Math 401, Section 201, Spring 2007
GREEN'S FUNCTIONS AND VARIATIONAL METHODS

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Office hours: Tuesdays 12-1, Fridays 1-2

Text: There is no required text for the course. A selection of notes will be posted on the web page. Some references:

- R.Habermann, Applied partial Differential Equations
- Courant and Hilbert: Methods of Math. Physics Vol. 1 & 2
- E. Zauderer, Partial Differential Equations of Applied Mathematics, Wiley.
- G. Carrier and Pearson. PDE's, Academic.
- Gelfand and Fomin: Calculus of variations
- Stakgold: Green's functions and Boundary value problems.

Marks: 50% final, midterm worth 20%, homework worth 30%.

Midterm Date: Friday, March 2

Assignments: Will be posted Monday and collected Monday after two weeks. **Note:** No late assignments will be accepted but the lowest assignment score will be dropped when calculating the assignment average.

Course description:

A: Green's functions for ODEs and PDEs

1. Introduction to generalized functions
2. Green's functions for ODEs
3. Self-adjoint differential operators
4. Methods to construct Green's functions
5. Green's functions for PDE:
6. Free space Green's functions
7. Method of images
8. Eigenfunction expansions
9. Boundary integral equation methods

B: Calculus of variations

1. Euler Lagrange Equations for unconstrained problems
2. Rayleigh-Ritz method
3. The finite element method
4. The Galerkin approximation
5. Variational statement of eigenvalue problems
6. Rayleigh's quotient
7. Bounds on eigenvalues

C: Perturbation Methods, as time permits

Math 401 Notes - I.

Three main topics:

1. Green's functions
2. Variational principles
3. perturbation methods.

These are techniques for studying differential equations (DEs).

+ understanding behaviour of solutions.
+ finding approximate solutions (analytic, numerical).

My interests:

+ numerical methods
+ modelling physical systems.

will be reflected in the material.

1. Introduction.

Let's look at simple examples of 1 - 3.

§ 1.1 Consider the linear system

$$\mathbb{A}\,\underline{u} = \underline{f}$$

$n \times n$ matrix $\mathbb{A}$; Solution $\underline{u}$ in $\mathbb{R}^n$; data $\underline{f}$ in $\mathbb{R}^n$

Assume $\mathbb{A}$ is invertible (there are many ways to determine this, even if finding A^{-1} is computationally intensive).

so $\quad \underline{u} = \mathbb{A}^{-1}\underline{f} = \mathbb{G}\underline{f}$

where we denote $\mathbb{A}^{-1}$ by $\mathbb{G}$ (think "Green's fn").

i'th row, j'th column of $\mathbb{G}$

$$u_i = \sum_{j=1}^{n} G_{ij}\, f_j \qquad (1)$$

solution at location i

sum over all source terms

encodes the influence of a source at j on the value at i.

source at location j

<u>Note</u>: force at one position affects the soln at all positions.

<u>Note</u>: The form (1) only results from a linear problem.

For linear DE's, same kind of idea - the sum in (1) is replaced by an integral, discrete G_{ij} replaced by continuous

$$G(s, x)$$

source position

solution position.

§1.2 Consider the ODE boundary value problem for $u(x)$, $x \in [0,1]$:

$$\boxed{\begin{array}{l} u'' = f(x) \quad \text{given data.} \\ u(0) = 0, \; u(1) = 0 \end{array}}$$

problem ⊛.

Q: Does this problem have a unique solution u for every given function f? [Ans: yes].

Consider the possibility of 2 solutions for a given f: u_1 & u_2.

$u_1'' = f$, $u_2'' = f$, u_1 & $u_2 = 0$ at $x = 0, 1$.

Let $w = u_1 - u_2$ [want to show $w \equiv 0$].

$\equiv 0$ for all x

$$\left.\begin{array}{l} w'' = u_1'' - u_2'' = f - f = 0 \\ w(0) = 0, \quad w(1) = 0 \end{array}\right\} \quad (2)$$

Two ways to show (2) implies $w \equiv 0$.

I. $w'' = 0 \Rightarrow w = a + bx$.

$w(0) = 0 \Rightarrow a = 0$

$w(1) = 0 \Rightarrow b = 0$.

