may take this fact as given). Show the following:

- For every point (x,t) with $t > 0$ the value of x can be written as $x(s,t)$ for some s. That is, every point is crossed by a characteristic.
- Show that for every fixed t, $x(s,t)$ is an increasing function of s.

Math 400 Assignment #4 solutions
Fall, 2012.

1. $$\underbrace{4u_{xx} - 4u_{xy} + u_{yy}}_{au_{xx} + 2bu_{xy} + cu_{yy}} + 4u_x = 0 \qquad (1)$$

$a = 4, \; b = -2, \; c = 1$

$ac - b^2 = 4 - 4 = 0$, so (1) is parabolic.

2. $$\underbrace{2u_{xx} - 3u_{xy} + u_{yy}} = 0 \qquad (2)$$

$a = 2, \; b = -3/2, \; c = 1$

$ac - b^2 = 2 - 9/4 < 0$, so (2) is hyperbolic.
Proceeding as in the notes part III, consider

$$A = \begin{bmatrix} 2 & -3/2 \\ -3/2 & 1 \end{bmatrix}.$$

The eigenanalysis of A yields $\lambda_1 = (3+\sqrt{10})/2$, $\lambda_2 = (3-\sqrt{10})/2$ (one positive and one negative as expected from a hyperbolic equation). The normalized eigenvectors are

$$\underline{V_1} = \frac{1}{\beta_1}\begin{bmatrix} 1-\sqrt{10} \\ 3 \end{bmatrix}, \quad \underline{V_2} = \frac{1}{\beta_2}\begin{bmatrix} 1+\sqrt{10} \\ 3 \end{bmatrix}$$

where $\beta_1 = \sqrt{9+(1-\sqrt{10})^2}$, $\beta_2 = \sqrt{9+(1+\sqrt{10})^2}$.
Note that $\underline{V_1} \cdot \underline{V_2} = 0$ as expected. From the notes, we know that under the transformation

$\begin{bmatrix} v \\ w \end{bmatrix} = \mathbb{P} \begin{bmatrix} x \\ y \end{bmatrix}$ where $\mathbb{P}$ is the matrix with $\underline{v_1}$ and $\underline{v_2}$ in columns, (2) becomes

$$\lambda_1 u_{vv} + \lambda_2 u_{ww} = 0.$$

Now considering the scaled variables,

$$s = \frac{v}{\sqrt{\lambda_1}}, \quad t = \frac{w}{\sqrt{|\lambda_2|}}$$

we obtain the desired form

$$u_{ss} - u_{tt} = 0.$$

The combined transformation $(x,y) \to (s,t)$ can be written as

$$\begin{bmatrix} s \\ t \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{1}{\sqrt{\lambda_1}} & 0 \\ 0 & \frac{1}{\sqrt{|\lambda_2|}} \end{bmatrix} \mathbb{P}}_{\text{linear transformation with this matrix}} \begin{bmatrix} x \\ y \end{bmatrix}.$$

3. $u_t + x^2 u_x = 0$

$$u(s,0) = \frac{1}{1+s^2}$$

characteristics $x(t)$ solve $\dot{x} = x^2$, $x(0) = s$, (separable) with solution

$$x(s,t) = \frac{s}{1-st}.$$

It is also convenient to solve for $s(x,t)$ here, since we'll need that later

$$s(x,t) = \frac{x}{1+xt}. \qquad (3)$$

Now along characteristics $u(t)$ with $\dot{u}=0$, so $u(s,t) = u_0(s) = \frac{1}{1+s^2}$. Now using (3) we can write the solution

$$u(x,t) = \frac{1}{1+\left(\frac{x}{1+xt}\right)^2}$$

4. With $\Phi = Vx$, Φ_{xx}, Φ_{yy}, Φ_{xy} are zero for all x and y, so the equation is satisfied. Consider

$$\Phi = Vx + \varepsilon\varphi(x,y).$$

Put this form into the equation and ignore terms of ε^2 and higher (this is the "recipe" for linearization)

$$M_x = (V + \varepsilon\varphi_x)/a$$

$$(M_x)^2 \approx \frac{V^2}{a^2} + \frac{2\varepsilon V}{a^2}\varphi_x.$$

$$M_y = \varepsilon\varphi_y/a$$

$$(M_y)^2 \approx 0 \qquad \text{(terms start at } \varepsilon^2\text{)}.$$

So the equation becomes

$$\left(1 - \frac{V^2}{a^2} + \overbrace{\frac{2\varepsilon V}{a^2}\varphi_{xx}}^{\varepsilon^2 \text{ term}}\right)\varepsilon\varphi_{xx} - 2\cdot 0$$

$$+ \varepsilon\varphi_{yy} = 0$$

or finally, dividing through by ε,

$$\left(1 - \frac{V^2}{a^2}\right)\varphi_{xx} + \varphi_{yy} = 0 \qquad (4)$$

In the form (4) it is clear that for $V > a$ the equation is hyperbolic and $V < a$ it is elliptic.

<u>Note</u>: Formally you can call (4) parabolic for $V = a$, but it is a bit of a special case since there are no lower order terms.

<u>Additional</u>: This equation is for a flow potential in transonic fluid flow. The term V/a is the mach number.

5. The characteristics $x(t)$ solve

$$\dot{x} = a(x,t) \qquad x(0) = s \qquad (5)$$

yielding $x(s,t)$. It is given that $x(s,t)$ exists and is differentiable for every s and t (can be proved under the given conditions on $a(x,t)$). Consider a given point (x^*, t^*). We need to show that $x^* = x(s, t^*)$ for some s. Let $M = \sup_{t \geq 0, x} |a(x,t)|$ (it is given that a is bounded). Consider the characteristic beginning at $s_0 = x^* - (M+1)t^*$. We can follow this characteristic to time t^* and show that

$$\underbrace{x(s_0, t^*) - s_0}_{\uparrow \; x(s_0,0)} = \int_0^{t^*} a(x,t)\,dt \leq M t^*$$

So $x(s_0, t^*) = s_0 + Mt^*$

$$= (x^* - (M+1)t^*) + Mt^*$$

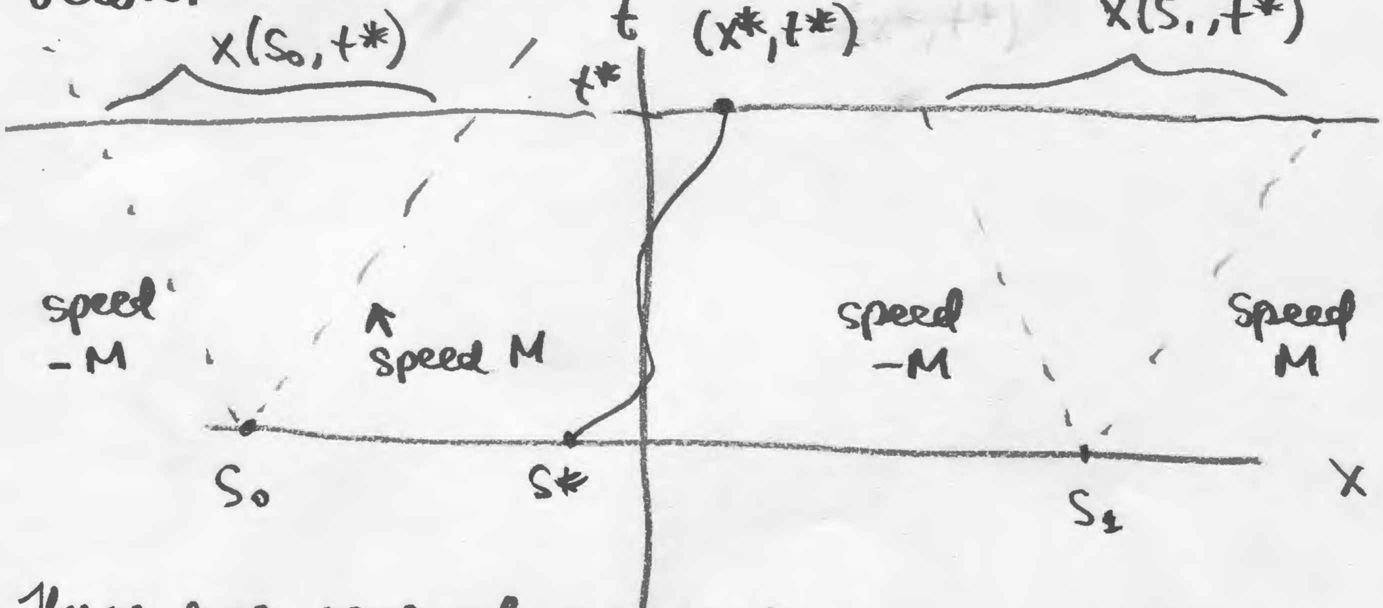
$$= x^* - t^* < x^* \qquad (6)$$

Similarly if we consider the characteristic starting at $s_1 = x^* + (M+1)t^*$ it can be shown that $x(s_1, t^*) > x^*$ (7).

Since $x(s, t^*)$ is differentiable in s it is necessarily continuous and using (6) and (7) and the intermediate value theorem,

$$x(s^*, t^*) = x^*$$

for some $s^* \in (s_0, s_1)$. This is the desired result. A diagram of the idea above is given below.



There are several ways to approach the next problem. Consider (5) with solution $x(s,t)$. It is known that $v = \frac{\partial x}{\partial s}$ exists for all s and $t \geq 0$. Take the derivative of (5) with respect to s, obtaining linear in V

$$\left.\begin{aligned} \dot{V} &= a_x(X(S,t),t)\,V \\ V(0) &= \frac{\partial X}{\partial S}(S,0) = 1. \end{aligned}\right\} \quad (8).$$

From (8) it is seen that V can also be determined along characteristics,

$$V(t) = e^{\int_0^t a_x(X(S,T),T)\,dT}$$

use the usual formula.

Note that this is well defined for all s and t since a_x is bounded (a given fact).

Thus $V(t) > 0$ for all s and t,

so $\frac{\partial X}{\partial S} > 0$ for all s and t and thus $X(S,t)$ is increasing in S at all S for fixed t.

MATH 400, Fall 2012, Wetton
Assignment #5 - due Monday, October 15

Part A Routine questions

1. [Review of phase plane] Consider the following autonomous ODE system for $x(t)$ and $y(t)$:

$$dx/dt = y \quad \text{and} \quad dy/dt = x$$

Make a phase plane diagram of the velocity vectors (y, x). Sketch some trajectories (solutions) of the system.

2. [Review of equilibria for autonomous systems] Consider the following autonomous ODE system for $x(t)$ and $y(t)$:

$$dx/dt = x - x^2 + xy \quad \text{and} \quad dy/dt = xy.$$

Note that $x = 1$, $y = 0$ is a constant solution of the system (an equilibrium point). Linearize the system about this equilibrium point. From the linearization, determine whether this equilibrium point is stable or unstable.

3. Consider the following PDE for $u(x, y)$:

$$yu_x + xu_y = xy^3$$

with data given $u = s^2$ on the line segment $x = s$, $y = 0$, $1 \le s \le 2$. Find the solution $u(x, y)$ and the region where it is defined.

4. Consider the following PDE for $u(x, y)$:

$$yu_x - xu_y = -u$$

with data given $u = e^s$ on the line segment $x = s$, $y = 0$, $1 \le s \le 2$. Find the solution $u(x, y)$. Find the region where the solution is defined (there is a small subtlety here).

5. Consider the following PDE for $u(x, y)$:

$$u_y + \frac{1}{1 + x^2} u_x = -u.$$

Consider data given on lines through the origin $y = cx$. For what values of c do the lines satisfy the data and characteristic compatibility condition (notes IVa, top of page 10) for all points on the line?

6. Consider the following Cauchy problem for Burger's equation:

$$u_t + (\frac{1}{2}u^2)_x = 0$$

with $u(x, 0) = 1/(1 + x^2)$. Find the time t_* of singularity formation, that is the first time at which the solution is no longer differentiable.

7. Show that $u(x, t) \equiv U$ (that is, $u = U$ for all x and t) with U constant solves Burger's equation. Linearize Burger's equation around this constant solution.

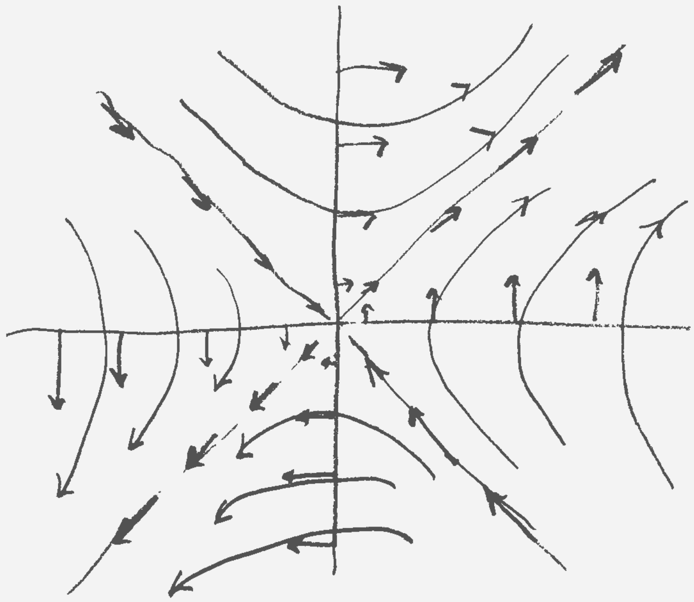
8. Consider the following problem for $\rho(x, t)$:

$$\rho_t + (f(\rho))_x = 0$$

where $f(\rho) = V\rho(1-\rho/R)$ where V and R are given positive constants. Scale and non-dimensionalize this problem.

Math 400 Assignment #5 solutions
Fall, 2012

1.



$y=x$ is the image of a trajectory, as is $y=-x$.

2. Take $x = 1 + \varepsilon u(t)$, $y = \varepsilon v(t)$.

$$\varepsilon \frac{du}{dt} = (1+\varepsilon u(t)) - (1+\varepsilon u(t))^2 + (1+\varepsilon u(t))(\varepsilon v)$$

$$\approx -\varepsilon u + \varepsilon v.$$

$$\varepsilon \frac{dv}{dt} = (1+\varepsilon u(t))(\varepsilon v) \approx \varepsilon v.$$

so

$$\frac{d}{dt}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} -u+v \\ v \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}}_{A} \begin{bmatrix} u \\ v \end{bmatrix}$$

Note that A is the Jacobian matrix of the right hand functions evaluated at the equilibrium $(1,0)$, as you might have expected. You could have started with that result. The eigenvalues of A are ± 1, so $(1,0)$ is a saddle point, unstable.

3. $y\,u_x + x\,u_y = x y^3$

characteristics $x(T)$, $y(T)$ have initial data $x(0) = S$, $y(0) = 0$ $(1 \le S \le 2)$ and solve

$$\frac{dx}{dT} = y, \quad \frac{dy}{dT} = x \qquad (1).$$

Note that this is the same system as problem 1. Do some algebra on (1),

$$\frac{d^2x}{dT^2} = \frac{dy}{dT} = x$$

so

$$x(T) = A\cosh T + B\sinh T$$

$$y(T) = \frac{dx}{dT} = A\sinh T + B\cosh T$$

(algebra is much easier with hyperbolic trig functions).

matching initial conditions gives $A = S$, $B = 0$ so

$$\left.\begin{array}{l} x(T) = S\cosh T \\ y(T) = S\sinh T \end{array}\right\} (2).$$

Now along characteristics,

$$\frac{du}{dT} = x y^3 = S^4 \cosh T \sinh^3 T \qquad u(0) = S^2$$

which is solved by integrating using substitution

$$u(T) = \frac{S^4}{4}\sinh^4 T + S^2. \qquad (3)$$

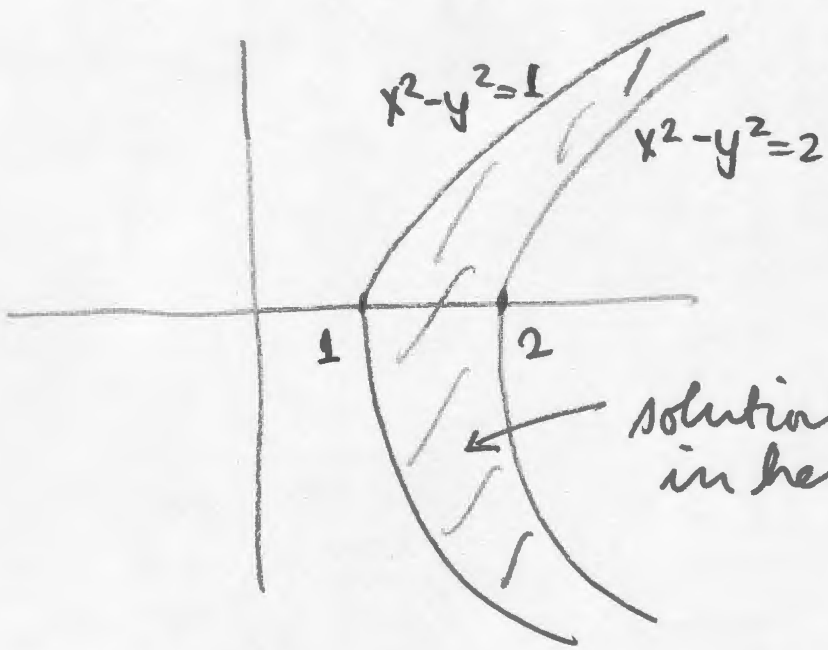
Note that $S^4 \sinh^4 T = y^4$ using (2) and that $S^2 = x^2 - y^2$ using the identity $\cosh^2 T - \sinh^2 T = 1$.

Now (3) can be written

$$u(x,y) = \frac{y^4}{4} + (x^2 - y^2).$$

Can verify that u solves the PDE and matches the data.

The solution is defined between the hyperbolas $x^2-y^2=1$ and $x^2-y^2=2$ in the region $x \geq 0$.



4. $y u_x - x u_y = -u$.

As in the lecture notes, the characteristics solve

$$\frac{dx}{d\tau} = y, \quad \frac{dy}{d\tau} = -x \qquad x(0)=s,\ y(0)=0,\quad 1 \leq s \leq 2$$

with solution

$$x(\tau) = s\cos\tau, \quad y(\tau) = s\sin\tau. \Rightarrow \tau = \arg(x+iy),\quad s = \sqrt{x^2+y^2}$$

On the characteristics, (polar coordinates)

$$\frac{du}{d\tau} = -u \qquad u(0) = e^s$$

so $u(\tau) = e^s e^{-\tau}$. Note that the solution is only defined in a neighborhood of the data line. There must be a discontinuity in the solution at some $\tau \in (-\pi, \pi]$. I will put the discontinuity at $\tau = \pi$, leading to

$$u(x,y) = e^{\sqrt{x^2+y^2}}\, e^{-\theta}, \quad \text{for } \begin{matrix} 1<\sqrt{x^2+y^2}<2 \\ -\pi<\theta<\pi, \end{matrix}$$

where $\theta = \arg(x+iy)$. Note that on the line segment $y=0$, $-2 \le x \le -1$ the solution is discontinuous. This is like a branch cut in an analytic function.

5. $u_y + \dfrac{1}{1+x^2} u_x = -u.$

characteristics $\dfrac{dx}{d\tau} = \dfrac{1}{1+x^2}$, $\dfrac{dy}{d\tau} = 1.$

data line $x = s$, $y = cs$, so the tangent direction to the data line is $(1, c)$. On the data line, the tangent vector to the characteristic is $(\frac{1}{1+s^2}, 1)$. For the data line to be compatible, these vectors cannot be parallel for any s, i.e.

$$\det\begin{pmatrix} 1 & c \\ \frac{1}{1+s^2} & 1 \end{pmatrix} = 1 - \frac{c}{1+s^2} \neq 0 \quad \text{for any } s.$$

or $c \neq 1+s^2$ for any s. That is, any line with $c < 1$ is compatible, any line with $c \geq 1$ is not.

6. $u_t + \left(\frac{u^2}{2}\right)_x = 0$, $u_t + uu_x = 0$ before the discontinuity forms. Characteristics are given by

$$x(t) = u_0(s)\,t + s = \frac{t}{1+s^2} + s.$$

as shown in the notes, we need to find the minimum M of $u_0'(s)$.

$$u_0(s) = \frac{1}{1+s^2}, \quad u_0'(s) = \frac{-2s}{1+s^2}$$

$$u_0''(s) = \frac{-2(1+s^2) + 4s^2}{(1+s^2)^2} = \frac{2(s^2-1)}{(1+s^2)^2}.$$

Minimum of u_0' occurs where u_0'' is zero, $s = 1$ ($s = -1$ is a maximum).

$$M = u_0'(1) = -1, \quad \text{so } t^* = -\frac{1}{M} = 1.$$

<u>additional</u> For a general problem $u_t + (f(u))_x = 0$, discontinuities will form if $f''(u_0(s))\,u_0'(s) < 0$ some s at time $t^* = \dfrac{-1}{\min_s f''(u_0(s))\,u_0'(s)}$.

Make sure you see how this can be derived.

7. if $u(x,t) \equiv U$ then $\frac{\partial u}{\partial t} = 0$ and $\frac{\partial}{\partial x}\left(\frac{U^2}{2}\right) = 0$ so this trivially solves Burger's equation.

Take $u = U + \varepsilon V(x,t)$.

$$u_t = \varepsilon V_t$$

$$\frac{\partial}{\partial x}\left(\frac{1}{2}u^2\right) = \frac{\partial}{\partial x}\left(\frac{1}{2}U^2 + \varepsilon U V + \frac{1}{2}\varepsilon^2 V^2\right)$$

$$\approx \varepsilon U \frac{\partial V}{\partial x}.$$

so $V_t + U V_x = 0$ is the linearized equation.

Additional Note that linear disturbances move at constant speed U, the characteristic speed.

8. $\rho_t + (f(\rho))_x = 0.$

Let $t = \tau T$, τ dimensionless time, T constant scale

$x = y X$, y dimensionless space

$\rho = r Q$, r dimensionless density.

$$\frac{Q}{T} r_\tau + \frac{1}{X}\left(V Q r \left(1 - r Q/R\right)\right)_y = 0.$$

$$r_\tau + \frac{VT}{X}\left(r\left(1 - \frac{rQ}{R}\right)\right)_y = 0.$$

take $Q = R$, density scale determined, and

$$\frac{VT}{X} = 1. \qquad (4).$$

Note that (4) only determines $\frac{X}{T}$ not X and T individually, due to an invariance in the problem. We can take $X = 1$ (with units) and $T = \frac{1}{V}$ ← with length units. to obtain

$$r_\tau + (\underbrace{r(1-r)}_{g(r)})_y = 0$$

$g(r)$, still in conservation form.

Math 400 Assignment #5, Fall 2012
corrected solution to problem 6.

$$u_0(s) = \frac{1}{1+s^2}, \quad u_0' = \frac{-2s}{(1+s^2)^2}$$

$$u_0''(s) = \frac{-2}{(1+s^2)^2} + \frac{8s^2}{(1+s^2)^3}$$

$$= \frac{4s^2 - 2(1+s^2)}{(1+s^2)^3} = \frac{6s^2-2}{(1+s^2)^3}$$

Therefore the minimum of u_0' occurs at $s = \frac{1}{\sqrt{3}}$ and the minimum value is

$$M = u_0'\left(\frac{1}{\sqrt{3}}\right) = \frac{-2/\sqrt{3}}{(1+1/3)^2} = \frac{-3\sqrt{3}}{8}.$$

Now $t^* = -\frac{1}{M} = \frac{8}{3\sqrt{3}}$

MATH 400, Fall 2012, Wetton
Assignment #7 - due Monday, October 29

Part A Routine questions

1. Find the general solution of the Riemann problem for Burger's equation. That is, solve

$$u_t + (\frac{1}{2}u^2)_x = 0$$

with initial data

$$u(x,0) = \begin{cases} u_l & \text{if } x < 0 \\ u_r & \text{if } x > 0 \end{cases}$$

for any constant values u_l and u_r. *Hint:* Just like we saw in class, there are separate cases depending on whether $u_l < u_r$ or $u_l > u_r$.

2. Use the results of #1 above to sketch (in the $x-t$ plane) the solution of the problem:

$$u_t + (\frac{1}{2}u^2)_x = 0$$

with

$$u(x,0) = \begin{cases} 3 & \text{if } |x| > 1 \\ -1 & \text{if } |x| < 1 \end{cases}$$

3. Solve the general Riemann problem for the traffic flow model

$$u_t + [u(1-u)]_x = 0$$

4. As defined in question #1, solve the general Riemann problem for

$$u_t + (u^4)_x = 0.$$

5. Find an analytic formula to the solution to Burger's equation with initial data given below up to time $t = 2$.

$$u(x,0) = \begin{cases} 1 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Hint: Up to time $t = 2$ this is just two Riemann problems (one shifted from $x = 0$ to $x = 1$) that do not interact.

6. In problem #4 above describe in words what happens to the solution for $t > 2$. Draw a sketch of the solution, showing characteristic lines and any shock locations in time.

Part B More challenging questions

7. Consider the Riemann problem for the following conservation law:

$$u_t + (u^4 - 2u^2)_x = 0$$

with

$$u(x,0) = \begin{cases} u_l & \text{if } x < 0 \\ u_r & \text{if } x > 0 \end{cases}$$

Find the solution of this problem for two cases: $u_l = 0, u_r = 2$ and $u_l = 2, u_r = 0$. *Note:* The flux function in this case is not convex and this will make the solution for one of the cases above more complex. In that case, the solution will be a combination of a shock wave and a rarefaction. Any shock waves introduced must satisfy the Rankine-Hugoniot and Lax Entropy conditions. *Hint:* It may be helpful to consider the initial data smoothed out as we did in class when rarefaction waves were introduced.

8. In class notes we considered the following problem:

$$u_t + (\frac{1}{2}u^2)_x = 0$$

with

$$u(x,0) = e^{-x^2}.$$

We showed that a shock would form at a specific time and location. Consider the shock curve $X(t)$ that starts here and assume no other shocks form (true in this problem). Find an ODE for $X(t)$ (it is OK to have an implicitly defined right hand side). How does the shock curve behave as $t \to \infty$?

Bonus: Continuing #8, calculate $X(t)$ numerically using your ODE and the known point of shock formation as initial data. Describe your numerical approach in words and show your results.

Math 400, Fall 2012
Assignment #7 Solutions

1. There are three cases.

(i) $U_L = U_R = U$, the solution is then just the constant U for all x and t.

(ii) $U_L > U_R$. In this case, a shock can form with speed $S = \frac{1}{2}(U_L + U_R)$.

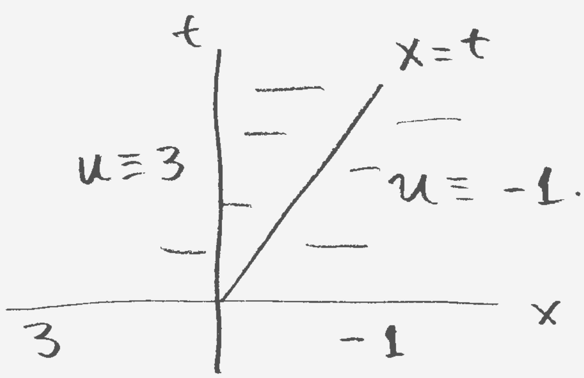
$$u(x,t) = \begin{cases} U_L & x < St \\ U_R & x > St. \end{cases}$$

(iii) $U_L < U_R$. In this case a rarefaction wave (expansion fan) forms.

$$u(x,t) = \begin{cases} U_L & x < U_L t \\ c & x = ct \qquad U_L < c < U_R \\ U_R & x > U_R t \end{cases}$$

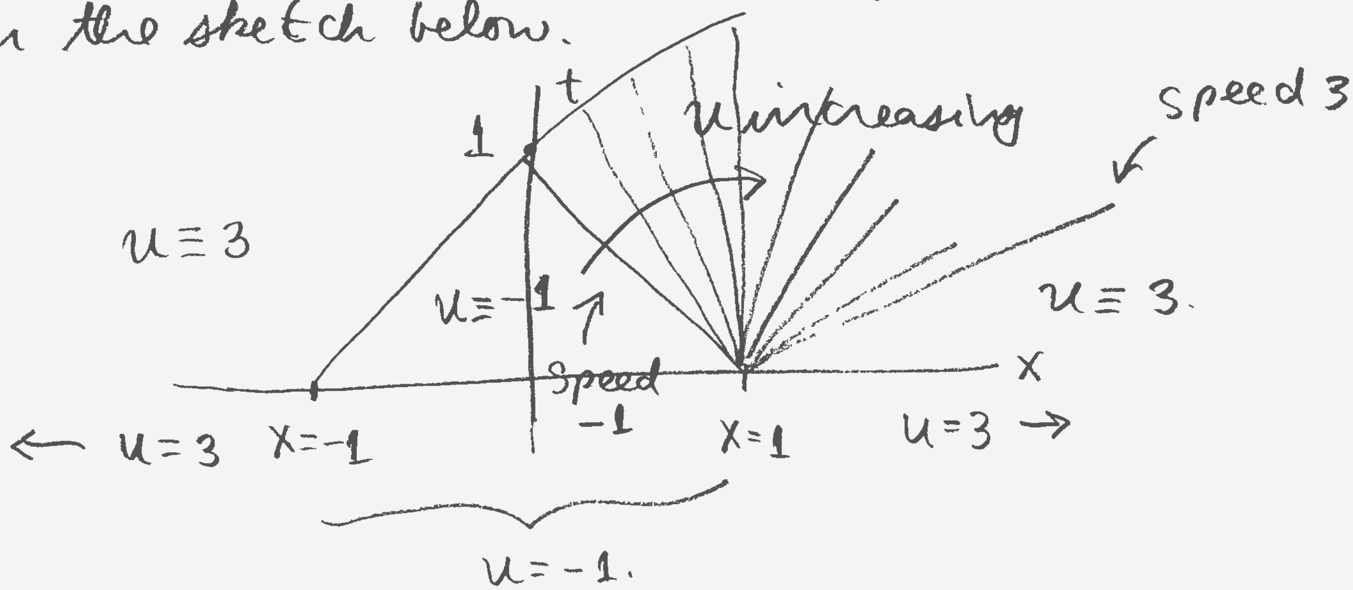
2. $\boxed{\text{Modified version}}$ had $u(x,0) = \begin{cases} 3 & \text{if } x<0 \\ -1 & \text{if } x>0 \end{cases}$

as shown in problem 1, a shock forms with speed 1



$\boxed{\text{Original version}}$ had $u(x,0) = \begin{cases} 3 & \text{if } |x|>1 \\ -1 & \text{if } |x|<1. \end{cases}$

In this case, the shock wave seen in the modified version appears at $x=-1$ and a rarefaction wave from $U_L=-1$ to $U_R=3$ appears at $x=1$. The left edge of this rarefaction moves with speed -1 and so the two waves will begin to interact at $t=1$, $x=0$ as shown in the sketch below.



Note that after time 1 the shock wave will speed up since it starts to see larger values of u on its right side.

3. $$\left.\begin{aligned} f(u) &= u - u^2 \\ f'(u) &= 1 - 2u. \\ s &= 1 - (U_L + U_R) \end{aligned}\right\} \quad (1)$$

again there are three cases,

(i) $U_L = U_R = U$, leading as in problem 1 to the constant U solution.

(ii) $U_L > U_R$ is a rarefaction case here. u decreases through the wave from left to right and this corresponds to an increase in characteristic speeds (consistent).

$$u(x,t) = \begin{cases} U_L, & x < (1-2U_L)t \\ \frac{1}{2}(1-c) & x = ct, \quad (1-2U_L) < c < (1-2U_R) \\ U_R, & x > (1-2U_R)t \end{cases}$$

$f'(u) = c, \quad 1-2u = c \Rightarrow u = \frac{1}{2}(1-c)$

(iii) $U_L < U_R$ leads to a shock wave with speed $S = 1-(U_L+U_R)$,

$$u(x,t) = \begin{cases} U_L & x < St \\ U_R & x > St. \end{cases}$$

4. $f(u) = u^4$ convex

$f'(u) = 4u^3$ increasing (2)

$$S = \frac{U_R^4 - U_L^4}{U_R - U_L} = (U_R^2 + U_L^2)(U_R + U_L)$$

as before, three cases.

(i) $U_L = U_R = U \Rightarrow$ constant solution U

(ii) $U_L > U_R$, $f'(U_L) > f'(U_R)$ since f' is increasing. It can be shown directly that if $U_L > U_R$ that

$$4U_L^3 > \underbrace{U_R^3 + U_L^3 + U_R U_L^2 + U_L U_R^2}_{S} > 4U_R^3$$

so the entropy conditions would be satisfied with a shock wave in this case.

as usual then, the solution is

$$u(x,t) = \begin{cases} U_L & x < \S t \\ U_R & x > \S t. \end{cases}$$

(iii) $U_L < U_R$. a rarefaction in which u is increasing and characteristic speed is increasing is consistant. On a characteristic with speed c, the value of u satisfies

$$f'(u) = c \Rightarrow 4u^3 = c \Rightarrow u = \sqrt[3]{c/4}$$

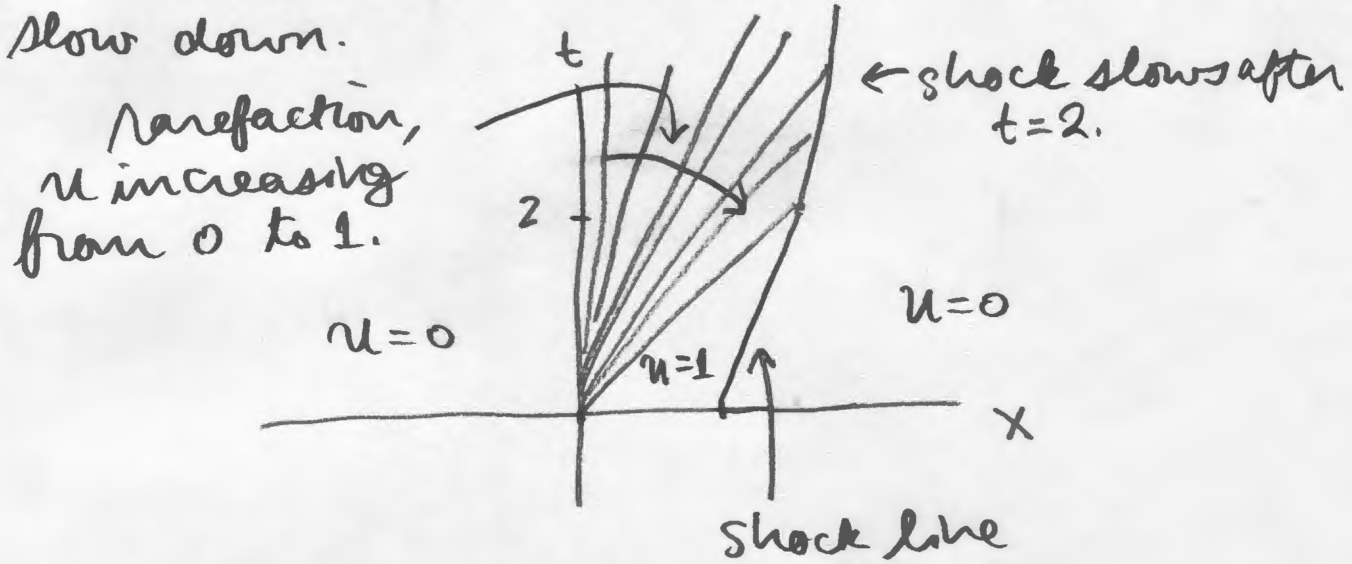
(cube root so well defined for c both positive and negative).

$$u(x,t) = \begin{cases} U_L & x < 4U_L^3 t \\ \sqrt[3]{c/4} & x = ct, \quad 4U_L^3 < c < 4U_R^3 \\ U_R & x > 4U_R^3 t \end{cases}$$

5. At $x=0$ we have a Riemann problem with $U_L=0$ and $U_R=1$ that leads to a rarefaction wave. The right edge of the rarefaction moves with speed 1. At $x=1$ we have a Riemann problem with $U_L=1$ and $U_R=0$ that leads to a shock wave moving with speed 1/2. At $t=2$, the rarefaction wave begins to interact with the shock wave at $x=2$, before that the waves are disjoint.

$$u(x,t) = \begin{cases} 0 & x < 0 \\ c & 0 < x < ct \qquad 0 \le c \le 1 \\ 1 & x > t, \ x < 1 + t/2 \\ 0 & x > 1 + t/2. \end{cases}$$

6. After $t > 2$ the shock will interact with the rarefaction wave. The shock will always have $U_R = 0$ but U_L will decrease with time and so the shock with speed $\dot{s} = \frac{1}{2}(U_R + U_L)$ will slow down.



7. $F(u) = u^4 - 2u^2$

$$F'(u) = 4u^3 - 4u = 4(u^3 - u).$$

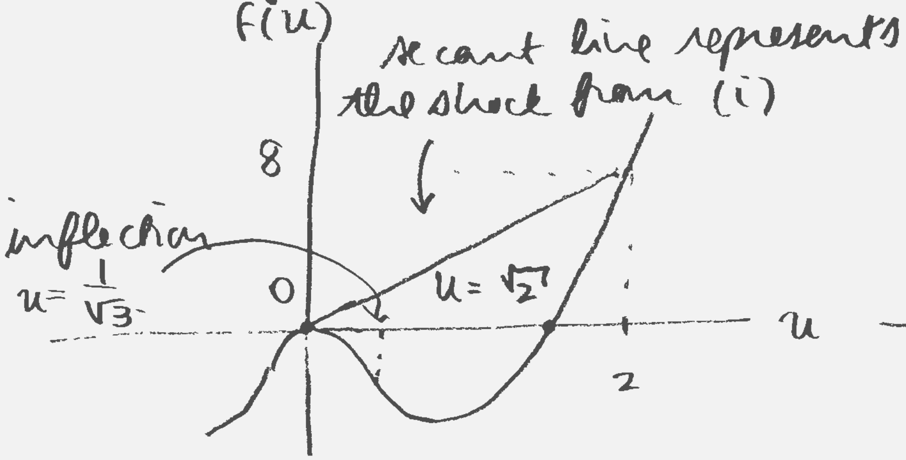
$$\dot{s} = \frac{(U_R^4 - 2U_R^2) - (U_L^4 - 2U_L^2)}{U_R - U_L}$$

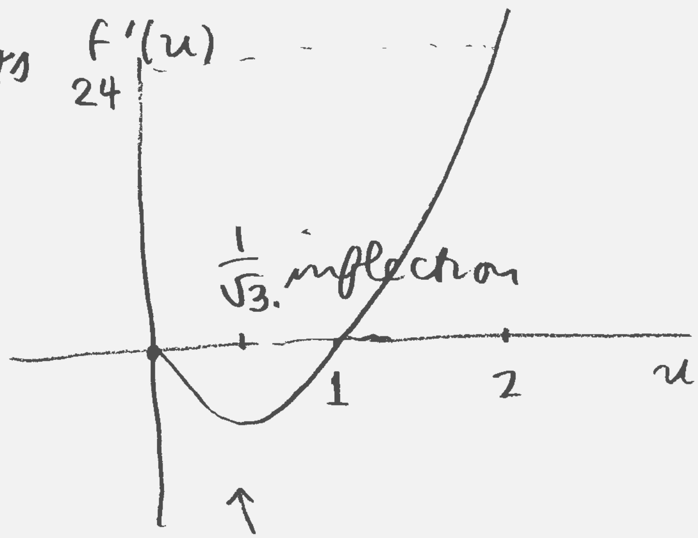
$$= (U_R^2 + U_L^2 - 2)(U_R + U_L)$$

$F'(0) = 0, \quad F'(2) = 24$ so the only case that can lead to a pure shock is

(i) $U_L = 2$, $U_R = 0$. Here, $\dot{s} = 4$ so the shock would satisfy the Lax entropy conditions. This is the easy case.

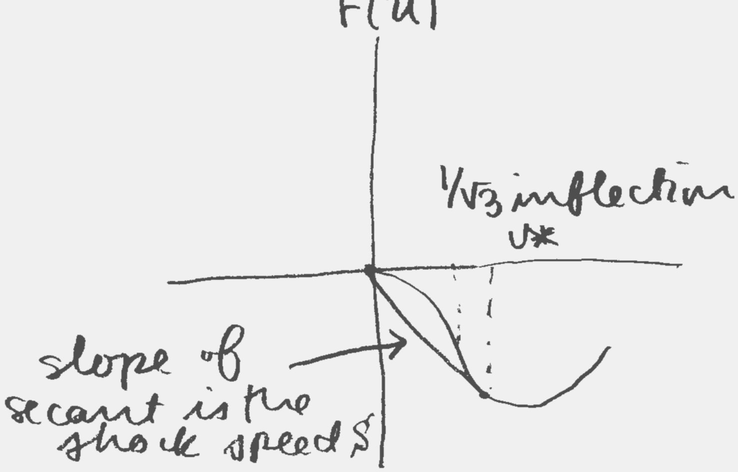
(ii) $U_L = 0$, $U_R = 2$. Look more closely at the graphs of $F(u)$ and $F'(u)$. A useful observation is that the shock speed from the R-H condition is the slope of the secant line joining $(U_R, f(U_R))$ and $(U_L, f(U_L))$ on the graph of $f(u)$ versus u. To satisfy the entropy condition, the tangent slope at U_L must be larger than this secant slope and the tangent slope at U_R must be smaller. For a rarefaction from U_L to U_R, $f'(u)$ must be increasing as $U_L \to U_R$. This last condition is not satisfied for the current case of $U_L = 0$, $U_R = 2$ as seen by the diagrams below:





$f''(u) = 4(3u^2 - 1)$
inflection at $u = \frac{1}{\sqrt{3}}$

So the solution in this case cannot be a pure rarefaction since f' is decreasing from 0 to $u = \frac{1}{\sqrt{3}}$. The resolution is to introduce an intermediate state $\frac{1}{\sqrt{3}} < U_M < 2$. U_M can be connected to $U_L = 2$ by a rarefaction wave because $f'(u)$ is increasing from $u = \frac{1}{\sqrt{3}}$ to 2. Considering the graph of $f(u)$ more closely indicates that for $U_M \in [\frac{1}{\sqrt{3}}, U^*]$, $U_L = 0$ can be connected to U_M by a shock since $f'(0) = 0 > S \geq f'(U_M)$ for U_M in this interval.



from state $U_L = 0$ to U_M.