II. $w''=0 \Rightarrow ww''=0$ for all x.

integrate $\int_0^1 ww'' = \underbrace{ww'\Big|_0^1}_{} - \int_0^1 (w')^2 = 0.$

by parts

zero since $w(0)=w(1)=0$

so $\int_0^1 (w')^2 = 0$, $w'=0$ for all $x \Rightarrow w=$ constant

since $w(0)=w(1)=0$, $w \equiv 0$.

So we have shown (★) has unique solutions when it has solutions. Expect, from finite dimensional linear algebra, that (★) must have a unique solution for all f [true but because of ∞ dimensionality some extra work is needed].

One can write the solution of (★) as

$$u(x) = \int_0^1 G(s,x)\, f(s)\, ds \qquad (3)$$

in the form predicted in §1.1. <u>Note</u>: force at one point affects the solutions at all points.

$$G(s,x) = \begin{cases} s(x-1) & 0 \le s < x \\ x(s-1) & x \le s < 1. \end{cases}$$

Take as given, we'll learn how to construct G later.

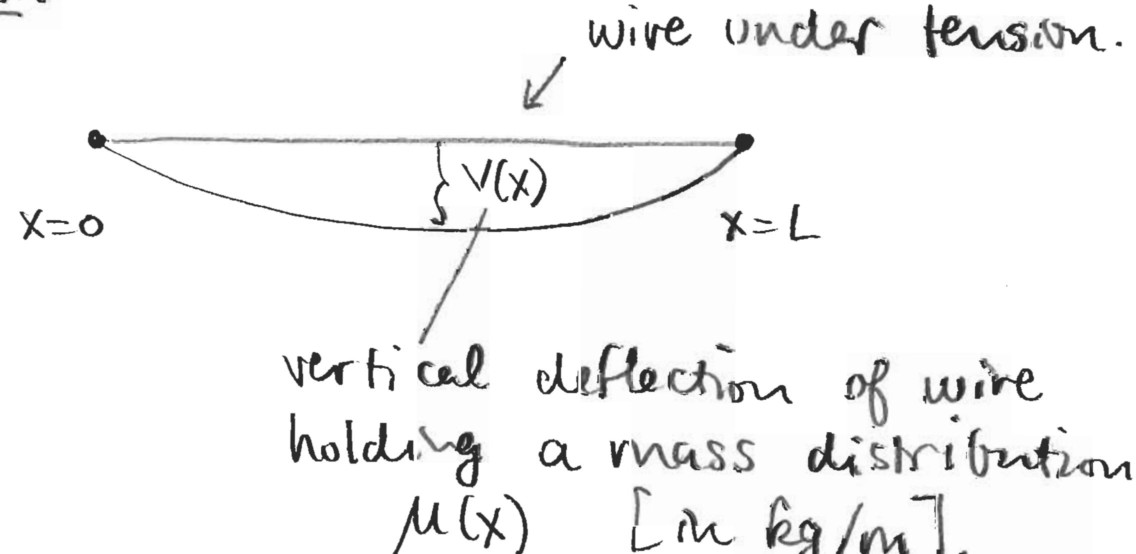
Note: $G(s,x) = G(x,s)$.

Like a symmetric matrix. We say ⊛ is self adjoint.

Advantages of (3):

1. It is transparent how forcing at one point (s) affects solutions at all points (x).
2. Numerical improvement - easier to do numerical integration in (3) than to solve the DE⊛ if high accuracy is needed at only a few points x.
3. The form (3) can be used to show the completeness of the eigenvalues of ⊛.

§1.3 Where did ⊛ come from? Consider a massless wire strung between 2 poles under tension.



Note: in the picture above, V(x) is negative.

Increasing the length of the string by ΔL requires a tension $k\Delta L$

spring constant N/m.

The energy to stretch the string is
$\frac{1}{2}k(\Delta L)^2$.

Energy of a steady configuration:

$$E = \frac{k}{2}\Big[\underbrace{\int_0^L \sqrt{1+[V'(x)]^2} - L_0}_{\Delta L}\Big]^2 + \int_0^L \mu g V \, dx. \quad (4)$$

potential energy of the stretched wire.

minus work done by the gravitational force moving through a distance $-V$.

Here $L_0 < L$

unstretched length of wire

and the wire without the mass is under tension $k(L - L_0)$.

The system will move to the configuration with the least energy.

Note: $E[v]$ is called a (nonlinear) functional.

real valued function of a function

Q: What $v_0(x)$ (with $v_0(0) = v_0(L) = 0$) minimizes E? $v_0(x)$ would be called a minimizer.