At u^* the secant line has the same slope as the tangent line,

$$f(u^*) = f'(u^*)$$

quartic equation for u^* that can be solved numerically.

For $U_M > U^*$, U_L cannot be joined to U_M by a shock that satisfies the entropy conditions since here $f'(U_M) > S$.

For $U_M < \frac{1}{\sqrt{3}}$, U_M cannot connect to U_R by a rarefaction since $f'(u)$ is decreasing for $u < \frac{1}{\sqrt{3}}$. It can be shown that in fact U_M must be equal to U^*. Considering smoothed initial data, U_M has to "come from" the origin, so its characteristic speed must equal the shock speed.

We have $u(x,t) = \begin{cases} 0 & x < St \\ u(c) & x = ct, \quad S < c < 24 \\ 2 & x > 24t. \end{cases}$

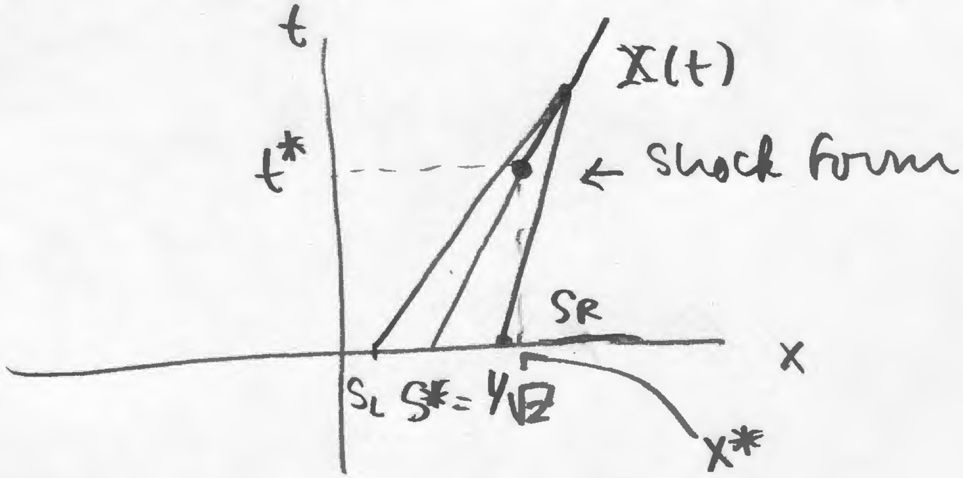
where $S = f(U^*)$ determined afove above and $u(c)$ solves $f'(u) = c$ uniquely for $u > u^* > \frac{1}{\sqrt{3}}$

8. $u(s,t) = e^{-s^2}$

$x(s,t) = te^{-s^2} + s.$

Shock formed at $s^* = \frac{1}{\sqrt{2}}$, and $t^* = \frac{1}{\sqrt{2}} e^{1/2}$ so at $x^* = \frac{1}{\sqrt{2}} e^{1/2} \cdot e^{-1/2} + \frac{1}{\sqrt{2}} = \sqrt{2}$

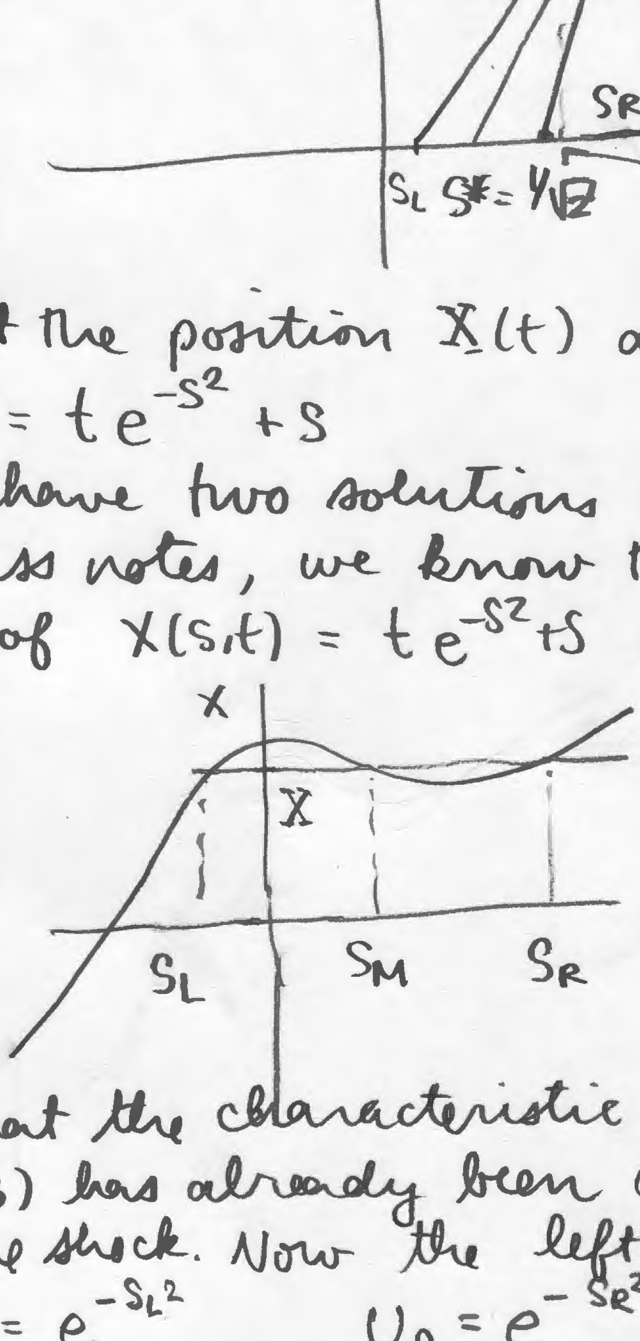
For $t > t^*$, the shock will have left and right states coming from different characteristics beginning at s_L and s_R with $s_L < \frac{1}{\sqrt{2}}$ and $s_R > \frac{1}{\sqrt{2}}$. This is shown in the diagram below



Now at the position $X(t)$ at time t,

$$X = t e^{-s^2} + s \tag{3}$$

must have two solutions s_L and s_R. From the class notes, we know that for $t > t^*$ the graph of $x(s,t) = t e^{-s^2} + s$ looks like.



for $t > t^*$, $\frac{\partial x}{\partial s} < 0$ in an interval.

Note that the characteristic starting at the root s_M of (3) has already been absorbed previously into the shock. Now the left and right states are

$$U_L = e^{-s_L^2}, \quad U_R = e^{-s_R^2}$$

and $\dot{S} = \frac{1}{2}\left(e^{-s_L^2} + e^{-s_R^2}\right)$

so $$\frac{dX}{dt} = \frac{1}{2}\left(e^{-S_L^2} + e^{-S_R^2}\right)$$

where S_L and S_R are the two extreme roots of (3).

MATH 400, Fall 2012, Wetton
Assignment #8 - due Monday, November 5

Part A Routine questions

1. Find the eigenvectors and eigenvalues of the following Sturm-Liouville problem for $\phi(x)$ for $x \in [0,1]$ with $\phi(0) = 0$ and $\phi'(1) = 0$

$$\phi''(0) = \lambda\phi.$$

For this problem, show that there are no eigenvalues with $\lambda \geq 0$. Do this problem carefully as it will be used in many of the subsequent problems.

2. Use the results of #1 above write a series solution to the problem for $u(x)$ given below

$$u'' = f(x) \quad \text{with } u(0) = 0 \text{ and } u'(1) = 0$$

where $f(x)$ is a given function. Your series will have coefficients that involve integrals of f.

3. Use the results of #1 above write a series solution to the problem for $u(x,t)$ given below

$$\begin{aligned} &u_t = u_{xx} \\ &u(0,t) = 0 \quad \text{and} \quad u_x(1,t) = 0 \quad \text{for all } t \text{ (boundary conditions)} \\ &u(x,0) = u_0(x) \quad \text{for } 0 \leq x \leq 1 \text{ (initial conditions)} \end{aligned}$$

where $u_0(x)$ is a given function. Your series will have coefficients that involve integrals of u_0.

4. What is the slowest decay rate in the problem above? How does it compare to the slowest decay rate in the problem in which the right boundary condition is replaced by $u(0,t) = 0$ (in this case, the decay rate is $-\pi^2$ as shown in the lectures). Describe briefly (one or two sentences) why you expect the change you see in the slowest decay rate between these two cases.

5. Use the results of #1 above write a series solution to the problem for $u(x,t)$ given below

$$u_{tt} = u_{xx}$$
$$u(0,t) = 0 \quad \text{and} \quad u_x(1,t) = 0 \quad \text{for all } t \text{ (boundary conditions)}$$
$$u(x,0) = u_0(x) \quad \text{and} \quad u_t(x,0) = u_1(x) \quad \text{for } 0 \le x \le 1 \text{ (initial conditions)}$$

where $u_0(x)$ and $u_1(x)$ are given functions. Your series will have coefficients that involve integrals of u_0 and u_1.

6. Use the results of #1 above write a series solution to the problem for $u(x,t)$ given below

$$u_t = u_{xx} + g(x,t)$$
$$u(0,t) = 0 \quad \text{and} \quad u_x(1,t) = 0 \quad \text{for all } t \text{ (boundary conditions)}$$
$$u(x,0) = 0 \quad \text{for } 0 \le x \le 1 \text{ (initial conditions)}$$

where $g(x,t)$ is a given function. Your series will have coefficients that involve integrals of g.

7. On your midterm, question B2 involved a left side boundary condition of

$$u - \alpha u_x = 0$$

for the wave equation (on the midterm, the value of α was 1). Describe a physical situation in which this boundary condition could apply for $\alpha > 0$, where u is the vertical displacement of a stretched wire as derived in the lectures. In this case, u_x is proportional to the vertical component of the the tension force to the right.

Part B More challenging questions

8. Show that if $g(x)$ (independent of t) is taken as the forcing term in question #6 above, then as $t \to \infty$ the solution tends to the time independent solution of question #2 with $f = -g$.

9. Use the results of #1 above write a series solution to the problem for $u(x,t)$ given below

$$u_{tt} = u_{xx} + g(x,t)$$
$$u(0,t) = 0 \quad \text{and} \quad u_x(1,t) = 0 \quad \text{for all } t \text{ (boundary conditions)}$$
$$u(x,0) = 0 \quad \text{and} \quad u_t(x,0) = 0 \quad \text{for } 0 \le x \le 1 \text{ (initial conditions)}$$

where $g(x, t)$ is a given function. Your series will have coefficients that involve integrals of g. The ideas in Assignment #1, question #9 (variation of parameters) may be useful here.

10. Show that the series solution of #5 above can be written in D'Alembert form, that is that the series can be rewritten in the form

$$u(x, t) = f(x - t) + g(x + t)$$

for scalar functions f and g. Note that the f and g will also be written as series. Some trigonometric identities are needed here.

Math 400, Assignment #8 Solutions
Fall, 2012.

1. $\phi'' = \lambda\phi$

Try $\lambda=0$, $\phi = A + Bx$ $\quad \phi(0)=0 \Rightarrow A=0$, $\phi'(1)=B, \phi'(1)=0$
so $\lambda=0$ only gives the trivial solution. $\Rightarrow B=0$.

Try $\lambda>0$, $\lambda=\mu^2$, $\phi = A\cosh\mu x + B\sinh x$
$\phi(0)=0 \Rightarrow A=0$, $\phi'(1) = B\cosh 1$, $\phi'(1)=0 \Rightarrow B=0$.
so $\lambda>0$ also only gives trivial solutions.

Move on to $\lambda<0$, $\lambda=-\mu^2$.
$\phi(x) = A\cos\mu x + B\sin\mu x$.
$\phi(0)=0 \Rightarrow A=0$.
$\phi'(1) = B\cos\mu$, $\phi'(1)=0 \Rightarrow \cos\mu=0$.

So $\mu_n = \frac{\pi}{2} + n\pi \qquad n=0,1,2\ldots$

Thus $\lambda_n = -\mu_n^2 = -(n+\frac{1}{2})^2\pi^2$ and corresponding eigenfunctions are $\phi_n(x) = \sin\mu_n x$

2. We will need to determine the normalization constant for the $\phi_n(x)$:

$$\int_0^1 \sin^2\mu_n x\, dx = \frac{1}{2}\int_0^1 (1-\cos 2\mu_n x)\, dx$$

$$= \frac{1}{2} + \frac{1}{4\mu_n}\sin 2\mu_n x \Big|_0^1$$

$$= \frac{1}{2} + \frac{1}{4\mu_n}\sin\left[(2n+1)\pi\right] = 1/2.$$

So the normalization constant is the same as the usual sine series but this will not always be the case and needs to be checked.

Now $f(x) = \sum_{n=0}^{\infty} f_n \sin \mu_n x$ (1)

with $f_n = 2\int_0^1 f(x) \sin \mu_n x \, dx$. Assume that $u(x)$ has the same series form (1) and it is easily shown that

$$u_n = -\frac{1}{\mu_n^2} f_n.$$

In summary,

$$u(x) = -\sum_{n=0}^{\infty} \frac{f_n}{(n+\frac{1}{2})^2\pi^2} \sin\left[(n+\tfrac{1}{2})\pi x\right]$$

3. Following similar steps as above,

$$u(x,t) = \sum_{n=0}^{\infty} A_n e^{-(n+1/2)^2\pi^2} \sin\left[(n+\tfrac{1}{2})\pi x\right]$$

where $A_n = 2\int_0^1 u_0(x) \sin\left[(n+\frac{1}{2})\pi x\right] dx$.

4. The slowest decay rate is for $\lambda_0 = -\pi^2/4$. The temperature decays more slowly than the $u(1,t)=0$ case. In the $u(1,t)=0$ both ends are exposed to $u=0$ but in the current case, $x=1$ is insulated so temperature can only equilibrate to zero at the $x=0$ end so deviations persist longer.

5. The general solution to the problem can be written in series form $\mu_n = (n+1/2)\pi$ 2

$$u(x,t) = \sum_{n=0}^{\infty} (A_n \cos \mu_n t + B_n \sin \mu_n t) \sin \mu_n x$$

To match the initial conditions,

$$A_n = 2\int_0^1 u_0(x) \sin \mu_n x \, dx$$

$$B_n = \frac{2}{\mu_n} \int_0^1 u_1(x) \sin \mu_n x \, dx$$

6. Proceeding exactly like question #2 on assignment #2, we can write

$$u(x,t) = \sum_{n=0}^{\infty} u_n(t) \sin \mu_n x \qquad \mu_n = (n+\tfrac{1}{2})\pi$$

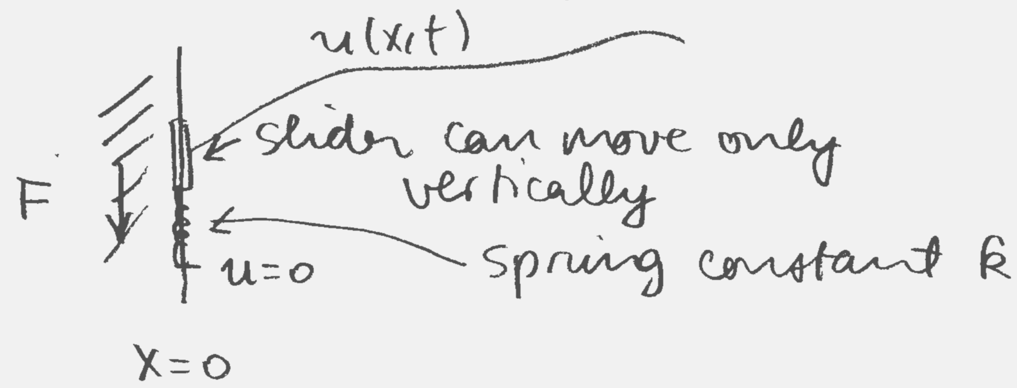
where $u_n(t) = \int_0^t e^{-\mu_n^2 (t-s)} g_n(s)\,ds$ where

$$g_n(t) = 2 \int_0^1 g(x,t) \sin \mu_n x \, dx.$$

7. Since u_x is proportional to the vertical component of the right tension force, we can assume that

$$u = \alpha u_x \qquad (2)$$

represents a force balance. Let c represent the constant of proportionality, so that $c u_x$ is the vertical component of the right tension force. Thus, u must be proportional to a downward vertical force. This could be due, for example, to a spring force if the $x=0$ end was constrained to move only vertically with a (massless, frictionless) slider system:



Downward force F is $k u(0,t)$. This leads to (2) with $\alpha = c/k$.

8. Start from #6,

$$g_n = 2\int_0^1 g(x) \sin \mu_n x \, dx$$

with g_n independent of t. Now

$$u_n(t) = \int_0^t e^{-\mu_n^2 (t-s)} g_n \, ds$$

$$= \frac{1}{\mu_n^2} \left(e^{-\mu_n^2 (t-s)} \right) \Big|_{s=0}^{s=t} g_n$$

$$= \frac{g_n}{\mu_n^2} \underbrace{\left(1 - e^{-\mu_n^2 t} \right)}$$

$\to 1$ with t uniformly in n.

so $u_n(t) \to \dfrac{g_n}{\mu_n^2}$ as $t \to \infty$

and $u(x,t) \to \displaystyle\sum_{n=0}^{\infty} \frac{g_n}{\mu_n^2} \sin \mu_n x$ as $t \to \infty$

for all n

Note this has the same form as the series in problem 2 with $f_n = -g_n$ and so solves the same problem with $f(x) = -g(x)$.

9. We start with $u(x,t) = \displaystyle\sum_{n=0}^{\infty} u_n(t) \sin \mu_n x$, putting into the equations gives

$$\ddot{u}_n = -\mu_n^2 u_n + g_n(t), \quad u_n(0) = 0, \quad \dot{u}_n(0) = 0 \qquad (3)$$

where $g_n(t) = 2\int_0^1 g(x,t) \sin \mu_n x \, dx$.

Consider (3) for fixed n, and let $y = u_n(t)$.

$$\ddot{y} = -\mu_n^2 y + g(t) \qquad (4)$$

This problem has homogeneous solutions

$$y_1 = \cos \mu_n t, \quad y_2 = \frac{1}{\mu_n} \sin(\mu_n t)$$

These form the matrix exponential for the homogeneous problem written as a system

$$e^{At} = \begin{bmatrix} y_1 & y_2 \\ \dot{y}_1 & \dot{y}_2 \end{bmatrix} = \begin{bmatrix} \cos\mu_n t & \boxed{\frac{1}{\mu_n}\sin\mu_n t} \\ -\mu_n \sin\mu_n t & \cos\mu_n t \end{bmatrix}.$$

Following assignment #1, question 9, we write (4) as

$$\frac{d}{dt}\begin{bmatrix} y \\ \dot{y} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\mu_n^2 & 0 \end{bmatrix}}_{A} \begin{bmatrix} y \\ \dot{y} \end{bmatrix} + \begin{bmatrix} 0 \\ g(t) \end{bmatrix}.$$

So $$y(t) = \frac{1}{\mu_n}\int_0^t \sin(\mu_n(t-s))\, g(s)\, ds \qquad (5).$$

where the entry in the rectangular box in the matrix above is the only one that comes into play. (5) can also be written from a variation of parameters formula for second order equations you can find in text books.

Recalling now that $y = u_n(t)$ with $g = g_n(t)$ we have

$$u_n(t) = \frac{1}{\mu_n}\int_0^t \sin(\mu_n(t-s))\, g_n(s)\, ds.$$

Summary: Questions 1, 2, 3, 5, 6, and 9 are a roadmap for techniques you can use for other problems that lead to other S-L forms.

10. Start with

$$u(x,t) = \sum_{n=0}^{\infty} \left(A_n \cos \mu_n t \sin \mu_n x + B_n \sin \mu_n t \sin \mu_n x\right)$$

Use the trig identities

$$\left.\begin{aligned} &\sin(a+b) + \sin(a-b) = 2 \sin a \cos b. \\ \text{and } &\cos(a-b) - \cos(a+b) = 2 \sin a \sin b. \end{aligned}\right\} \qquad a = \mu_n x \quad b = \mu_n t$$

To write $\sin(\mu_n x)\cos(\mu_n t) =$

$$\frac{1}{2}\left(\sin(\mu_n(x+t)) + \sin(\mu_n(x-t))\right)$$

and $\sin \mu_n t \sin \mu_n x =$

$$\frac{1}{2}\left(\cos(\mu_n(x-t)) - \cos(\mu_n(x+t))\right)$$

so now the series at the start can be written

$$u(x,t) = f(x-t) + g(x+t)$$

where $f(s) = \frac{1}{2}\sum_{n=0}^{\infty}\left(A_n \sin(\mu_n s) + B_n \cos(\mu_n s)\right)$

$$g(s) = \frac{1}{2}\sum_{n=0}^{\infty}\left(A_n \sin(\mu_n s) - B_n \cos(\mu_n s)\right)$$

MATH 400, Fall 2012, Wetton
Assignment #9 - due Wed, November 14

Part A Routine questions

1. Consider heat diffusion with homogeneous Dirichlet boundary conditions in a composite bar of length 1 made of two materials joined together at the midpoint. Suppose that on the left side $c\rho = 1$ and $K = 1$ as in class and on the right, $c\rho = 1$ and $K = 3/2$. Find the μ value corresponding to the slowest decaying term in a series solution (that has time behaviour $e^{\lambda t}$ with $\lambda = -\mu^2$). You will need to use a root finder (or Newton's method) to find this value accurately.