Before we proceed, let's simplify E under the assumption that $|v'(x)|$ is "small".

$$\sqrt{1+[v'(x)]^2} \approx 1 + \frac{1}{2}[v'(x)]^2$$

$$\int_0^1 \sqrt{\;}\,dx - L_0 \approx \frac{1}{2}\int_0^L [v'(x)]^2\,dx + (L-L_0)$$

$$[\quad]^2 \approx (L-L_0)^2 + (L-L_0)\int_0^L [v'(x)]^2\,dx.$$

So (4) simplifies to

$$E \approx \frac{k}{2}(L-L_0)^2 + \frac{k}{2}(L-L_0)\int_0^L [v'(x)]^2\,dx + \int_0^L \mu g v\,dx \qquad (5).$$

To find a DE for the minimizer of (5) use a technique of Euler.

Let $\eta(x)$ be an arbitrary differentiable fn with $\eta(0) = \eta(L) = 0$. Think of $\eta(x)$ being fixed for now.

Consider $V(x) = V_0(x) + \varepsilon\eta(x)$.

minimizer that we're trying to find.

$E[V(x)]$ for fixed η is only a function of ε. Since $V_0(x)$ is a minimizer,

$$\left.\frac{dE}{d\varepsilon}\right|_{\varepsilon=0} = 0.$$

Let's follow this approach for our example and see what happens...

$$E[V] = \frac{k}{2}(L-L_0)^2 + \frac{k}{2}(L-L_0)\int_0^L [V_0' + \varepsilon\eta']^2\,dx + \int_0^L \mu g\,(V_0 + \varepsilon\eta)\,dx.$$

$$\frac{dE}{d\varepsilon} = k(L-L_0)\int_0^L [V_0' + \varepsilon\eta']\,\eta'\,dx + \int_0^L \mu g\eta\,dx$$

$$\left.\frac{dE}{d\varepsilon}\right|_{\varepsilon=0} = \int_0^L \left\{k(L-L_0)V_0'\eta' + \mu g\eta\right\}dx.$$

integrate by parts.

$$= \left.k(L-L_0)V_0'\eta\right|_0^L + \int_0^L \left\{-k(L-L_0)V_0'' + \mu g\right\}\eta\,dx.$$

boundary terms zero since $\eta(0) = \eta(L) = 0$.

$$\left.\frac{dE}{d\varepsilon}\right|_{\varepsilon=0} = 0 \Rightarrow$$

$$\int_0^L \left\{ -k(L-L_0)V_0'' + \mu g \right\} \eta \, dx = 0. \quad (6)$$

Now this must be true for all functions η.
Consider $\eta = -k(L-L_0)V_0'' + \mu g$. Now (6) is

$$\int_0^L \left\{ -k(L-L_0)V_0'' + \mu g \right\}^2 dx = 0$$

so
$$\left.\begin{aligned} -k(L-L_0)V_0'' + \mu g &= 0. \\ V_0(0) = V_0(L) &= 0 \end{aligned}\right\} \quad (7)$$

(7) is the DE derived from the functional (5).

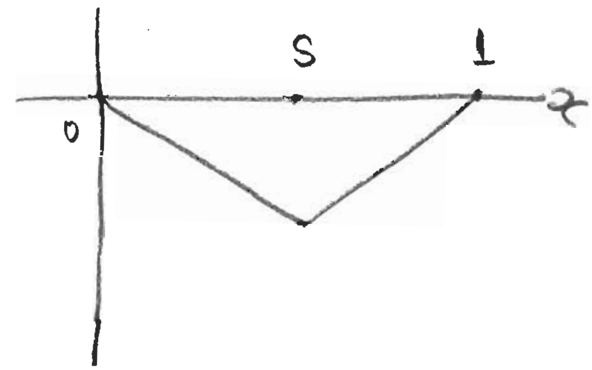
Using scaled distance $y = x/L$,
$u(y) = V_0(Ly)$ we have

$$\left.\begin{aligned} u'' &= \frac{L^2 g}{k(L-L_0)} \mu(yL) \\ u(0) &= u(1) = 0 \end{aligned}\right\}$$ in the form (A) & units OK.

Note: it is possible to go back and verify what parameters are consistent with the assumptions of $V'(x)$ "small".

§1.4 With the physical motivation of §1.3 above, Let's revisit the Green's function for this problem

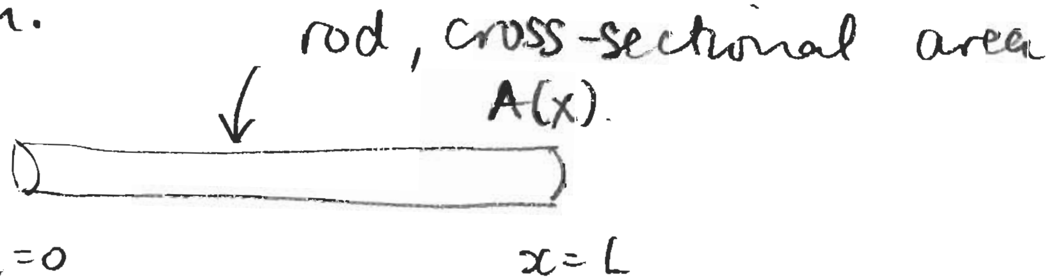
$$G(s,x) = \begin{cases} s(x-1) & s < x \\ x(s-1) & s > x. \end{cases}$$



This would be the solution to the physical problem if a unit scaled mass were put at s.

Our problem was posed in terms of a continuous mass distribution $\mu(x)$.
Having a mass concentrated at a single point does not fit into this framework.
We'll resolve this by considering generalized ("delta") functions next week.

§1.5 Problem ⊛ also describes a heat conduction problem.



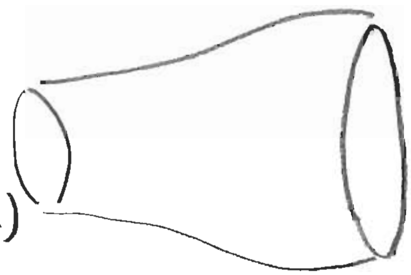
Temperature assumed uniform across the cross-section $T(x)$ to be determined.
↑ in °C.

Heat flux in the rod (W) to the right is

$$-\sigma A(x) T'$$

↑ thermal conductivity $W/(m°C)$.

External heating $h(x)$ in W/m.



Note: $T(x)$ minimizes

$$\int_0^L \{\sigma A(x) [T']^2 - hT\} dx$$

but this is <u>not</u> energy in this case.

Energy conservation

$$-\sigma A(x+\Delta x) T'(x+\Delta x) = -\sigma A(x) T'(x) + h\Delta x.$$

divide by Δx and take limits as $\Delta x \to 0$

$$\sigma (A T')' = -h. \qquad (8).$$

Letting $u(y) = T(yL)$ and taking A constant gives

$$u'' = \underbrace{-\frac{L^2}{\sigma A} h(x)}_{F(x)}.$$

both ends of rod in an ice-water bath.

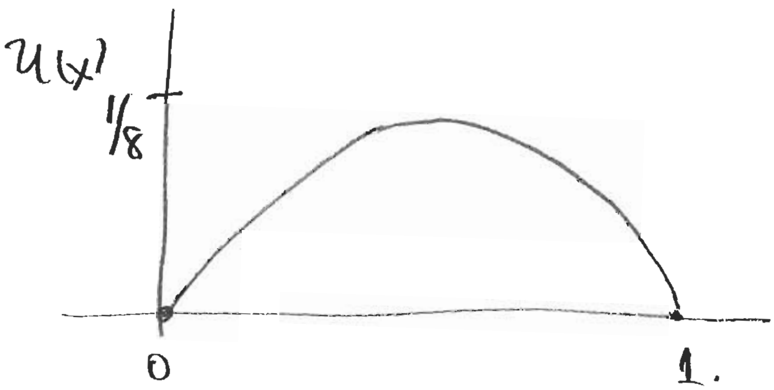
We could assume $u(0) = u(1) = 0$, obtaining ★ again.