2. Show how a general second order eigenvalue problem for $y(x)$ of the form

$$P(x)y'' + Q(x)y' + R(x)y = \lambda y$$

with $P(x) > 0$ can be written in standard Sturm-Liouville form

$$(a(x)y')' + c(x)y = \lambda b(x)y$$

with $a(x) > 0$ and $b(x) > 0$. *Hint:* Use an integrating factor.

3. Consider the following generalized Sturm-Liouville problem for $\phi(x)$ with $x \in [0, 1]$ that allows ϕ to be discontinuous at $x = 1/2$:

$$\begin{aligned}
\phi'' &= \lambda\phi \\
\phi'(0) &= 0 \\
\phi'(1) &= 0 \\
\phi'(\frac{1}{2}_-) &= \phi(\frac{1}{2}_+) - \phi(\frac{1}{2}_-) \\
\phi'(\frac{1}{2}_+) &= \phi(\frac{1}{2}_+) - \phi(\frac{1}{2}_-).
\end{aligned}$$

It can be shown that there are no positive eigenvalues. There is an eigenvalue $\lambda = 0$ with $\phi(x) \equiv 1$. Find the next eigenvalue (negative with smallest magnitude) and corresponding eigenvector.

4. Show that in problem #5 above, eigenfunctions satisfy

$$\int_0^1 \phi_n(x)\phi_m(x)dx$$

when they correspond to different eigenvalues $\lambda_n \neq \lambda_m$. There is some work to do here to handle the discontinuity at $x = 1/2$.

5. Consider the following diffusion problem for $u(x,t)$ with $x \in [0,1]$. In this problem, $u(x,t)$ can be discontinuous in x at $x = 1/2$.

$$\begin{aligned}
u_t &= u_{xx} \\
u(0,t) &= 0 \\
u(1,t) &= 0 \\
u_x(\frac{1}{2}_-,t) &= u(\frac{1}{2}_+,t) - u(\frac{1}{2}_-,t) \\
u_x(\frac{1}{2}_+,t) &= u(\frac{1}{2}_+,t) - u(\frac{1}{2}_-,t) \\
u(x,0) &= 1 \text{ for } x < 1/2 \text{ and } 0 \text{ for } x > 1/2.
\end{aligned}$$

Find the first two terms in a series solution to the problem. The results of questions #5 and #6 above will be useful here. As time $t \to \infty$, your solution should tend to the constant $1/2$ for all x. Using your series solution, sketch a graph of the solution at some intermediate time.

6. Consider a bar of length 1 with constant $K = 1$ but $(c\rho)(x) = 1+\epsilon e^x$ with ϵ small. Use homogeneous Dirichlet boundary conditions and initial data $u_0(x) = 1$ for all x. Approximate the dominant (slowest decaying) term in a series expansion for this problem using perturbation analysis.

Part B More challenging questions

7. Suppose a long, thin bar of length 1 is made of uniform material but due to the nature of the manufacturing process the cross sectional area changes slowly from one end to the other. Assume $A(x) = 1 + \epsilon x$ with ϵ small. Use asymptotic analysis to determine if the slowest decaying mode in this bar (with homogeneous Dirichlet boundary conditions) with $\epsilon \neq 0$ decays more or less slowly than the uniform bar with $\epsilon = 0$. *Note:* In this case, the perturbation to the eigenvalue begins at order ϵ^2.

8. Show that for the composite bar problem (length 1, with one material to the left of $x = 1/2$ and a different material to the right $x > 1/2$) with homogeneous Dirichlet boundary conditions, the μ value corresponding to the slowest decaying mode of the solution satisfies

$$\pi\sqrt{\alpha_l} < \mu < \pi\sqrt{\alpha_r}$$

where $\alpha_l = K_l/(c_l\rho_l)$ and $\alpha_r = K_r/(c_r\rho_r)$ are the thermal diffusivities on the left and right sides and it is assumed that $\alpha_l < \alpha_r$. That is, the μ value (and so the decay rate) of the slowest decaying mode is intermediate between the ones for a bar made solely from the left and right materials.

Bonus: Show that the same fact is true if the division between the two materials is at any location $x_* \in (0, 1)$.

Math 400
Assignment #9 Solutions.

#1 Proceeding as in the course notes, we reach

$$\varphi(x) = \begin{cases} A \sin \mu x & x < 1/2 \\ B \sin\left(\sqrt{\tfrac{2}{3}}\, \mu (x-1)\right) & x > 1/2. \end{cases}$$

The junction conditions are applied at $x = 1/2$

$[\varphi] = 0 \qquad A \sin \frac{\mu}{2} + B \sin\left(\sqrt{\frac{2}{3}}\, \frac{\mu}{2}\right) = 0$

$[K\varphi'] = 0 \qquad A\mu \cos \frac{\mu}{2} - \frac{3}{2} B \mu \sqrt{\frac{2}{3}} \cos\left(\sqrt{\frac{2}{3}}\, \frac{\mu}{2}\right) = 0$

For a nontrivial solution to the above system, the determinant must be zero, so

$$\frac{3}{2}\sqrt{\frac{2}{3}} \sin \frac{\mu}{2} \cos\left(\sqrt{\frac{2}{3}}\, \frac{\mu}{2}\right) + \sin\left(\sqrt{\frac{2}{3}}\, \frac{\mu}{2}\right) \cos\left(\frac{\mu}{2}\right) = 0$$

The smallest root, $\mu \approx 3.494$, is found with a root finder

#2. $P(x)\, y'' + Q(x)\, y' + R(x)\, y = \lambda y$

Divide by $P(x) > 0$,

$$y'' + \frac{Q(x)}{P(x)} y' + \frac{R(x)}{P(x)} y = \lambda \cdot \frac{1}{P(x)} y.$$

Now multiply by the integrating factor

$$I(x) = e^{\int^x \frac{Q(x)}{P(x)} dx}$$

to get $\quad \left(I(x)\, y'\right)' + I(x) \dfrac{R(x)}{P(x)} y = \lambda \dfrac{I(x)}{P(x)} y$

This is in the required form, with

$$a(x) = I(x), \quad b(x) = \frac{I(x)}{p(x)}, \quad c(x) = \frac{I(x)R(x)}{P(x)}$$

Note that since $I(x)$ is an exponential, $I(x) > 0$, so $a(x) > 0$ and $b(x) > 0$ as required.

#3. Take $\lambda = -\mu^2$. To satisfy the boundary conditions at $x=0$ and $x=1$, ϕ has the form

$$\phi(x) = \begin{cases} A \cos(\mu x) & x < 1/2 \\ B \cos(\mu(x-1)) & x > 1/2. \end{cases}$$

Set $\phi'(\frac{1}{2}-) = \phi'(\frac{1}{2}+)$ to obtain

$$-\mu A \sin \frac{\mu}{2} = \mu B \sin \frac{\mu}{2}$$

so either

(i) $\mu = 0$ (this leads to the know eigenvalue of $\lambda = 0$, $\phi \equiv 1$).

(ii) $\sin \frac{\mu}{2} = 0 \Rightarrow \mu = 2n\pi$, $n = 1, 2, \dots$

Now $\phi'(\frac{1}{2}\pm) = 0$ in this case, so $\phi'(\frac{1}{2}\pm) = \phi(\frac{1}{2}+) - \phi(\frac{1}{2}-)$ is satisfied

only if $A\cos(n\pi) = B\cos(n\pi)$
$\Rightarrow A = B$. In this case, ϕ simplifies to $\cos(2\pi x)$ for all x.

(iii) $B = -A$, so $\phi'(\frac{1}{2}\pm) = \phi(\frac{1}{2}+) - \phi(\frac{1}{2}-)$ is satisfied if

$$-\mu A \sin\frac{\mu}{2} = -2A\cos\frac{\mu}{2}.$$

$$\Rightarrow \mu\sin\frac{\mu}{2} - 2\cos\mu/2 = 0$$

Numerical solution gives $\mu \simeq 1.721$ as the smallest root, and since this value is smaller than 2π, this is the desired root. The corresponding eigenfunction is

$$\phi(x) = \begin{cases} \cos\mu x & x < 1/2 \\ -\cos\mu(x-1) & x > 1/2. \end{cases}$$

#4. This follows a variant of the argument used to show the orthogonality for the S-L problems with smooth solutions and arbitrary conditions at $x=0$ and $x=1$.

Consider $\lambda_n \int_0^1 \varphi_n(x)\varphi_m(x)\,dx$

$$= -\int_0^1 \varphi_n''\varphi_m\,dx \quad \text{since } \lambda_n\varphi_n = \varphi_n''$$

integrate by parts twice separately in $[0, 1/2]$ and $[1/2, 1]$ to obtain

$$\dots = \int_0^1 \varphi_m''\varphi_n\,dx + \varphi_n'\varphi_m\Big|_{1/2^-}^{1/2^+} - \varphi_n\varphi_m'\Big|_{1/2^-}^{1/2^+}$$

$$= \underbrace{\lambda_m \int_0^1 \varphi_m \varphi_n \, dx}_{\text{since } \lambda_m \varphi_m = \varphi_m''} + \varphi_n'(\tfrac{1}{2}) \left[\varphi_m(\tfrac{1}{2}+) - \varphi_m(\tfrac{1}{2}-) \right]$$

$$- \varphi_m'(\tfrac{1}{2}) \left[\varphi_n(\tfrac{1}{2}+) - \varphi_n(\tfrac{1}{2}-) \right]$$

using the fact that
$\varphi_n'(\tfrac{1}{2}+) = \varphi_n'(\tfrac{1}{2}-)$ and
similarly $\varphi_m'(\tfrac{1}{2}+) = \varphi_m'(\tfrac{1}{2}-)$

$$= \lambda_m \int_0^1 \varphi_m \varphi_n dx + \varphi_n'(\tfrac{1}{2}) \varphi_m'(\tfrac{1}{2}) - \varphi_m'(\tfrac{1}{2}) \varphi_n'(\tfrac{1}{2})$$

$\varphi_m'(\tfrac{1}{2}) = \varphi_m(\tfrac{1}{2}+) - \varphi_m(\tfrac{1}{2}-)$
by the interface conditions.

similarly

$$= \lambda_m \int_0^1 \varphi_m \varphi_n dx$$

So we have shown that

$$(\lambda_n - \lambda_m) \int_0^1 \varphi_m \varphi_n \, dx = 0$$

which gives the desired result.

#5 From #3 above,

$$u(x,t) \approx A_0 + A_1 \varphi(x) e^{-\mu^2 t}$$

with $\mu \approx 1.721$ and

$$\varphi(x) = \begin{cases} \cos \mu x & x < 1/2 \\ -\cos(\mu(x-1)) & x > 1/2 \end{cases}$$

Using # 4 above,

$$A_0 = \int_0^1 u_0(x)\, dx = 1/2.$$

so as $t \to \infty$, $u \to 1/2$ as indicated in the question. (for all x).

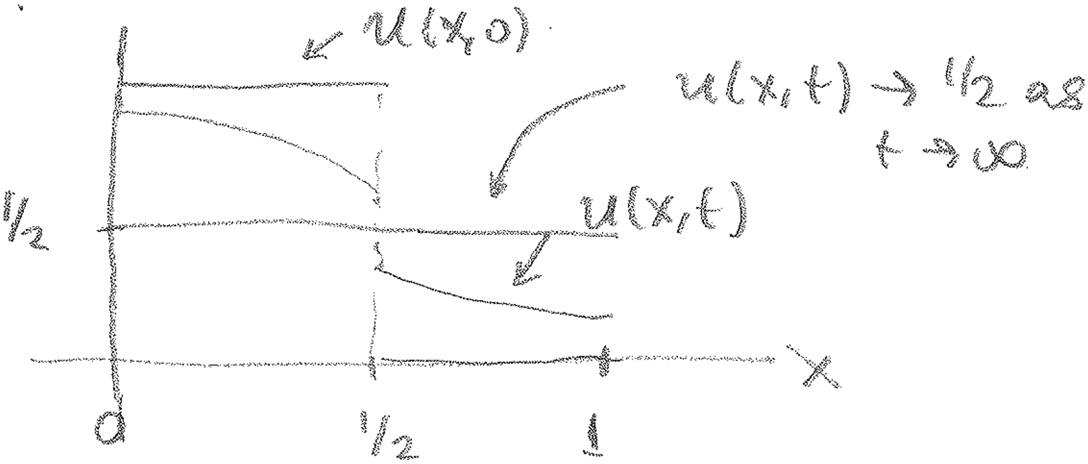
$$A_1 = \frac{\int_0^1 u_0(x)\, \varphi(x)\, dx}{\int_0^1 [\varphi(x)]^2 dx}$$

$$= \frac{\int_0^{1/2} \cos \mu x \, dx}{\int_0^{1/2} [\cos \mu x]^2 dx + \int_{1/2}^1 [\cos \mu (x-1)]^2 dx}$$

$$= \ldots = \frac{\sin \mu/2}{\cos(\mu/2)\sin(\mu/2) + \mu/2} \approx 0.560$$

not necessary

a sketch of the solution $u(x,t)$ at various times:



#6. $u(x,t)$ solves

$$u_t = \frac{1}{1+\varepsilon e^x}\, u_{xx} \qquad u(0,t) = u(1,t) = 0.$$

This leads to the S-L problem

$$\lambda \varphi = \frac{1}{1+\varepsilon e^x}\, \varphi''$$

$$\varphi'' = \lambda(1+\varepsilon e^x)\varphi. \qquad \varphi(0)=0,\ \varphi(1)=0 \quad (1)$$

Consider $\varphi \approx \sin\pi x + \varepsilon v(x)$, $\lambda \approx -\pi^2 + \varepsilon\beta$.
Plug into (2), neglect terms of order ε^2 and higher.

← form is correct for $\varepsilon = 0$

$$\cancel{-\pi^2 \sin\pi x} + \varepsilon v'' \approx \cancel{-\pi^2 \sin\pi x} - \varepsilon\pi^2 v - \varepsilon\pi^2 e^x \sin\pi x + \beta \sin\pi x.$$

Divide by ε,

$$v'' + \pi^2 v = -\pi^2 e^x \sin\pi x + \beta \sin\pi x \quad (2)$$
$$v(0)=0, \quad v(1)=0.$$

Look for a particular solution to this problem using the method of undetermined coefficients

$$v_p(x) = a x \sin\pi x + b x \cos\pi x \qquad \left(\begin{matrix}\sin\pi x\\ \text{resonant}\end{matrix}\right)$$
$$+ c e^x \sin\pi x + d e^x \cos\pi x.$$

Putting into (2) and equating coefficients gives $a=0$, $b = -\beta/2\pi$, $c = \dfrac{-\pi^2}{1+4\pi^2}$, $d = \dfrac{+2\pi^3}{1+4\pi^2}$

Now the full solution of (2) is

$$V(x) = \underbrace{A \sin \pi x + B \cos \pi x}_{} + V_p(x)$$

homogeneous solution

we can take $A=0$ without loss of generality since this just scales the original eigenfunction as discussed in the notes. Apply the boundary conditions

$$V(0)=0 \Rightarrow B + d = 0 \Rightarrow B = -d = \frac{-2\pi^3}{1+4\pi^2}$$

$$V(1)=0 \Rightarrow -b - d\,e - B = 0$$

exp(1)

$$\Rightarrow \quad \frac{\beta}{2\pi} = \frac{2(e-1)\pi^3}{1+4\pi^2}, \quad \beta = \frac{4\pi^4(e-1)}{1+4\pi^2} \qquad (3)$$

So now $\lambda \approx -\pi^2 + \varepsilon\beta$ with β given above, and $\varphi(x) \approx \sin \pi x +$

$$\varepsilon \left\{ \frac{-\pi^2}{1+4\pi^2} e^x \sin \pi x + \cos \pi x \left[-\frac{\beta}{2\pi} x + \frac{2\pi^3}{1+4\pi^2}(e^x - 1) \right] \right\}$$

The series solution for the given problem will have as first term,

$$A\, \varphi(x)\, e^{-\lambda t}$$

with φ and λ as above and

$$A \approx \frac{4}{\pi} + \varepsilon \underline{\qquad\qquad}$$

a painful calculation, I will provide this in an appendix later.

7. The problem is

$$A(x)\, u_t = (A(x)\, u_x)_x, \qquad A(x) = 1 + \varepsilon x$$

leading to the eigenvalue problem

$$((1+\varepsilon x)\varphi')' = (1+\varepsilon x)\lambda\varphi \qquad (4)$$

Let
$$\left.\begin{aligned} \varphi &= \sin\pi x + \varepsilon v + \varepsilon^2 w \\ \lambda &= -\pi^2 + \varepsilon\beta + \varepsilon^2\gamma. \end{aligned}\right\} \qquad (5)$$

The question suggests that $\beta = 0$, which we'll see below. Rewrite (4) as

$$\varphi'' + \frac{\varepsilon}{1+\varepsilon x}\varphi' - \lambda\varphi = 0$$

or $$\varphi'' + \varepsilon(1 - \varepsilon x + \varepsilon^2 x^2 + \dots)\varphi' - \lambda\varphi = 0.$$

Put in the terms (5) and equate terms of powers of ε up to ε^2.

1: $\quad -\pi^2 \sin\pi x + \pi^2 \sin\pi x = 0 \quad \checkmark$

ε: $\quad v'' + \underbrace{\pi\cos\pi x}_{(\sin\pi x)'} + \pi^2 v - \beta\sin\pi x = 0 \qquad (6)$

$$\varepsilon^2: \quad W'' + V' - \pi x \cos \pi x + \pi^2 W - \beta V - \gamma \sin \pi x = 0 \quad (7).$$

The following particular solutions will be needed. They can be found using the method of undetermined coefficients.

$$y'' + \pi^2 y = f(x)$$

$f(x)$	particular solution
$\sin \pi x$	$-\frac{1}{2\pi} x \cos \pi x$
$\cos \pi x$	$\frac{1}{2\pi} x \sin \pi x$
$x \cos \pi x$	$\frac{1}{4\pi} x^2 \sin \pi x - \frac{1}{4\pi^2} x \cos \pi x$

Rewrite (6) as

$$V'' + \pi^2 V = \beta \sin \pi x - \pi \cos \pi x$$

giving $V(x) = A \sin \pi x + B \cos \pi x$ ← homogeneous solution

$$- \frac{\beta}{2\pi} x \cos \pi x - \frac{1}{2} x \sin \pi x$$ ← particular solution using the table above.

As in the notes we can set $A = 0$ since it just adjusts the scale of the underlying eigenfunction. Then applying the boundary conditions $V(0) = 0$ and $V(1) = 0$ gives

$B=0$ and $\beta = 0, 10$

$$v(x) = -\frac{1}{2} x \sin \pi x.$$

Moving on to (7) we have

$$W'' + \pi^2 W = \underbrace{\frac{1}{2}\sin \pi x + \frac{\pi}{2} x \cos \pi x}_{-v'} + \pi x \cos \pi x + \gamma \sin \pi x$$

$$= \left(\frac{1}{2} + \gamma\right) \sin \pi x - \frac{3\pi}{2} x \cos \pi x.$$

Solution using the table on the previous page gives as before

$$W(x) = A \cancel{\sin \pi x} + B \cos \pi x - \frac{1}{2\pi}\left(\frac{1}{2} + \gamma\right) x \cos \pi x - \frac{3\pi}{2}\left(\frac{1}{4\pi} x^2 \sin \pi x - \frac{1}{4\pi^2} x \cos \pi x\right)$$

$W(0) = 0 \Rightarrow B = 0.$

$W(1) = 0 \Rightarrow +\frac{1}{2\pi}\left(\frac{1}{2} + \gamma\right) - \frac{3\pi}{2} \cdot \frac{1}{4\pi^2} = 0.$

So $\frac{1}{2} + \gamma = \frac{3}{16\pi^2}$

$$\gamma = \frac{3}{16\pi^2} - \frac{1}{2} = \frac{3 - 8\pi^2}{16\pi^2} \approx -0.4810 < 0$$

So $\lambda \approx -\pi^2 - \varepsilon^2 \gamma$, so the non uniformity will cause the bar to cool more quickly.