If we considered the simple case of uniform scaled heating

$$u'' = -1 \qquad u(0)=0, \; u(1)=0.$$

and integrated twice, applying the BC's we find

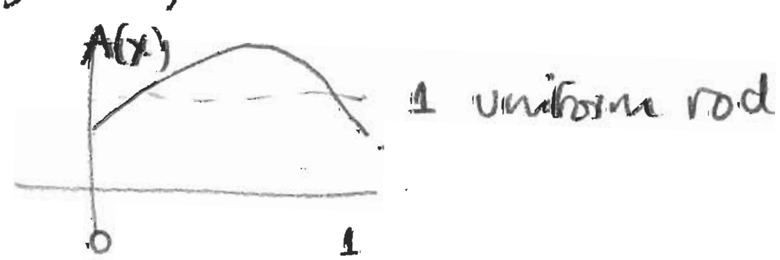
$$u(x) = \frac{-x^2}{2} + \frac{x}{2} = \frac{1}{2}x(1-x)$$



§1.6. What if $A(x)$ is not constant? What if the rod was manufactured so that the middle was thicker or thinner than the ends? Could this reduce the maximum temperature in a uniformly heated rod?

test this idea...

$$\Bigg(\underbrace{\left\{ \frac{1+\varepsilon \sin \pi x}{1+2\varepsilon/\pi} \right\}}_{A(x)} u' \Bigg)' = -1 \qquad (9).$$



Want to see how $u(x)$ depends on ε.

Note: $\underbrace{\int_0^1 A(x)\,dx}_{\text{Volume of the rod.}} = 1$ — so this is a fair comparison to the uniform rod.

Could compute (9) numerically for each ε, but there is a way to approximate solutions that apply for all ε (small).

Make the assumption:

$$u(x) \approx \underbrace{-\tfrac{1}{2}x^2 + \tfrac{1}{2}x}_{u_0(x),\ \text{solution when } \varepsilon = 0} + \varepsilon \underbrace{u_1(x)}_{\text{to be determined}} + \underbrace{\varepsilon^2 u_2(x) + \cdots}_{\text{Could add higher order corrections.}}$$

This is a perturbation series. Idea: plug into (9), equate coefficients of powers of ε, get solvable equations for $u_1(x)$ [and $u_2(x), \ldots$].

$$A(x) = \frac{1+\varepsilon \sin \pi x}{1 + 2\varepsilon/\pi} \approx (1+\varepsilon \sin\pi x)(1 - 2\varepsilon/\pi + \cdots)$$

$$\approx 1 + \varepsilon\left(\sin \pi x - \frac{2}{\pi}\right) + \cdots$$

$$A(x)u' \approx \left(1 + \varepsilon\left(\sin\pi x - \frac{2}{\pi}\right)\right)\left(\tfrac{1}{2} - x + \varepsilon u_1'\right)$$

$$\approx \tfrac{1}{2} - x + \varepsilon\left\{\left(\sin\pi x - 2/\pi\right)\left(\tfrac{1}{2} - x\right) + u_1'\right\}$$

$$(A(x)u')' \approx -1 + \varepsilon\left\{(\pi\cos\pi x)(\tfrac{1}{2}-x) - (\sin\pi x - 2/\pi) + u_1''\right\}.$$

plug into (9) $(Au')' = -1$ and equate powers of ε:

1: $\quad -1 = -1 \qquad$ ($u_0(x)$ was the soln for $\varepsilon = 0$ ✓).

$$\left.\begin{aligned} \varepsilon: \quad & u_1'' = \sin\pi x - \frac{2}{\pi} + \pi\cos\pi x\left(x - \tfrac{1}{2}\right). \\ & u_1(0) = 0, \quad u_1(1) = 0. \end{aligned}\right\} (10).$$

<ins>Note</ins>: $u_1(x)$ solves the $\varepsilon = 0$ problem with forcing term coming from $A(x)$ and $u_0(x)$.

Integrate (10) twice (I used Maple)