#8 Proceeding as in the class notes and question #1 we have

$$\varphi(x) = \begin{cases} A \sin\left(\frac{\mu}{2\sqrt{\alpha_\ell}} x\right) & x < 1/2 \\ B \sin\left(\frac{\mu}{2\sqrt{\alpha_r}}(x-1)\right) & x > 1/2 \end{cases}$$

and the junction conditions give

$$A \sin\left(\frac{\mu}{2\sqrt{\alpha_\ell}}\right) + B \sin\left(\frac{\mu}{2\sqrt{\alpha_r}}\right) = 0$$

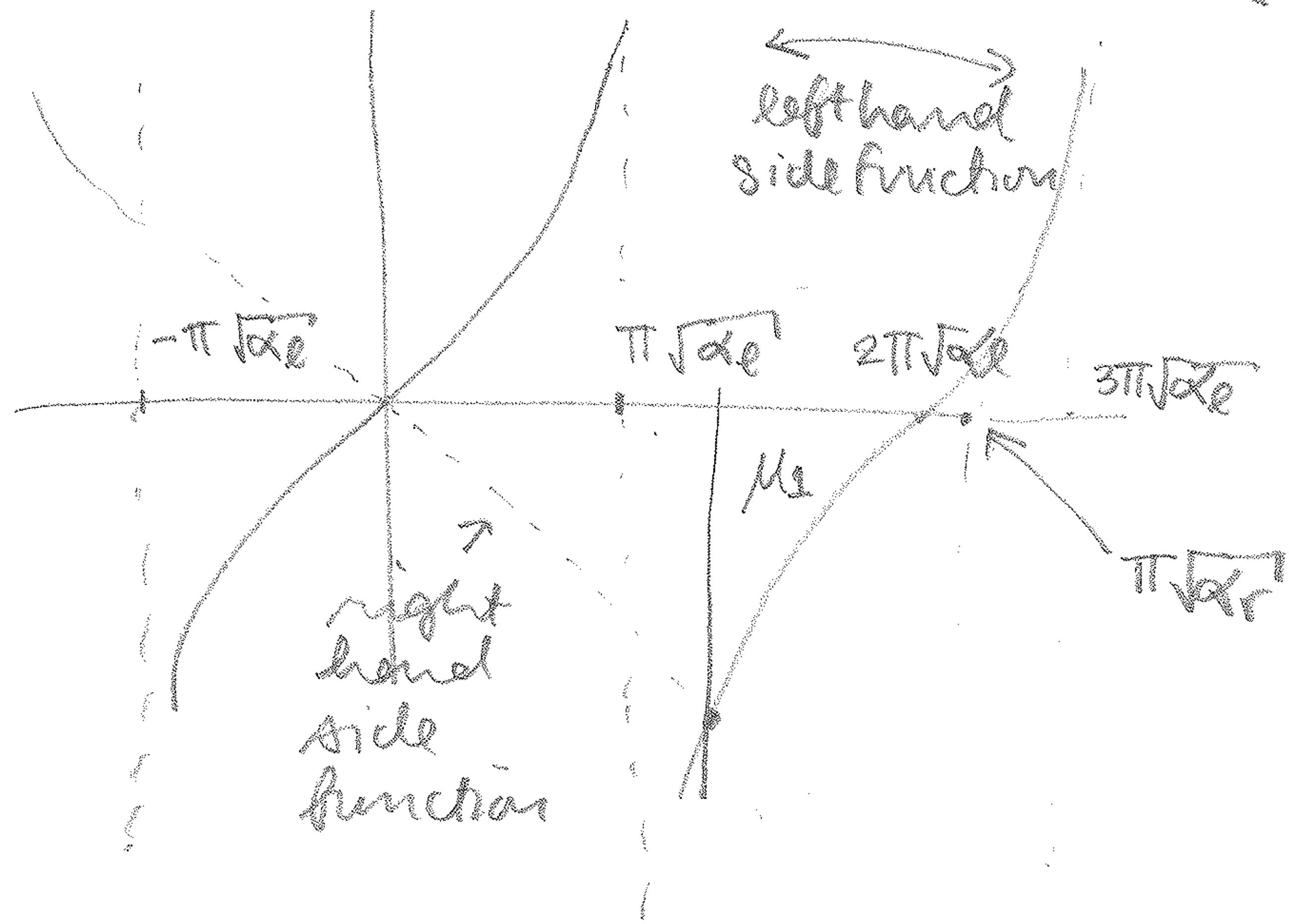
$$A K_r \frac{\mu}{\sqrt{\alpha_\ell}} \cos \frac{\mu}{2\sqrt{\alpha_\ell}} - B K_\ell \frac{\mu}{\sqrt{\alpha_r}} \cos \frac{\mu}{2\sqrt{\alpha_r}} = 0.$$

so
$$\sin\left(\frac{\mu}{2\sqrt{\alpha_\ell}}\right) K_\ell \frac{\mu}{\sqrt{\alpha_r}} \cos \frac{\mu}{2\sqrt{\alpha_r}} + \sin\left(\frac{\mu}{2\sqrt{\alpha_r}}\right) K_r \frac{\mu}{\sqrt{\alpha_\ell}} \cos \frac{\mu}{2\sqrt{\alpha_\ell}} = 0.$$

Assume that neither $\cos \frac{\mu}{2\sqrt{\alpha_\ell}}$ nor $\cos \frac{\mu}{2\sqrt{\alpha_r}}$ are zero and divide through by these terms to obtain

$$K_\ell \frac{\mu}{\sqrt{\alpha_r}} \tan \frac{\mu}{2\sqrt{\alpha_r}} = -K_r \frac{\mu}{\sqrt{\alpha_\ell}} \tan \frac{\mu}{2\sqrt{\alpha_\ell}}$$

Graph these functions and consider intersections.



Note that the graph shows that μ_1 must lie in

$$\pi\sqrt{\alpha_\ell} < \mu_1 < \pi\sqrt{\alpha_r}$$

as required.

MATH 400, Fall 2012, Wetton
Assignment #10 - due Monday, November 19

Part A Routine questions

1. (Review) The following definite integral is not easily done with analytic methods (might not be possible in terms of familiar functions).

$$\int_0^1 e^{\cos x} dx$$

Using numerical integration (Trapezoidal rule or Simpson's rule) evaluate the integral accurate to 3 significant figures (error less than 0.05). Describe how you are convinced yourself the answer you have has the desired accuracy (this discussion does not have to be mathematically rigorous).

2. (Cooling of a sphere) Find a series solution $u(r,t)$ for $t > 0$, $0 < r < 1$ to the problem

$$u_t = u_{rr} + \frac{2}{r} u_r$$

with initial conditions $u(r,0) = 1$ and boundary conditions $u_r = -u$ at $r = 1$. You may leave the series coefficients in terms of definite integrals.

3. Find an approximate solution at larger time to the problem above (the slowest decaying term in the series solution). Your answer should be in the form

$$u(r,t) \approx Av(r)e^{-ct}$$

for specified function $v(r)$ and constants A and c. In this case, evaluate the integral for the A value.

4. Consider the following 2D wave equation problem for $u(r,\theta,t)$ for $t > 0$ and $0 \le r \le 1$:

$$u_{tt} = \Delta u = \frac{1}{r}(ru_r)_r + \frac{1}{r^2} u_{\theta\theta}$$

with boundary conditions $u_r = 0$ at $r = 1$. What are the three lowest frequencies of vibration for this model? You will need the roots of the derivatives of Bessel functions provided in the appendix to the notes, part V.

5. Consider the following 3D heat conduction problem for $u(r, \theta, z, t)$ for $t > 0$ in a cylinder $0 \le r \le R$ and $0 \le z \le L$.

$$u_t = \Delta u = \frac{1}{r}(ru_r)_r + \frac{1}{r^2}u_{\theta\theta} + u_{zz}$$

with boundary conditions $u_r = 0$ at $r = R$ and $u = 0$ at $z = 0$ and L. Scale r and z so that they both lie in the interval 0 to 1. Write down the equation for u in the scaled coordinates.

Part B More challenging questions

6. Write a series solution to problem 5. Consider the solution when R is much smaller than L (a rod). Show that (except during a short transient) this justifies the approximation of this problem by the one dimensional problem in z. You will have to investigate some properties of Bessel functions to give a complete answer to this question.

Math 400, Assignment #10 solutions
Fall, 2012.

1. $I = \int_0^1 e^{\cos x}\,dx$

I used the trapezoidal rule using N equal subintervals of length $h = 1/N$, getting approximations T_N to I:

N	T_N
50	2.3415
100	2.3416
200	2.3416

I used the MATLAB "trapz" command

While it is not a proof, it is convincing that this answer is correct to 3 digits since the 5th digit is not changing as N increases. You could bound the second derivative of the integrand and use the error formula for Trapezoidal rule for a rigorous estimate.

2. Proceed following the class notes where it is shown that for $\varphi(r)$,

$$\varphi'' + \frac{2}{r}\varphi' = \lambda\varphi$$

has solutions

$$\varphi(r) = A\,\frac{\sin \mu r}{r} + B\,\frac{\cos \mu r}{r}$$

where $\lambda = -\mu^2$. To match the PDE problem, $\varphi(0_+)$ must be bounded, thus $B = 0$. At $r = 1$, we have $\varphi' = -\varphi$, and since $\varphi' = A\,\frac{\mu\cos\mu r}{r} - A\,\frac{\sin\mu r}{r^2}$

$\varphi' = -\varphi$ at $r=1$ leads to

$$A\left(\mu\cos\mu - \sin\mu\right) = A\sin\mu$$

or $\quad \mu\cos\mu = 0.$

Note: I did not realize this would lead to so simple a result.

Thus $\mu_n = (n+\frac{1}{2})\pi$, $\quad n = 0, 1, 2\ldots$

$$\lambda_n = -(n+\tfrac{1}{2})^2\pi^2$$

$$\varphi_n(r) = \frac{\sin\left[(n+\frac{1}{2})\pi r\right]}{r}$$

and the series solution to the problem is

$$u(r,t) = \sum_{n=0}^{\infty} u_n \frac{\sin\left[(n+\frac{1}{2})\pi r\right]}{r} e^{-(n+\frac{1}{2})^2\pi^2 t}$$

where $u_n = 2\int_0^1 r\sin\left[(n+\frac{1}{2})\pi r\right] dr.$

#3 Evaluate $A = 2\int_0^1 r\sin\left(\frac{\pi r}{2}\right) dr \xrightarrow{\text{by parts.}} = \frac{8}{\pi^2}$

So $$u(r,t) \approx \underbrace{\frac{8}{\pi^2}}_{A} \; \underbrace{\frac{\sin\left(\frac{\pi r}{2}\right)}{r}}_{V(r)} \; e^{-\underbrace{(n+\frac{1}{2})^2\pi^2}_{c} t}$$

4. Proceeding as in the class notes, we consider the eigenvalue problem

$$\frac{1}{r}(r\varphi_r)_r + \frac{1}{r^2}\varphi_{\theta\theta} = \lambda\varphi \qquad \varphi(0_+,\theta) \text{ bounded}$$

with $\lambda = -\mu^2$ and obtain

$$\varphi(r,\theta) = J_n(\mu r)\, e^{in\theta}$$

To satisfy $\varphi_r = 1$ at $r=1$ we need

$$\mu\, J_n'(\mu) = 0, \qquad \text{only } J_0(0) \neq 0$$

so either $\mu = 0$ or μ is a positive root $d_{n,s}$ for $s = 1, 2, \cdots$ of the derivative of the Bessel J_n function. Thus, $\mu_{n,s}$

$$\mu_{n,s} = \begin{cases} 0 & \text{if } s=0 \text{ and } n=0 \\ d_{n,s} & \text{if } s>0. \end{cases}$$

Going back to the wave equation we see that the μ values $(\lambda = -\mu^2)$ are the angular frequencies of vibration. We can exclude $\mu = 0$ (zero frequency so no vibration) and look in the table for the three smallest roots, giving

$\mu_1 = d_{1,1} \approx 1.8412$
$\mu_2 = d_{2,1} \approx 3.0542$
$\mu_3 = d_{0,1} \approx 3.8317$

} smallest angular frequencies.

5. Let $r = Rs$, and $z = Lw$ (so s and w are scaled r and z respectively), then

$$u_t = \frac{1}{R^2}\left(\frac{1}{s}(s\,u_s)_s + \frac{1}{s^2}u_{\theta\theta}\right) + \frac{1}{L^2}u_{ww}.$$

6. Let $t = L^2 T$ (note that there is a thermal diffusivity in the equation of unit size that would make the units work out) to further simplify the form to

$$u_t = \beta^2\left(\frac{1}{s}(s\,u_s)_s + \frac{1}{s^2}u_{\theta\theta}\right) + u_{ww}.$$

and $\beta = L/R$.

with $0 \le s \le 1$ and $0 \le z \le 1$. Consider eigenvalues of the RHS operator, $\varphi(s,\theta,w)$, that is

$$\varphi_{ww} + \beta^2\left(\frac{1}{s}(s\,\varphi_s)_s + \frac{1}{s^2}\varphi_{\theta\theta}\right) = \lambda\varphi. \quad (1)$$

Based on question 4, it is clear to consider φ of the form

$$\varphi(s,\theta,w) = \phi(w)\, e^{in\theta}\, J_n(\mu_{n,j}\, s) \quad (2)$$

$\uparrow$
to be determined

It is convenient to call $\mu_{n,j}$ the j'th root of the n'th Bessel J function derivative and renumber the $n=0$ roots so that $\mu_{0,1} = 0$.
Putting (2) into (1) gives

$$\phi'' = (\lambda + \beta^2 \mu_{n,j}^2)\phi \quad , \phi(0)=0, \ \phi(1)=0.$$

we know the eigenanalysis of this problem well, $\phi_m(W) = \sin m\pi W$ with

$$\lambda + \beta^2 \mu_{n,j}^2 = -m^2\pi^2, \quad \lambda = -m^2\pi^2 - \beta^2 \mu_{n,j}^2.$$

Thus, the series solution to the problem is

$$u(s,\theta,W,t) = \sum_{m=1}^{\infty} \sum_{n=-\infty}^{\infty} \sum_{j=1}^{\infty} u_{mnj} \sin m\pi W \, e^{in\theta} J_n(\mu_{n,j} s)$$
$$e^{-(m^2\pi^2 + \beta^2 \mu_{n,j})t}$$

Notice that under the assumption of L much bigger that R, $\beta = L/R$ is large. Thus all the terms in the series above decay very quickly in time <u>except</u> the ones for $\mu_{n,j}=0$ which only occurs when $n=0$ and $j=1$.
Here, $J_0(0) = 1$ and $e^{in\theta} = 1$ when $n=0$ so

$$u(s,\theta,W,t) \approx \sum_{m=1}^{\infty} u_{m01} \sin m\pi W \, e^{-m^2\pi^2 t},$$

our old 1D solution.

MATH 400, Fall 2012, Wetton
Assignment #11 - due Monday, November 26

Note: This is the last assignment that will be collected and graded.

Part A Routine questions

1. Find the Fourier transform of

$$f(x) = \begin{cases} 1+x & \text{if } -1 \le x \le 0 \\ 1-x & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

2. Find the Fourier transform of

$$f(x) = 1/(x^4 + a^4)$$

for a real, $a > 0$. *Hint:* remember contour integration from Math 300.

3. Consider the following Cauchy problem for $u(x,t)$ defined for $t \ge 0$ and all x:

$$u_{tt} + a u_t = u_{xx}$$

where a is a given non-zero constant and initial data $u_0(x)$ are given at $t = 0$. Find an expression for the Fourier transform in x of u at each t. From your result, determine if this problem is well posed for $a > 0$ and separately for $a < 0$. Justify your conclusion. *Note:* If the Fourier coefficients of the solution grow exponentially in time with unbounded growth rates, the problem is ill-posed.

4. Consider the following *vector* PDE for $u(x,t)$ and $v(x,t)$:

$$\begin{aligned} u_t &= v + u_{xx} \\ v_t &= u + v_{xx}. \end{aligned}$$

Is the Cauchy problem for this PDE well posed? If the problem is well-posed, determine the maximum growth rate and the minimum decay rate (if any). *Hint:* take the Fourier Transform in x of both u and v.

Part B More challenging questions

5. Consider the Cauchy problem for the heat equation in three spatial dimensions. That is, consider $u(x, y, z, t)$ for $t \geq 0$ and all x, y and z with initial data $u(x, y, z, 0) = u_0(x, y, z)$ satisfying

$$u_t = \Delta u := u_{xx} + u_{yy} + u_{zz}.$$

Derive a Green's function representation for the solution to this problem. *Hint:* Start with a Fourier transform representation in x, y and z of u at a fixed time.

Math 400, Assignment #11 solutions

Fall, 2012

1. $$\hat{f}(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{-i\alpha x}\, dx$$

$$= \frac{1}{2\pi}\underbrace{\int_0^1 (1-x)\, e^{-i\alpha x}\, dx}_{A} + \frac{1}{2\pi}\underbrace{\int_{-1}^0 (1+x)\, e^{-i\alpha x}\, dx}_{B}$$

$$A = (1-x)\frac{e^{-i\alpha x}}{-i\alpha}\Big|_0^1 + \int_0^1 \frac{e^{-i\alpha x}}{-i\alpha}\, dx \quad \text{(by parts)}$$

$$= \frac{1}{i\alpha} - \frac{e^{-i\alpha x}}{\alpha^2}\Big|_0^1 = \frac{1}{i\alpha} + \frac{1}{\alpha^2}\left(1 - e^{-i\alpha}\right)$$

similarly $B = -\frac{1}{i\alpha} + \frac{1}{\alpha^2}\left(1 - e^{i\alpha}\right)$

combining gives

$$\hat{f}(\alpha) = \frac{1}{2\pi\alpha^2}\left(2 - \left(e^{i\alpha} + e^{-i\alpha}\right)\right)$$

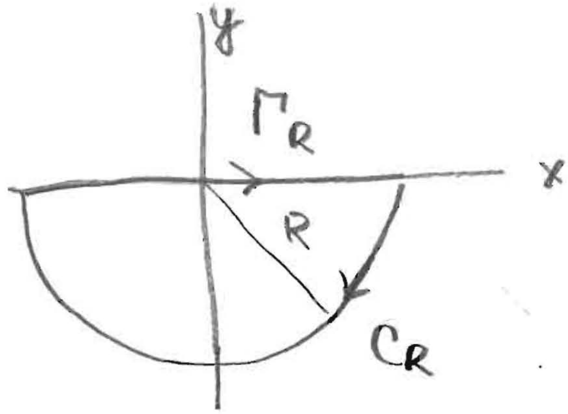
$$= \frac{1}{\pi\alpha^2}\left(1 - \cos\alpha\right).$$

Note that $\hat{f}(0) = \frac{1}{2\pi}$ can be computed separately and agrees with the limit of the expression above.

Note that your answer may differ by a power of 2π if you used a different definition of the transform.

2. Consider $f(z) = \dfrac{e^{-i\alpha z}}{z^4 + a^4}$ for complex $z = x + iy$.

Suppose that $\alpha \geq 0$ so that $|e^{-i\alpha z}| \leq 0$ in the lower half plane. Consider the following contour $C_R = \Gamma_R + A_R$



We want $\hat{f}(\alpha) = \dfrac{1}{2\pi} \lim\limits_{R\to\infty} \displaystyle\int_{\Gamma_R} f(z)\,dz$

$$= \frac{1}{2\pi} \lim_{R\to\infty} \int_{C_R} f(z)\,dz$$

since $\lim\limits_{R\to\infty} \displaystyle\int_{A_R} f(z)\,dz = \lim\limits_{R\to\infty} \dfrac{\pi R}{O(R^4)} = 0.$

So $\hat{f}(a) = \underset{\substack{\uparrow \\ \text{clockwise}}}{-}\dfrac{2\pi i}{2\pi}$ times residues of poles in the lower half plane. (1)

The poles of $z^4 + a^4$ in the lower half plane are

$z_1 = a e^{-3i\pi/4} = -\dfrac{a}{\sqrt{2}}(1+i)$

$z_2 = a e^{-i\pi/4} = \dfrac{a}{\sqrt{2}}(1-i)$

Since $f(z) = \dfrac{p(z)}{q(z)}$ with $q'(z_1) \neq 0$ and
$q'(z_2) \neq 0$
(simple poles)

we can use the formula $\dfrac{p(z_i)}{q'(z_i)}$ for the residues at the two poles.

$$q'(z) = 4z^3.$$

$$q'(z_1) = 4a^3 e^{-i9\pi/4} = 4a^3 e^{-i\pi/4}$$

$(e^{i\pi} = -1)$

$$q'(z_2) = 4a^3 e^{-3i\pi/4} = -4a^3 e^{+i\pi/4}$$

$$p(z_1) = e^{i\alpha a(1+i)/\sqrt{2}} = e^{i\alpha a/\sqrt{2}} \cdot e^{-\alpha a/\sqrt{2}}$$

$$p(z_2) = e^{-i\alpha a(1-i)/\sqrt{2}} = e^{-i\alpha a/\sqrt{2}} \cdot e^{-\alpha a/\sqrt{2}}$$

So $R_1 + R_2 = \dfrac{e^{-\alpha a/\sqrt{2}}}{4a^3} \left(2i \sin\left(\alpha a/\sqrt{2} + \pi/4\right)\right)$

Thus going back to (1) we have for $\alpha \geq 0$

$$\hat{f}(\alpha) = \frac{1}{2a^3} e^{-\alpha a/\sqrt{2}} \sin\left(\alpha a/\sqrt{2} + \pi/4\right).$$

The similar calculation for $\alpha < 0$ leads to the form that holds for all α,

$$\hat{f}(\alpha) = \frac{1}{2a^3} e^{-|\alpha| a/\sqrt{2}} \sin\left(|\alpha| a/\sqrt{2} + \pi/4\right).$$

3. $u_{tt} + a u_t = u_{xx}$

Take the Fourier transform to obtain

$$\frac{d^2\hat{u}}{dt^2} + a\frac{d\hat{u}}{dt} = -\alpha^2 \hat{u}.$$

This is a second order constant coefficient linear ODE. Look at the characteristic equation

$$r^2 + ar + \alpha^2 = 0$$

(i) $a > 0$, $\quad r = -\frac{a}{2} \pm \sqrt{\frac{a^2}{4} - \alpha^2} \qquad (2)$

For $\alpha > |\frac{a}{2}|$, the r values have real part $-a/2$ and imaginary part. For $\alpha < |\frac{a}{2}|$ the value of the square root term is smaller than $a/2$. Thus all r values have non-positive real parts. No Fourier component of the solution grows exponentially. The problem is well posed.

(ii) $a < 0$. The same formula (2) applies. The maximum growth rate is $-a$ that occurs at $\alpha = 0$. This case has exponential growth but with rates bounded by $-a$ and so is also well posed.

4. $u_t = v + u_{xx}$

$v_t = u + v_{xx}$

Take fourier transforms

$$\frac{d}{dt}\begin{bmatrix}\hat{u}\\ \hat{v}\end{bmatrix} = \underbrace{\begin{bmatrix}-\alpha^2 & 1\\ 1 & -\alpha^2\end{bmatrix}}_{A(\alpha)}\begin{bmatrix}\hat{u}\\ \hat{v}\end{bmatrix}.$$

do the eigenanalysis of A.

$$\det\begin{vmatrix}-\alpha^2-\lambda & 1\\ 1 & -\alpha^2-\lambda\end{vmatrix} = 0$$

$$\Rightarrow \quad (\alpha^2+\lambda)^2 = 1, \qquad \alpha^2+\lambda = \pm 1$$

$$\lambda = 1-\alpha^2,\ -1-\alpha^2$$

In this case the λ values are real and are smaller that 1. Thus, there can be exponential growth at rate e^t in the problem but the problem is well posed (maximum growth rate 1, no minimum decay rate since $\lambda_{1,2} \to -\infty$ as $|\alpha| \to \infty$).

5. $u_t = u_{xx} + u_{yy} + u_{zz}$.

Take the fourier transform in x, y and z, using α, β, and γ as the corresponding transform values.

$$\frac{d\hat{\hat{\hat{u}}}}{dt} = -(\alpha^2+\beta^2+\gamma^2)\,\hat{\hat{\hat{u}}}$$

so $\hat{\hat{\hat{u}}}(\alpha,\beta,\gamma,t) = e^{-(\alpha^2+\beta^2+\gamma^2)t}\,\hat{\hat{\hat{u}}}_0(\alpha,\beta,\gamma)$

Rewrite as (3)

$$\hat{\hat{\hat{u}}}(\alpha,\beta,\gamma,t) = e^{-\alpha^2 t}\,e^{-\beta^2 t}\,e^{-\gamma^2 t}\,\hat{\hat{\hat{u}}}_0(\alpha,\beta,\gamma).$$

as in the class notes,

$$F^{-1}\left(e^{-\alpha^2 t}\right) = \sqrt{\frac{\pi}{t}}\,e^{-x^2/(4t)}$$

and similarly $F^{-1}\left(e^{-\beta^2 t}\right) = \sqrt{\frac{\pi}{t}}\,e^{-y^2/(4t)}$

$$F^{-1}\left(e^{-\gamma^2 t}\right) = \sqrt{\frac{\pi}{t}}\,e^{-z^2/4t}.$$

Apply the convolution property of inverse Fourier transforms successively in x, y, and z to obtain

$$u(x,y,z,t) = \frac{1}{(4\pi t)^{3/2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u_0(x-a,\,y-b,\,z-c)\,e^{-(a^2+b^2+c^2)/4t}\,da\,db\,dc$$

Math 400, Fall 2012

Assignment #12.

- Not to be handed in, solutions will be posted.
- Some questions on the Fourier & Laplace Transform and time harmonic solutions to PDE problems (material after the last assignment).

1. Show that convolutions of f and g commute, that is that

$$\int_{-\infty}^{\infty} f(x-s)\,g(s)\,ds = \int_{-\infty}^{\infty} g(x-s)\,f(s)\,ds.$$

2. Show the result stated in class, that

$$\underbrace{F}_{\text{Fourier transform}}\big(\overbrace{f\cdot g}^{\text{pointwise multiplication}}\big) = 2\pi\ \hat{f} \underbrace{*}_{\text{convolution}} \hat{g}.$$

3. Decide which sign in the equation below leads to an ill-posed problem.

$$u_t = \pm\, u_{xxxx}$$

4. Consider the time-harmonic signal

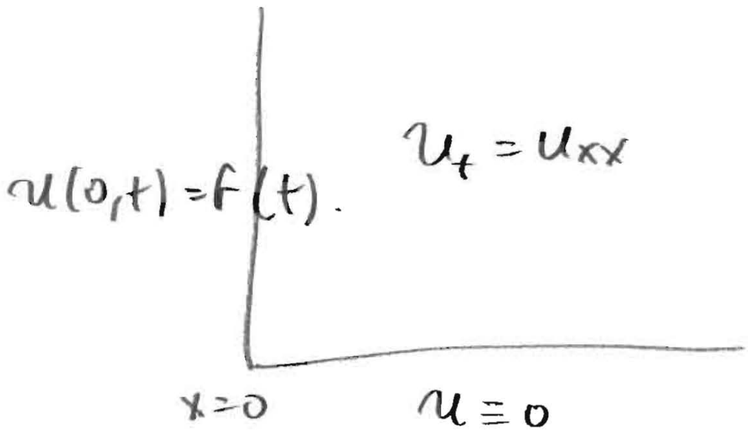
$$f(t) = 3\cos 2t - \sin 2t$$

write this signal in the two alternate forms (ie find a, δ and complex A below)

$$f(t) = a\cos(2t-\delta)$$

$$f(t) = \operatorname{Re}\left\{A e^{2it}\right\}.$$

5. Consider the following heat conduction problem for $u(x,t)$; defined for $x \geq 0$, $t \geq 0$.



(a) Use Laplace transforms to write the transform of u at each x in the following way

$$\tilde{u}(x,s) = \check{g}(x,s)\,\hat{f}(s)$$

(b) Find the function $g(x,t)$. You will have to look in tables of Laplace transforms.

(c) write $u(x,t)$ as a convolution

$$u(x,t) = \int_0^t f(t-T)\, g(x,T)\, dT$$

6. Consider the wave equation with time harmonic forcing, $u(x,t)$, $0 \leq x \leq 1$, $t \geq 0$.

$$u_{tt} = u_{xx} + \cos \omega t$$

$$u(0,t) = 0, \quad u(1,t) = 0$$

Find a time harmonic solution to this problem. Note that a time harmonic solution does not exist when $\omega = n\pi$ for any $n = 1, 2, \ldots$ Discuss what happens in this case.

Math 400, Fall 2012

Assignment #12 solutions

1. $\int_{-\infty}^{\infty} f(x-s)\, g(s)\, ds.$

Introduce the change of variables

$$y = x - s \Rightarrow s = x - y, \quad ds = -dy.$$

Note that as $s \to \infty$, $y \to -\infty$ and as $s \to -\infty$, $y \to +\infty$, so the integral above becomes

$$-\int_{\infty}^{-\infty} f(y)\, g(x-y)\, dy = \int_{-\infty}^{\infty} f(y)\, g(x-y)\, dy$$

which is in the desired form.

2. $$\widehat{f \cdot g}(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, g(x)\, e^{i\alpha x}\, dx$$

and $$f(x) = \int_{-\infty}^{\infty} \hat{f}(\beta)\, e^{-i\beta x}\, d\beta.$$

so $$\widehat{f \cdot g}(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{f}(\beta)\, g(x)\, e^{i(\alpha-\beta)x}\, dx\, d\beta.$$

bring the x integral sign here and factor of $\frac{1}{2\pi}$
substitute $x = \alpha - \beta$

$$= \int_{-\infty}^{\infty} \hat{f}(\beta)\, \hat{g}(\alpha-\beta)\, d\beta.$$

$$= \hat{f} * \hat{g}$$

Sorry about my confusion over the factor of 2π, it is not present in this scaling in the transform and its inverse in this formulation.

3. Take fourier transforms,

$$\frac{d\hat{u}}{dt} = \pm (i\alpha)^4 \hat{u}$$

$$= \pm (\alpha^4)\hat{u}$$

so $\hat{u}(\alpha,t) = e^{\pm \alpha^4 t} \hat{u}_0(\alpha)$.

Clearly the − sign leads to a well-posed problem and the + sign to a ill-posed problem (α^4 is real and has arbitrarily large positive values).

4. From the notes,

$$a = \sqrt{3^2 + (-1)^2} = \sqrt{10}$$

and $\delta = \mathrm{Arg}(3-i) = \arctan(\frac{1}{3})$ since $3-i$ is in the right half plane.

For the other form,

$$A = \sqrt{10}\, e^{-i\delta} = 3+i$$

5. as in the class notes, the Laplace Transform is

$$\tilde{u}(s,x) = A(s)\, e^{-\sqrt{s}\,x}$$

to satisfy the equation and boundedness as

$x \to \infty$. To match the boundary condition at $x=0$, $A(s) = \hat{f}(s)$. So $\hat{u}(s,x) = \hat{f}(s) e^{-\sqrt{s}x}$.

From tables,

$$\mathcal{L}^{-1}\left(e^{-\sqrt{s}x}\right) = \frac{x}{2\sqrt{\pi}\, t^{3/2}} e^{-x^2/4t}$$

Using the convolution theorem then,

$$u(x,t) = \frac{x}{2\sqrt{\pi}} \int_0^t f(\tau) \frac{1}{(t-\tau)^{3/2}} e^{-x^2/4(t-\tau)} \, d\tau.$$

6. Consider

$$u(x,t) = \operatorname{Re}\{A(x) e^{i\omega t}\}.$$

$$-\omega^2 A = A'' + 1.$$

$$A'' - \omega^2 A = -1$$

Solution (homogeneous + particular).

$$A(x) = C \sin \omega x + D \cos \omega x + \frac{1}{\omega^2}.$$

$$A(0) = 0 \Rightarrow D = -\frac{1}{\omega^2}$$

$$A(1) = 1 \Rightarrow C \sin \omega = \frac{1}{\omega^2}(\cos \omega - 1).$$

Note that if $\omega = n\pi$ and n is odd, we have

$$C \cdot 0 = -2/\omega^2$$

which cannot be satisfied. If n is even,

$C \cdot 0 = 0$ which can be satisfied for every C.

Note that $\omega = n\pi$ is a natural frequency of the system and so this is a case where resonance can occur. The n odd modes are forced and will grow resonantly in time (so no time harmonic solution can be found).

Math 400, Fall 2012
Material for the Midterm.

Review: ODE solution techniques
eigenanalysis of symmetric matrices
sine series
concept of linearity

Introduction to:
Boundary value operators and their eigen-analysis.
weak solutions
well-posedness
quasi-linearity
Cauchy problem.

New topics:
Derivation of heat and wave equations (handling varying and discontinuous material parameters).
Scaling and non-dimensionalization
Linearization
Wave equation in 1D: D'Alembert's solution
Maximum principle
Classification and standard forms for linear, c.c. second order PDEs in two variables.
Linear, first order PDE's: solution by the method of characteristics
First order, quasi-linear conservation laws: shock formation.

end: Notes $\boxed{\text{IV}}$b, p.5

Math 400, Fall 2012
Assignment #6
For midterm review, not to be handed in.

Note: Some problems here are longer than actual midterm problems.

#1. Consider the following heat conduction problem for $u(x,t), x \in [0,1], t \geq 0$.

$$u_t = u_{xx} + F(x,t), \quad F \text{ given.}$$

$$\left.\begin{aligned} u_x(0,t) &= 0 \\ u_x(1,t) &= 0 \end{aligned}\right\} \text{ for all } t.$$

$$u(x,0) = u_0(x) \text{ given.}$$

(a) Draw a diagram of this problem in the x-t plane, labelling the initial and boundary conditions and interior forcing terms.

(b) Describe in words what physical condition leads to the boundary conditions in this scaled problem.

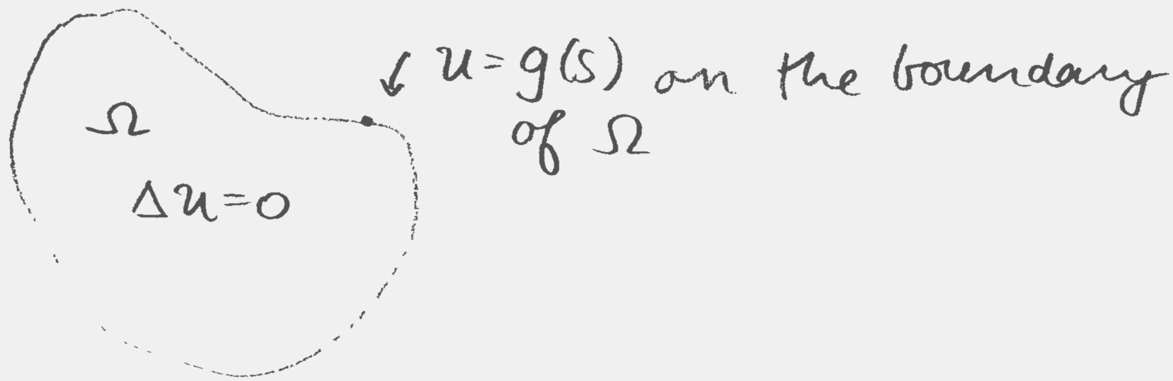
(c) Describe in words what the term $F(x,t)$ represents.

(d) Write the solution u in terms of the solutions u_1 and u_2 of two problems in each of which only one of the data $F(x,t)$ and $u_0(x)$ appears. It is only necessary to write the problems that u_1 & u_2 satisfy, not find solutions.

(e) Consider the operator $\mathcal{L}$ defined on functions $\varphi(x)$ for $x \in [0,1]$ with $\varphi'(0)=0$ and $\varphi'(1)=0$. Find the eigenvalues and eigenfunctions of $\mathcal{L}$.

(f) Use the result of (e) to construct a series solution to the original heat conduction problem. The series coefficients will involve integrals of f and u_0.

#2. Consider the solution $u(x,y)$ of the Laplace problem in a simply connected, bounded domain Ω as shown in the diagram below. The boundary of the domain is given by the parametrized curve $(x(s), y(s))$ for $0 \le s \le 1$.



Recall that $\Delta u := u_{xx} + u_{yy}$.

(a) Show that u obeys a maximum principle, that is

$$\max_{(x,y)\in\Omega} u(x,y) = \max_{s\in[0,1]} g(s)$$

(b) Show also that u obeys a minimum principle, i.e.

$$\min_{(x,y)\in\Omega} u(x,y) = \min_{s\in[0,1]} g(s)$$

and also that

$$\max_{(x,y)\in\Omega} |u(x,y)| = \max_{s\in[0,1]} |g(s)|$$

(c). Use the last fact above to show that solutions of the original problem are unique.

#3. Consider the following ODE for $u(t)$:

$$\frac{du}{dt} = ku(Q-u)$$

where k and Q are given positive constants.

(a) Show that $u \equiv 0$ and $u \equiv Q$ are equilibrium solutions of the ODE.

(b) Linearize the problem about the $u \equiv 0$ solution. Does the linearization indicate that the solution is stable or unstable?

(c) Repeat (b) for the $u \equiv Q$ solution.

(d) Scale u and t to make the problem have as few parameters as possible.

(e) Find the general solution of the scaled problem, i.e. the solution for any given $u(0) = u_0$ value.

#4 Explain briefly in words why characteristics can cross for nonlinear conservation laws but not for linear, first order problems.

#5 Consider the system of ODEs for $x(t)$ and $y(t)$:

$$\frac{dx}{dt} = y^2, \quad \frac{dy}{dt} = x.$$

(a) Sketch solutions to this problem starting at (1,1) and at (1,2). Do not try to find analytic solutions. [rough sketches are OK].

(b) use the information in (a) to sketch the region in which the solution to

$$y^2 u_x + x u_y = f(x,y)$$

$$u(1,y) = u_0(y) \text{ given for } 1 \leq y \leq 2$$

is defined.

#6 Consider the solution to

$$u_t + xt\,u_x = 0$$

with $u(x,0) = \dfrac{x}{1+x^2}$ for all x.

(a) Find the characteristic starting at $t=0$ and $x=s$ for every s.

(b) Show that the solution u is constant along each characteristic.

(c) Write down the expression for $u(s,t)$, that is, the solution on the characteristic that begins at $x=s$.

(d) Solve for $u(x,t)$.

#7 Consider the following nonlinear conservation law for $u(x,t)$:

$$u_t + (u^4)_x = 0$$

with initial data $u(0,x) = e^{-x^2}$ given.

(a) Find the characteristic starting at $t=0$ and $x=s$ for every s.

(b) Find the time $t^* > 0$ at which a discontinuity first forms in the solution.

Math 400, Fall 2012
Practice Midterm.

Note: The midterm will have the same format. No notes or calculators will be permitted.

part A: 5 short questions worth 2 marks each.

part B: 2 long questions worth 5 marks each.

Do all questions. Total 20 marks.

part A: Each question worth 2 marks.

A1. The symmetric matrix
$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$
has an eigenvalue $\lambda_1 = 3$ with eigenvector $\underline{v_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Find an ortho-normal basis of eigenvectors of A.

A2. Consider the following problems. Decide whether each problem is linear, quasi-linear, or fully nonlinear.

(a) $u(t)$: $\quad \dfrac{du}{dt} = t^2 u + \sin t$

(b) $u(x,t)$: $\quad \dfrac{\partial u}{\partial t} + \left(\dfrac{\partial u}{\partial x}\right)^2 = \sin(xt)$

(c) $x(t)$: $\quad \dot{x} = \sin(xt) + t^2$

A3. Determine whether the linear, c.c equation for $u(x,y)$ below is elliptic, hyperbolic, or parabolic.

$$u_{xx} + 2u_{xy} + 3u_{yy} + 10u_x - 3u_y = 0.$$

Both A4 and A5 below refer to the following ODE for $u(t)$:

$$\dddot{u} = (\dot{u})^2 + 4u + 8\sin(\ddot{u})$$

A4. Rewrite this ODE as a first order ODE system with 3 unknowns.

A5. Note that $u(t) = 0$ for all t solves the ODE. Linearize the ODE around this solution.

<u>part B</u> Each question is worth 5 marks.

B1. Consider the following Cauchy problem for $u(x,t)$, $t \geq 0$:

$$u_t + \frac{1}{1+x^2} u_x = 0$$

$$u(s,0) = u_0(s).$$

a) [1 mark] Write the ODE for the characteristics $x(t)$.

b) [2] Solve for the characteristics $x(s,t)$.

c) [2] Consider the data $u_0(s)$ for $s \geq 1$. ≤ 3
This data effects the solution $u(x,t)$ in a region of the x-t plane. Describe this region analytically and make a sketch of it.

B2. Consider the following wave equation problem for $u(x,t)$:

$$u_{tt} = u_{xx}, \quad t \geq 0, \ x \geq 0$$

$$u_x(0,t) = 0 \quad t \geq 0 \quad \text{(boundary condition)}$$

$$\left.\begin{aligned} u(x,0) &= u_0(x), \text{ given}, \ x \geq 0 \\ u_t(x,0) &= 0, \quad x \geq 0 \end{aligned}\right\} \text{(initial conditions)}$$

a) [1] Write the D'Alembert solution of the 1D wave equation.

b) [1] Make a labelled diagram of the problem in the x-t plane.

c) [3] Find the solution $u(x,t)$ in terms of the given initial data function $u_0(x)$.

Math 400, Fall 2012
Practice Midterm Solutions.

A1. A is symmetric so vectors orthogonal to $\underline{v_1}$ must be eigenvectors also, take $\underline{v_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.
Normalizing,

$$\left\{ \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} \right\}$$

is the desired o.n. basis.

A2. (a) linear
(b) fully nonlinear
(c) quasi-linear.

A3. $u_{xx} + 2u_{xy} + 3u_{yy} + \cdots$

$a = 1$ $2b = 2$, $b = 1$ $c = 3$.

$ac - b^2 = 2$, so the problem is elliptic.

A4. $\dddot{u} = (\dot{u})^2 + 4u + 8\sin(\ddot{u})$
introduce $v = \dot{u}$, $w = \ddot{u}$, then

$$\frac{d}{dt}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} v \\ w \\ v^2 + 4u + 8\sin(w) \end{bmatrix}.$$

A5. Let $u = \varepsilon v \quad (0 + \varepsilon v)$.

$$\varepsilon \dddot{v} = (\varepsilon \dot{v})^2 + 4\varepsilon v + 8\sin(\varepsilon \ddot{v})$$

$\downarrow$ ε^2 term $\qquad$ $\downarrow$ $\sin(x) \approx x$ for x small

$$\approx 0 + 4\varepsilon v + 8\varepsilon \ddot{v}$$

so $\dddot{v} = 4v + 8\ddot{v}$ is the linearized equation.

B1. actually, the problem I gave has algebra too difficult for a midterm question. Consider instead the problem

$$u_t + x^2 u_x = 0.$$

$$u(s,0) = u_0(s).$$

(a) $\dfrac{dx}{dt} = x^2 \qquad x(0) = s.$

(b) $\dfrac{dx}{x^2} = dt, \qquad -\dfrac{1}{x} = t - \dfrac{1}{s}. \quad s \neq 0.$

$$x(s) = \frac{1}{\frac{1}{s} - t} = \frac{s}{s-t} \qquad s \neq 0$$

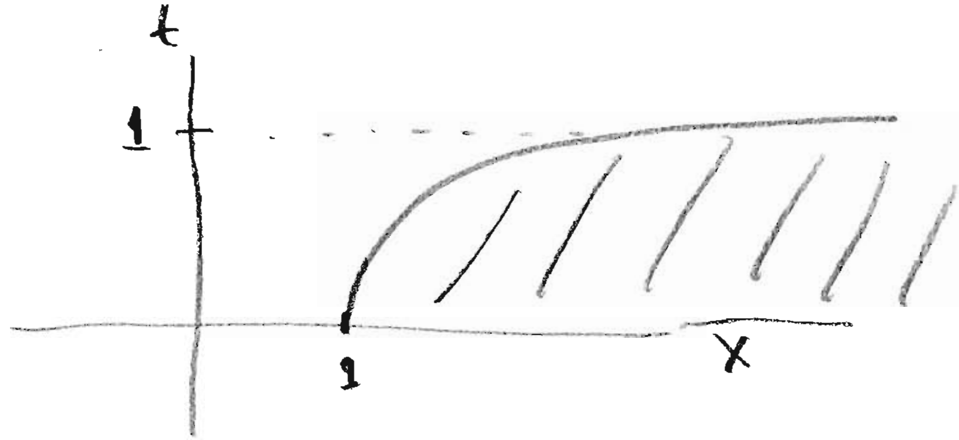
Considering the $s=0$ case separately leads to $x \equiv 0$ for $s=0$.

(c) The $s=1$ curve from (b) is

$$x = \frac{1}{1-t}$$

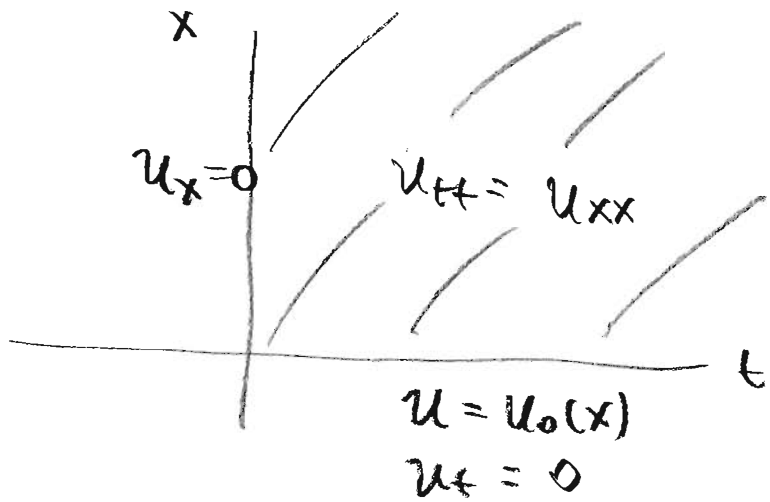
This is defined for $t<1$ and $\lim_{t \to 1-} x(t) = +\infty$

Thus the region affected by the data for $s \geq 1$ is shown below:



B2. (a) $u(x,t) = f(x-t) + g(x+t)$.

(b)



(c) $f(s) = g(s) = \frac{1}{2}u_0(s)$ for $s \geq 0$ by the usual argument. Still need to determine $f(s)$ for $s < 0$. Use the boundary condition

$$u_x(x,t) = f'(x-t) + g'(x+t).$$

$$u_x(0,t) = f'(-t) + g'(t) = 0 \quad \text{for all } t \geq 0$$

integrating w.r.t. t gives

$$-f(-t) + g(t) = C \quad \text{for } t \geq 0.$$

considering $t=0$ gives $C=0$. since $f(0) = g(0) = \frac{1}{2}u_0(0)$

This is promising since it will allow us to fill in the negative arguments of f.

$$f(s) = g(-s) \qquad \text{for } s<0.$$
$$= \tfrac{1}{2} u_0(-s)$$

Summarizing

$$g(s) = \tfrac{1}{2} u_0(s) \qquad s \geq 0.$$

$$f(s) = \begin{cases} \tfrac{1}{2} u_0(s) & s \geq 0 \\ \tfrac{1}{2} u_0(-s) & s \leq 0. \end{cases}$$

<u>Additional note</u>: This boundary gives an uninverted reflected wave.

Name: __________________ Student #: __________________

Math 400 (Wetton) Midterm, Fall 2012

Instructions:

- No notes, no calculators.
- Show all work. Justify your answers.
- Do all five questions in part A, each worth 2 points.
- Do all two questions in part B, each worth 5 points.
- total 6 pages, 20 marks.

part A: do all questions

A1. [2 marks] Determine whether each of the following problems for $u(x,t)$ is linear, quasi-linear or fully nonlinear:

(a) $(u_t)^2 + (u_x)^2 = 1$.

(b) $u_t + [u(1-u)]_x = 0$.

(c) $x^2 u_t + t^2 u_x = \sin(xt)$.

A2. [2] Write the differential equations for the characteristics for the following problem for $u(x,y)$:

$$(x^2 - y^2)u_x + (x^2 + y^2)u_y + u\sin(xy) - e^{\cos x} = 0$$

Do not solve the equations.

A3. [2] Consider the PDE for $u(x,t)$ below:

$$u_t + [f(u)]_x = 0$$

where the function f is given. Note if U is a constant, $u(x,t) = U$ for all x and t is a solution of the PDE. Find the linearization of this PDE about this solution.

A4. [2] Consider Burger's Equation for $u(x,t)$ below:

$$u_t + \left(\frac{u^2}{2}\right)_x = 0$$

with initial data

$$u(x,0) = e^x.$$

Write the characteristics for this problem. Will shock waves (discontinuities) appear in solutions for $t \geq 0$? Justify briefly (mark not awarded without justification).

A5. [2] Scale and non-dimensionalize the following problem for $u(x,t)$, with x in the interval $[0, L]$ and $t \geq 0$:

$$\rho(u_t + uu_x) = \nu u_{xx}$$

where ρ, ν, L and d are given positive constants. Leave as few dimensionless parameters as possible.

Part B: do both questions

B1. Consider the following first order PDE problem for $u(x,t)$ for $t \geq 0$ and all x:

$$u_t + xu_x = x$$
$$u(x,0) = e^{-x^2}$$

(a) [1 mark] Write the differential equation for the characteristics $x(t)$.

(b) [1] Solve for the characteristics $x(s,t)$ starting at every point $x = s$ at $t = 0$.

(c) [2] Solve for the solution $u(s,t)$, that is the solution at time t on the characteristic that started at $x = s$.

(d) [1] Solve for $u(x,t)$.

B2. Consider the following wave equation problem for $u(x,t)$ for $x \geq 0$ and $t \geq 0$:

$$
\begin{aligned}
&u_{tt} = u_{xx} \\
&u(x,0) = 0 \quad \text{for all } x \geq 0, \text{ initial condition} \\
&u_t(x,0) = 0 \quad \text{for all } x \geq 0, \text{ initial condition} \\
&u(0,t) - u_x(0,t) = a(t) \quad \text{for all } t \geq 0, \text{ boundary condition}
\end{aligned}
$$

where a is a given function for $t \geq 0$. It is known that $u(x,t) = f(x-t) + g(x+t)$ (D'Alembert's solution) and that the non-negative arguments of f and g are zero from the initial conditions. The negative arguments of f are still to be determined.

(a) [1] Draw a labelled diagram of the PDE problem in the $x-t$ plane.

(b) [2] Write the equation that results when the D'Alembert form of the solution is put into the boundary condition.

(c) [2] Solve for $f(s)$ for $s \leq 0$. Your answer will involve integrals of the given function a.

Name: Solutions Student #: ______________

Math 400 (Wetton) Midterm, Fall 2012

Instructions:

- No notes, no calculators.
- Show all work. Justify your answers.
- Do all five questions in part A, each worth 2 points.
- Do all two questions in part B, each worth 5 points.
- total 6 pages, 20 marks.

part A: do all questions

A1. [2 marks] Determine whether each of the following problems for $u(x,t)$ is linear, quasi-linear or fully nonlinear:

(a) $(u_t)^2 + (u_x)^2 = 1$. fully nonlinear
(b) $u_t + [u(1-u)]_x = 0$. quasi-linear
(c) $x^2 u_t + t^2 u_x = \sin(xt)$. linear.

A2. [2] Write the differential equations for the characteristics for the following problem for $u(x,y)$:

$$(x^2 - y^2)u_x + (x^2 + y^2)u_y + u\sin(xy) - e^{\cos x} = 0$$

Do not solve the equations.

$x(T)$: $\frac{dx}{dT} = (x^2 - y^2)$

$y(T)$: $\frac{dy}{dT} = (x^2 + y^2)$

Note: since $x^2 + y^2 > 0$ it is possible to make y a time-like variable and have characteristics $x(y)$ satisfying

$$\frac{dx}{dy} = \frac{x^2 - y^2}{x^2 + y^2}$$

but this is extra work...

A3. [2] Consider the PDE for $u(x,t)$ below:

$$u_t + [f(u)]_x = 0$$

where the function f is given. Note if U is a constant, $u(x,t) = U$ for all x and t is a solution of the PDE. Find the linearization of this PDE about this solution.

form $u = U + \varepsilon v(x,t)$.

$$f(u) \approx f(U) + \varepsilon f'(U)\, v.$$

$$u_t + [f(u)]_x = 0 \Rightarrow \varepsilon v_t + \varepsilon f'(U)\, v_x = 0$$

so linear equation is $v_t + f'(U)\, v_x = 0$.

A4. [2] Consider Burger's Equation for $u(x,t)$ below:

$$u_t + \left(\frac{u^2}{2}\right)_x = 0 \Rightarrow u_t + \boxed{u}\, u_x = 0$$

with initial data

$$u(x,0) = e^x.$$

Write the characteristics for this problem. Will shock waves (discontinuities) appear in solutions for $t \geq 0$? Justify briefly (mark not awarded without justification).

characteristics speed $u_0(s) = e^s$ so

$$x(t) = e^s t + s.$$

Speed is increasing with s $\left(\frac{d}{ds} e^s = e^s > 0\right)$ so characteristics won't cross, no shocks will form.

Note: The problem with $u_0(x)$ unbounded here is not physically reasonable.

A5. [2] Scale and non-dimensionalize the following problem for $u(x,t)$, with x in the interval $[0, L]$ and $t \geq 0$:

$$\rho(u_t + uu_x) = \nu u_{xx}$$

where ρ, ν and L ~~and d~~ are given positive constants. Leave as few dimensionless parameters as possible.

$u = vU$ U, T scales to be determined,

$t = \tau T$ v, τ, y dimensionless.

$x = yL$

$$\frac{\rho U}{T} v_\tau + \frac{\rho U^2}{L} v v_y = \frac{\nu U}{L^2} v_{yy}$$

$$v_\tau + \frac{TU}{L} v v_y = \frac{\nu T}{\rho L^2} v_{yy}$$

$U = \frac{L}{T} = \frac{\nu}{\rho L}$. then take $T = \frac{L^2 \rho}{\nu}$

final form $v_\tau + v v_y = v_{yy}$, $y \in [0, 1]$

Part B: do both questions

B1. Consider the following first order PDE problem for $u(x,t)$ for $t \geq 0$ and all x:

$$u_t + xu_x = x$$
$$u(x,0) = e^{-x^2}$$

(a) [1 mark] Write the differential equation for the characteristics $x(t)$.

(b) [1] Solve for the characteristics $x(s,t)$ starting at every point $x = s$ at $t = 0$.

(c) [2] Solve for the solution $u(s,t)$, that is the solution at time t on the characteristic that started at $x = s$.

(d) [1] Solve for $u(x,t)$.

(a) $\frac{dx}{dt} = x \quad , \quad x(0) = s.$

(b) $x(s,t) = se^t$

(c) $\frac{du}{dt} = x = se^t, \quad u(0) = e^{-s^2}$

$u = se^t - s + e^{-s^2}$

(d) $s = x/e^t$ from (b),

$u = x - x/e^t + e^{-(x/e^t)^2}$

B2. Consider the following wave equation problem for $u(x,t)$ for $x \geq 0$ and $t \geq 0$:

$$
\begin{aligned}
&u_{tt} = u_{xx} \\
&u(x,0) = 0 \quad \text{for all } x \geq 0, \text{ initial condition} \\
&u_t(x,0) = 0 \quad \text{for all } x \geq 0, \text{ initial condition} \\
&u(0,t) - u_x(0,t) = a(t) \quad \text{for all } t \geq 0, \text{ boundary condition}
\end{aligned}
$$

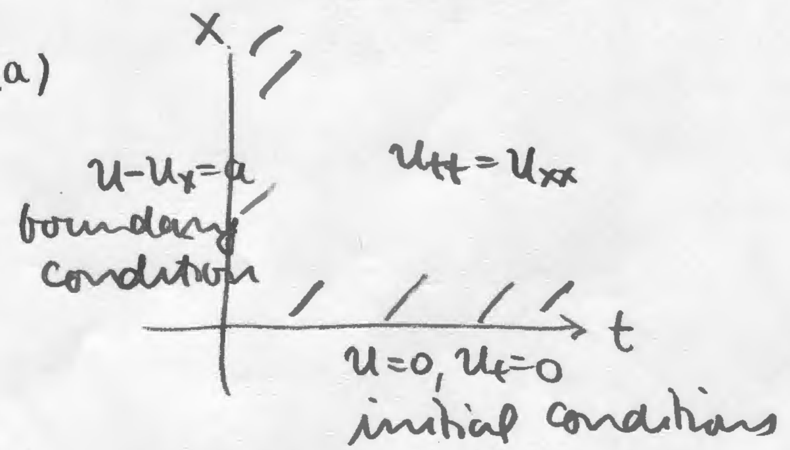
where a is a given function for $t \geq 0$. It is known that $u(x,t) = f(x-t) + g(x+t)$ (D'Alembert's solution) and that the non-negative arguments of f and g are zero from the initial conditions. The negative arguments of f are still to be determined.

(a) [1] Draw a labelled diagram of the PDE problem in the $x-t$ plane.

(b) [2] Write the equation that results when the D'Alembert form of the solution is put into the boundary condition.

(c) [2] Solve for $f(s)$ for $s \leq 0$. Your answer will involve integrals of the given function a.

(a)



(b) $u = f(x-t) + g(x+t) \qquad u(x,0) = f(-t) + g(t)$

$u_x = f'(x-t) + g'(x+t) \qquad u_x(x,0) = f'(-t) + g'(t). \qquad (1)$

$u - u_x\big|_{x=0} = a(t) \Rightarrow f(-t) - f'(-t) = a(t) \underbrace{-g(t) + g'(t)}_{\text{zero from IC.}}$

(c) In (1) let $s = -t$, $s \leq 0$.

$f'(s) - f(s) = -a(-s) + 0$

integrating factor e^{-s}...

$\frac{d}{ds}\left(e^{-s} f(s)\right) = -a(-s)e^{-s}$ integrate...

$$e^{-s} f(s) - \underbrace{\cancel{f(0)}}_{\text{zero by IC's}} = - \int_0^s e^{-\tau} a(\tau)\, d\tau$$

$$f(s) = \int_s^0 e^{s-\tau} a(-\tau)\, d\tau.$$

inverted integration limits since $s<0$, but this is not necessary.

Math 400, Fall 2012
Material for the Final Exam.

ODE solution techniques:
- constant coefficient, linear, homogeneous – both scalar (any order) and vector first order.
- non homogeneous by variation of parameters and undetermined coefficients.

Theoretical ideas:
- weak solutions
- well-posedness
- boundary value operators
- linearity
- cauchy problem
- PDE classification
- a priori bounds: maximum principle
- Green's functions
- Shock waves: conservation and entropy

General Applied Mathematics Techniques:
- Scaling and nondimensionalization
- linearization
- Asymptotic analysis.
- Modelling

Techniques:
- Series solutions using Sturm–Liouville theory
- D'Alembert's solution for the wave equation in 1D.
- Fourier & laplace transforms
- characteristics for first order equations
- Conservation laws, Rankine–Hugoniot and Lax Entropy conditions, shock formation, Riemann problems.
- Time harmonic problems

Math 400, Fall 2012

Practice Final Exam solutions.

A1. (a) linear

(b) fully nonlinear

(c) quasilinear.

A2. $$u_{xx} - 2u_{xy} + 5u_{yy} = 0$$

$$\underset{a=1}{\uparrow} \qquad \underset{b=-1}{\uparrow} \qquad \underset{c=5}{\uparrow}$$

$$ac - b^2 = 5 - 1 = 4$$

so equation is elliptic.

A3. Let $u = VU$ $x = SL$, $t = TT$

(V, S, T dimensionless).

$$\frac{U}{T} V_T + \frac{CU^2}{L} VV_S + \frac{KU}{L^3} V_{SSS} = Q$$

multiply by $\frac{T}{U}$

$$V_T + \frac{CTU}{L} VV_S + \frac{KT}{L^3} V_{SSS} = \frac{QT}{U}. \qquad (1).$$

It is possible to choose T, L and U so that all three coefficients above are unity. Note that all three coefficients above involve T linearly so a necessary condition is that

$$\frac{CU}{L} = \frac{K}{L^3} = \frac{Q}{U}$$

From the first two terms we have

$$U = \frac{K}{CL^2} \qquad (2).$$

and so from the second two terms we have

$$\frac{CQL^2}{K} = \frac{K}{L^3} \Rightarrow L^5 = \frac{K^2}{QC}, \quad L = \sqrt[5]{\frac{K^2}{QC}} \quad \checkmark$$

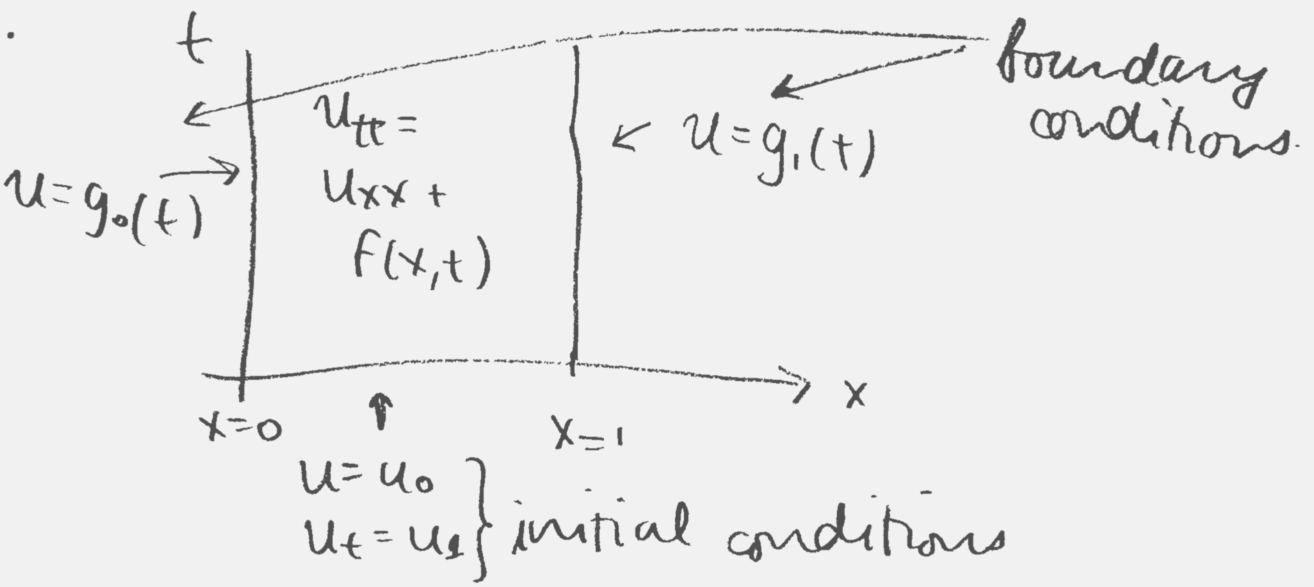
then U is obtained from (2) with this L and then $T = \frac{U}{Q}$ with that U.

A4. Let $u = 1 + \varepsilon v$, insert into the equation to obtain

$$v_t = v_{xx} - v - 4v = v_{xx} - 5v$$

$$-(1+\varepsilon v)^4 = -(1 + 4\varepsilon v + \varepsilon^2 + \cdots)$$

A5.



A6. Consider all problems of the form in the question.

	$f(x,t)$	u_0	u_1	g_0	g_1
u_1:	$f(x,t)$	0	0	0	0
u_2:	0	u_0	0	0	0
u_3:	0	0	u_1	0	0
u_4:	0	0	0	g_0	0
u_5:	0	0	0	0	g_1.

A7. $f(u) = u^4$ convex, so $U_L > U_R$ leads to a shock, $f'(u) = 4u^3$.

$$S = \frac{f(U_R) - f(U_L)}{U_R - U_L} = 1.$$

[check $f'(1) = 4 > S = 1 > f'(0) = 0$].

So $$u(x,t) = \begin{cases} 1, & \frac{x}{t} < 1 \\ 0, & \frac{x}{t} > 1. \end{cases}$$

A8. This will be a rarefaction. Solve for $u(c)$, c is the characteristic speed.

$C = f'(u) = 4u^3 \Rightarrow u = \sqrt[3]{c/4}$ [increasing ✓

$f'(1) = 4$, $f'(0) = 0$.

So $$u(x,t) = \begin{cases} 0 & x < 0 \\ \sqrt[3]{c/4} & \frac{x}{t} = c, \quad 0 < c < 4. \\ 1 & \frac{x}{t} > 4 \end{cases}$$

A9. $x(s,t)$ solves

$$\frac{\partial x}{\partial t} = \sin(x+t), \qquad x(s,0) = s.$$

[or $\dot{x} = \sin(x+t)$, $x(0) = s$ considering s fixed is OK].

A10. $\varphi'' + x\varphi' + e^x \varphi = \lambda \varphi$

Multiply by the integrating factor

$$e^{\int x\,dx} = e^{x^2/2}$$

$$(e^{x^2/2}\varphi')' + e^{x + x^2/2}\varphi = \lambda \underbrace{e^{x^2/2}}_{\text{orthogonality weight.}} \varphi.$$

so if $\lambda_n \neq \lambda_m$,

$$\int_0^1 e^{x^2/2} \varphi_n(x)\varphi_m(x)\,dx = 0$$

B1. By inspection

$\varphi_n(x) = \sin \mu_n x$ satisfies the S-L problem and $\varphi_n(0) = 0$.

$\varphi_n'(1) = 0 \Rightarrow \cos \mu_n = 0 \Rightarrow$

$\mu_n = (n - \frac{1}{2})\pi, \quad n = 1, 2 \dots$

so for (a),

$$u(x,t) = \sum_{n=1}^{\infty} u_n e^{-\mu_n^2 t} \sin \mu_n x.$$

with
$$u_n = \frac{\int_0^1 u_0(x) \sin \mu_n x}{\int_0^1 [\sin \mu_n x]^2 dx}$$

(b) The $n=1$ term is the slowest decaying mode in the series above.

$$u_1 = A = \frac{\int_0^1 \sin \frac{\pi x}{2} dx}{\int_0^1 \left[\sin \frac{\pi x}{2}\right]^2 dx} = \frac{-\frac{2}{\pi} \cos \frac{\pi x}{2} \Big|_0^1}{\frac{1}{2} - \underbrace{\int_0^1 \cos \pi x \, dx}_{\text{odd around } x = 1/2 \text{ so integral is zero}}} = \frac{4}{\pi}$$

$$[\sin \theta]^2 = \frac{1}{2}(1 - \cos 2\theta) \qquad (3)$$

so $u(x,t) \approx \frac{4}{\pi} e^{-(\pi^2/4)t} \sin\left(\frac{\pi x}{2}\right)$

Note: The identity (3) would be given if this question were asked on a final exam.

B2. We have the D'Alembert solution on either side of $x=0$.

$$u(x,t) = \begin{cases} f_0(x-t) + g_0(x+t) & x<0 \\ f_1(x-3t) + g_1(x+3t) & x>0. \end{cases}$$

Using the usual argument we have

$$f_0(s) = \frac{u_0(s)}{2}, \quad g_0(s) = \frac{u_0(s)}{2} \qquad s \leq 0.$$

$$f_1(s) = 0, \quad g_1(s) = 0 \qquad s \geq 0$$

However, we need the positive arguments of g_0 and the negative arguments of f_1 to complete the problem. Using the continuity of u and u_x at $x=0$ we have

$$\left.\begin{aligned} f_0(-s) + g_0(s) &= f_1(-3s) + \cancel{g_1(3s)} \quad (4) \\ f_0'(-s) + g_0'(s) &= f_1'(-3s) + \cancel{g_1'(3s)} \end{aligned}\right\} \ s>0$$

integrate

$$-f_0(-s) + g_0(s) = -\tfrac{1}{3} f_1(-3s) + \cancel{\tfrac{1}{3} g_1(3s)} + \cancel{c} \quad (5)$$

consider arguments at $t=0$, assuming $u_0(0)=0$ for continuity.

Now (4) & (5) are a linear system for $f_1(-3s)$ and $g_0(s)$ $s>0$ with data $f_0(-s) = u_0(-s)/2$ Subtracting (5) from (4) gives

$$\left.\begin{aligned} 2f_0(-s) &= \tfrac{4}{3} f_1(-3s) \\ \Rightarrow f_1(-3s) &= \tfrac{3}{4} u_0(-s) \end{aligned}\right\} \quad s>0$$

or $f_1(s) = \frac{3}{4} u_0\left(\frac{s}{3}\right)$ for $s<0$ ✓.

Returning to (4) gives

$g_0(s) = \frac{1}{4} u_0(-s)$. for $s>0$ ✓.

B3. Take Fourier transforms

$$\hat{u}(\alpha,t) = e^{[\pm(i\alpha)^4 - (i\alpha)^2]t}\,\hat{u}_0(\alpha).$$

$$= e^{[\pm\alpha^4 + \alpha^2]t}\,\hat{u}_0(\alpha).$$

(a)
Since $\alpha^4+\alpha^2$ has no upper bound, the + sign is ill-posed, the − sign is well posed.

(b) Consider $g(\alpha) = \alpha^2 - \alpha^4$. This has a maximum when $g'(\alpha) = 2\alpha - 4\alpha^3 = 0$

$$\Rightarrow \quad 1 - 2\alpha^2 = 0 \quad \Rightarrow \alpha = \pm\frac{1}{\sqrt{2}}.$$

$$g\left(\pm\frac{1}{\sqrt{2}}\right) = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}.$$

Thus, the maximum exponential growth rate is $\frac{1}{4}$.

B4. Consider the usual argument with

$v(x,t) = u(x,t) - \varepsilon t. \quad \Rightarrow \quad v_t - v_{xx} = -\varepsilon \qquad (6)$

The usual argument rules out maxima on the boundaries $x=0$ and $x=1$ and at the arbitrary time T, as well as in the interior away from $x = 1/2$. Consider the possibility of a maximum of V at $x = \frac{1}{2}+$ at time t. In this case,

$$V(\tfrac{1}{2}+, t) \geq V(\tfrac{1}{2}-, t)$$

so then by the interface condition

$$V_x(\tfrac{1}{2}+, t) \geq 0.$$

However, for $V(\frac{1}{2}+, t)$ to be a maximum,

$$V_x(\tfrac{1}{2}+, t) \leq 0 \quad \text{so} \quad V_x(\tfrac{1}{2}+, t) = 0.$$

In this case, $V_{xx}(\frac{1}{2}+, t) < 0$, $V_t(\frac{1}{2}+, t) = 0$ which violates (6). Similarly V cannot have a maximum at $x = \frac{1}{2}-$. The argument concludes as usual.

B5. $\lambda = -\pi^2/4 + \beta\varepsilon$

$$\varphi(x) \approx \sin \pi x/2 + \varepsilon V(x).$$

(a) $$V'' + \frac{\pi^2}{4} V = \left(\beta - \frac{\pi^2}{4} x\right) \sin \frac{\pi x}{2}. \qquad (7)$$

$$V(0) = 0, \quad V'(1) = 0.$$

(b) The algebra here is a bit too lengthy for an exam question, the solution to (7) is

$$V(x) = A \sin \frac{\pi x}{2} + B \cos \frac{\pi x}{2} \quad \leftarrow \text{homogeneous}$$

$$\left.\begin{aligned} &+ (ax + bx^2) \sin \frac{\pi x}{2} \\ &+ (cx + dx^2) \cos \frac{\pi x}{2} \end{aligned}\right\} \text{resonant MUC, complex form is easier.}$$

Solve for a, b, c and d by matching (7), and B by matching $V(0)=0$. A is arbitrary (set to zero) as discussed in class. Then only one value of β allows $V'(1)=0$ to be satisfied.

B6. $u_t = u_{xx} + x(1-x)\,e^{i3t}$

Time harmonic solution $u(x,t) = A(x)\,e^{i3t}$

(a) $3iA = A'' + x(1-x)$

(b) $A'' - 3iA = x^2 - x$ (8)

$r^2 = 3i \Rightarrow r = \underbrace{\pm\sqrt{\tfrac{3}{2}}\,(1+i)}_{r_1,\,r_2}$

$$A(x) = \underbrace{Ce^{r_1 x} + De^{r_2 x}}_{\text{homogeneous}} + \underbrace{a + bx + cx^2}_{\text{MUC}}$$

(c) MUC matching in (8).

$$A'' - 3iA = 2c - 3i(a + bx + cx^2) = x^2 - x$$

(i) $2c - 3ia = 0$

(ii) $-3ib = -1$

(iii) $-3ic = 1$

then $A(0)=0 \Rightarrow$

$C + D + a = 0$ (iv)

$A(1)=0 \Rightarrow$

$Ce^{r_1} + De^{r_2} + a + b + c = 0$ (v)

Math 400, Fall 2012
Practice Final Exam.

Note: The actual final will have the same format.
No notes or calculators permitted.

part A: 10 short questions worth 2 points each
part B: 6 long questions worth 5 points each

Do all questions. Total 50 marks.

part A Each question is worth 2 marks.

A1. Consider the following first order PDE problems. Decide whether each problem is linear, fully nonlinear, or quasi-linear.

(a) $u(x,t)$: $u_t + x^2t^2 u_x = 0$.

(b) $u(x,y)$: $\sin(u_x) + u_y = e^{xy}$

(b) $u(\theta,t)$: $u_t + u u_\theta = t^2+\theta^2$.

A2. Determine whether the following PDE for $u(x,y)$ is elliptic, hyperbolic, or parabolic:

$$u_{xx} - 2u_{xy} + 5u_{yy} = 0$$

A3. Scale x, t and u in the following problem to leave as few dimensionless parameters as possible:

$$u_t + C u u_x + K u_{xxx} = Q$$

The problem is posed for $t \geq 0$ and all x.
In the equation, C, K, and Q are given positive constants

A4. Consider the following equation for $u(x,t)$; $t \geq 0$ and all x.

$$u_t = u_{xx} - u - u^4 + 2.$$

Note that $u(x,t) \equiv 1$ is a solution to this problem. Linearize the equation about this solution.

A5. Consider the following wave equation problem for $u(x,t)$, $0 \leq x \leq 1$ and $t \geq 0$:

$$u_{tt} = u_{xx} + f(x,t).$$

$$u(x,0) = u_0(x)$$

$$u_t(x,0) = u_1(x)$$

$$u(0,t) = g_0(t)$$

$$u(1,t) = g_1(t).$$

Draw a labelled diagram of this problem in the x-t plane. Identify initial conditions and boundary conditions in your diagram.

A6. For problem A5 above, use linearity to write the solution as the sum of 5 problems each of which has only one of the data functions f, u_0, u_1, g_0, or g_1 in it. Do not solve any of the 5 problems, just state them.

A7. Consider the conservation law $u_t + (u^4)_x = 0$. Give the solution to the Riemann problem with $U_L = 1$ and $U_R = 0$.

A8. Repeat A7 with $U_R = 0$ and $U_L = 1$.

A9. Write the equation for the characteristics for the problem

$$u_t + \sin(x+t)\, u_x = 0.$$

Do not try to solve for the characteristics.

A10. Consider the eigen value problem for $\varphi(x)$

$$L\varphi = \lambda\varphi$$

where $L\varphi = \varphi'' + x\varphi' + e^x\varphi$

with $\varphi(0) = 0$, $\varphi(1) = 0$. Write the orthogonality relationship for eigenfunctions of this problem.

Part B Each question is worth 10 marks.

B1. Consider the following heat conduction problem for $u(x,t)$, $0 \le x \le 1$, $t \ge 0$.

$$u_t = u_{xx}$$

$$u(0,t) = 0, \quad u_x(1,t) = 0, \quad u(x,0) = u_0(x) \text{ given.}$$

(a) [3 marks] Write a series solution for the problem. Your series will have coefficients that involve integrals of $u_0(x)$.

(b) [2] If $u_0(x) \equiv 1$, write the slowest decaying term in the series above. Your answer should be of the form

$$u(x,t) \approx A e^{-ct}$$

with A and c determined.

B2. Consider the wave equation problem below for $u(x,t)$, $t \geq 0$, all x:

$$u_{tt} = \begin{cases} u_{xx} & x < 0 \\ 9u_{xx} & x > 0. \end{cases}$$

Initial data $u(x,0) = \begin{cases} u_0(x) & : x < 0 \\ 0 & x > 0. \end{cases}$

$$u_t(x,0) = 0 \quad \text{for all } x.$$

Find a solution for all x and t in terms of $u_0(x)$.

B3. Consider the following problem for $u(x,t)$, $t \geq 0$, all x:

$$u_t = \pm u_{xxxx} - u_{xx}$$

(a) [3 marks] Using the Fourier Transform, determine which sign in the equation above leads to a well-posed problem.

(b) [2] For the well-posed problem, the solution can still grow exponentially. Determine the maximum growth rate.

B4. On an assignment, you considered the diffusion problem for $u(x,t)$; $0 \leq x \leq 1$, $t > 0$:

$$u_t = u_{xx}$$

$$u_x(0,t) = u_x(1,t) = 0, \quad u(x,0) = u_0(x).$$

at $x = 1/2$ the following junction condition applies, where $u(x,t)$ can be discontinuous.

$$u_x\left(\tfrac{1}{2}-,t\right) = u_x\left(\tfrac{1}{2}+,t\right) = u\left(\tfrac{1}{2}+,t\right) - u\left(\tfrac{1}{2}-,t\right)$$

Show that this problem has a maximum principle, that is that

$$\max_{x,t} u(x,t) \le \max_{x} u_0(x).$$

B5. Consider the following Sturm-Liouville problem for $\varphi(x)$:

$$\varphi'' = (1+\varepsilon x)\lambda\varphi$$

$\varphi(0)=0$, $\varphi'(1)=0$. When $\varepsilon=0$, one eigenvalue is $\lambda = -\pi^2/4$ with $\varphi(x) = \sin \pi x/2$. Consider an asymptotic approximation of the problem for $\varepsilon > 0$,

$$\lambda \approx -\pi^2/4 + \beta\varepsilon$$

$$\varphi(x) \approx \sin \pi x/2 + \varepsilon V(x).$$

(a) [3] Write the equation and boundary conditions for the quantities β and $V(x)$.

(b) [2] Solve for β.

B6. Consider the following time-harmonic heat conduction problem for $u(x,t)$, $0 \le x \le 1$, $t > 0$.

$$u_t = u_{xx} + x(1-x)\cos 3t.$$

$u(0,t)=0$, $u(1,t)=0$.

Consider a time harmonic solution to this problem, after transients from initial conditions have decayed, that is,

$$u(x,t) = Re\{A(x)\, e^{i3t}\}.$$

(a) [2] Write the equation for $A(x)$. Do <u>not</u> try to solve this equation.

(b) [2] Write the form of the solution to the problem above, as a sum of homogeneous and particular solutions (using the method of undetermined coefficients). Your form should have 5 constants to be determined.

(c) [1] Write the 5 linear equations that determine the coefficients above. Do <u>not</u> try to solve these equations.

The University of British Columbia

Final Examination - December 5, 2012

Mathematics 400

Fall, 2012

Closed book examination Time: 2.5 hours

Special Instructions:

No books, notes, or calculators are allowed.

Part A		20
B1		5
B2		5
B3		5
B4		5
B5		5
B6		5
Total		50

Instructions:

- Do all ten questions in part A, each worth 2 points.
- Do all six questions in part B, each worth 5 points.
- total 50 marks.
- Justify all answers.
- Write your answers in booklets.

Part A: do all ten questions, each worth 2 marks

A1. [2marks] Write the D'Alembert form of the general solution to the wave equation for $u(x,t)$, $t \geq 0$ and all x:

$$u_{tt} = u_{xx}$$

A2. [2] Consider the following first order PDE problems. Decide whether each problem is linear, quasi-linear or fully nonlinear.

(a) $u(x,t)$: $u_t + \sin(u)u_x = 0$.

(b) $u(x,y)$: $x^2u_x + y^2u_y = e^{xy}$.

(c) $u(s,\theta)$: $u_s u_\theta - \log u_s = 0$.

A3. [2] Determine whether the following PDE for $u(x,y)$ is elliptic, hyperbolic or parabolic:

$$u_{xx} + 3u_{xy} + 2u_{yy} + 8u_x + 3u_y = 0$$

A4. [2] Consider the following heat conduction problem for $u(x,t)$, $0 \leq x \leq 1$, $t \geq 0$:

$$\begin{aligned} u_t &= u_{xx} + f(x,t) \\ u(0,t) &= g_0(t) \\ u_x(1,t) &= g_1(t) \\ u(x,0) &= u_0(x). \end{aligned}$$

Draw a labelled diagram in the $x-t$ plane of the problem. Identify the initial conditions and boundary conditions on your diagram.

A5. [2] Consider the problem in A4 above with $f \equiv 0$ and $u_0 \equiv 0$ but with given nonzero functions $g_0(t)$ and $g_1(t)$. Describe briefly (a sentence or two) how you would go about deriving a series or transform solution to the problem.

A6. [2] Consider the following problem for $u(x,t)$, $0 \leq x \leq L$ and $t \geq 0$:

$$u_t + Cuu_x + Ku = Q$$

where L, C, K and Q are given constants. Scale x to unit length and scale u and T so that the resulting equation has as few dimensionless parameters as possible.

A7. [2] Consider the following system of equations for $u(x,t)$ and $v(x,t)$

$$\begin{aligned} u_t &= u_{xx} + f(u,v) \\ v_t &= Dv_{xx} + g(u,v) \end{aligned}$$

where D is a positive constant and f and g are given functions with $f(0,0) = 0$ and $g(0,0) = 0$. Linearize the system about the solution $u \equiv 0$, $v \equiv 0$. Your answer will have coefficients that involve the partial derivatives of f and g.

A8. [2] Write the third order ODE for $u(t)$ below as a first order system.

$$\frac{d^3u}{dt^3} + \left(\frac{d^2u}{dt^2}\right)^2 - \sin(u) = 0.$$

A9. [2] Consider the heat conduction problem for $u(x,t)$, for $x \geq 0$ and $t \geq 0$:

$$\begin{aligned} &u_t = u_{xx} \\ &u_x(0,t) = 1 \quad \text{for all } t \geq 0 \\ &u(x,0) = 0 \quad \text{for all } x \geq 0. \end{aligned}$$

Find $U(x,s)$, where U is the Laplace transform of u in t. **Do not** try to solve for $u(x,t)$.

A10. [2] Consider the following Sturm-Liouville problem for $\phi(x)$, $0 \leq x \leq 1$:

$$\phi'' = \begin{cases} \lambda\phi & x < 1/2 \\ 4\lambda\phi & x > 1/2 \end{cases}$$

with $\phi(0) = 0$ and $\phi(1) = 0$ and ϕ and ϕ' continuous at $x = 1/2$. It is known that all eigenvalues λ are negative. Write $\lambda = -\mu^2$ and find a function $f(\mu)$ such that the roots of f correspond to values μ that lead to eigenvalues of the problem. **Do not** solve for the μ values.

Part B: do all five questions, each worth 5 points

B1. Consider the following linear, first order Cauchy problem for $u(x,t)$, $t \geq 0$ and all x:

$$u_t + xu_x = \sin t$$

with initial data $u(x,0) = e^{-x^2}$.

(a) [2 marks] Solve for the characteristics $x(s,t)$ for this problem.

(b) [2] Solve for $u(s,t)$ along these characteristics.

(c) [1] Write the solution $u(x,t)$.

B2. Consider the traffic flow model we developed in class for $\rho(x,t)$:

$$\rho_t + [f(\rho)]_x = 0$$

where $f(\rho) = \rho(1-\rho)$.

(a) [2] Solve the Riemann Problem for this equation with $\rho_L = 0$ and $\rho_R = 1$.

(b) [2] Solve the Riemann Problem for this equation with $\rho_L = 1$ and $\rho_R = 0$.

(c) [1] In the $x-t$ plane, sketch the solution of the Cauchy problem for this equation with initial data

$$\rho(x,0) = \begin{cases} 1 & \text{for } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

B3. Consider the following wave equation problem for $u(x,t)$, $0 \le x \le 1$ and $t \ge 0$, with time harmonic forcing

$$u_{tt} = u_{xx} + e^x \cos t$$

with boundary conditions $u(0,t) = 0$ and $u_x(1,t) = 0$. Consider a time harmonic solution of the form

$$u(x,t) = \mathrm{Re}\left\{A(x)e^{it}\right\}$$

(a) [2] Write the ODE that $A(x)$ satisfies.

(b) [1] Write the boundary conditions that $A(x)$ satisfies.

(c) [2] Solve for $A(x)$.

B4. Consider the heat conduction problem for $u(x,y,z,t)$, for $t \ge 0$, $0 \le x \le 1$, $0 \le y \le 1$ and $0 \le z \le 1$:

$$u_t = \Delta u = u_{xx} + u_{yy} + u_{zz}$$

with homogeneous Dirichlet boundary conditions on all boundaries.

(a) [3] Write the general solution to the problem as a series using the eigenvalues of the spatial operator. *Hint:* use sine series for x, y and z.

(b) [2] What is the slowest decaying term in your series in part (a)? Your answer should be in the form

$$Ae^{-ct}v(x,y,z)$$

where A and c are constants you give values for and $v(x,y,z)$ is specified.

B5. Consider the following Sturm-Liouville problem for $\phi(x)$ with parameter $\epsilon \geq 0$:

$$\phi'' + (1 + \epsilon x)\phi' = \lambda\phi$$

with $\phi(0) = 0$ and $\phi(1) = 0$. It is known that all eigenvalues of this problem are negative.

(a) [2] What is the orthogonality relationship for eigenfunctions of this problem?

(b) [2] For $\epsilon = 0$ find the largest (least negative) eigenvalue λ and its corresponding eigenfunction.

(c) [1] Discuss briefly (a sentence or two) how you could approach problem (b) with $\epsilon > 0$ but small.

B6. Use the Fourier transform to determine which of the following Cauchy problems for $u(x,t)$, $t \geq 0$ and all x, are well posed.

(a) [2] $u_t = u_{xxx}$

(b) [3] $u_{tt} + u_{txx} = u_{xx}$