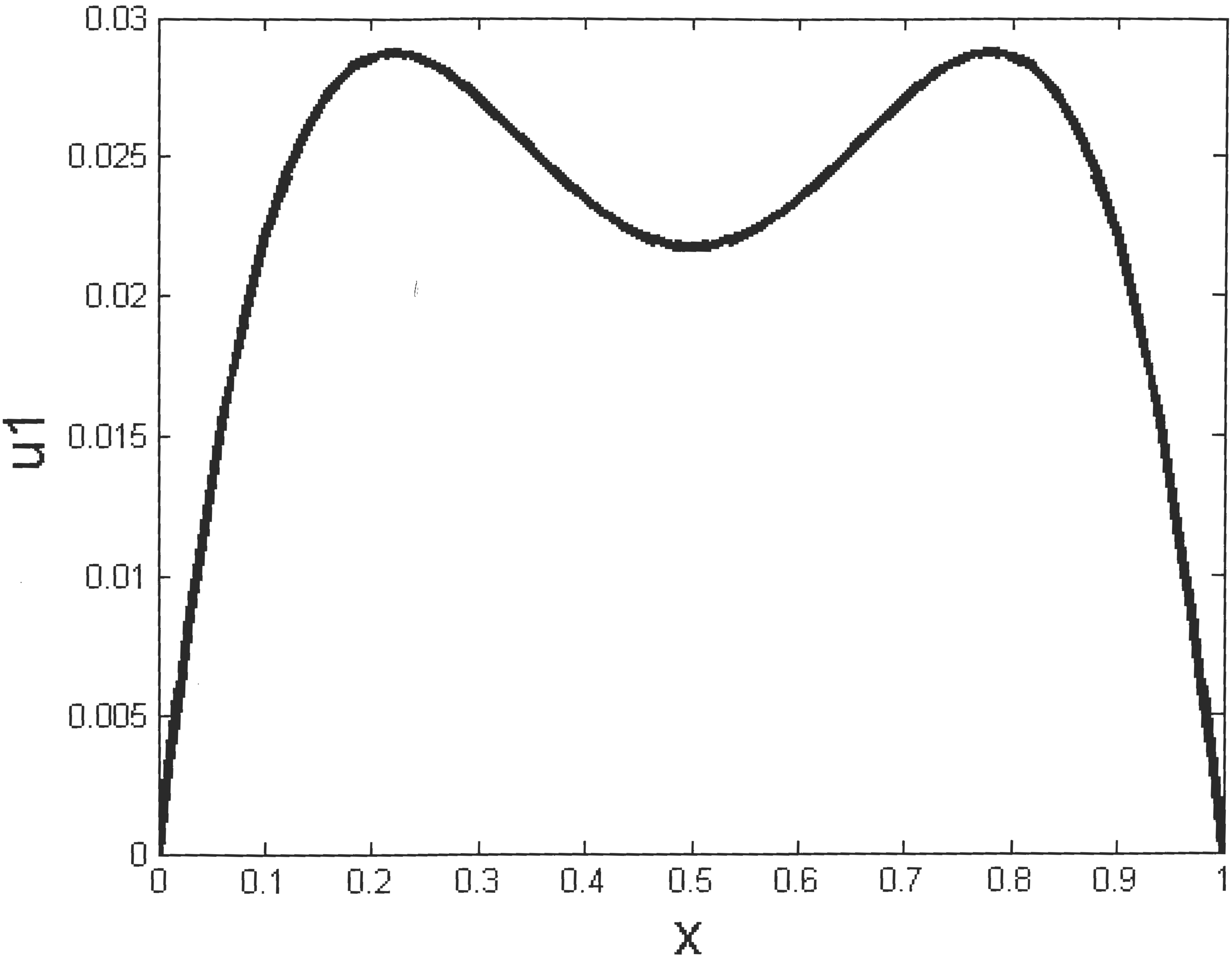
$$u_1(x) = \frac{1}{\pi^2}\sin\pi x + \frac{1}{2\pi}\cos\pi x - \frac{1}{\pi}x\cos(\pi x) - \frac{1}{\pi}x^2 + bx + a.$$

$$u_1(0) = 0 \Rightarrow a = -\frac{1}{2\pi}$$

$$u_1(1) = 0 \Rightarrow b = 1/\pi.$$

See p. 15 for a graph of $u_1(x)$.

15/



So by making the rod thinner in the middle ($\varepsilon<0$) you can lower its maximum temperature in the uniformly heated case.

Note: It is often difficult to prove that $u(x)$ is approximated accurately by its perturbation series or to know how small ε must be for reasonable accuracy. However, this is an amazingly useful technique that often works well.

Math 401 Notes - II

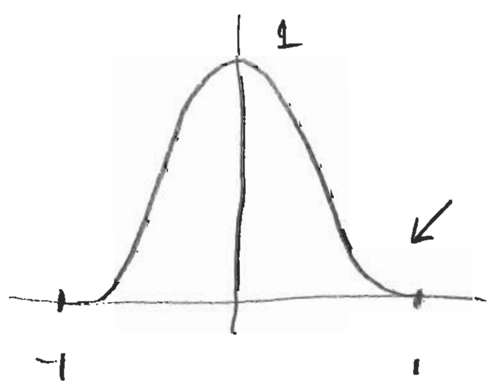
Green's Functions for ODE BVP's.

§2.1 Generalized Functions.

Defn A test function ϕ is infinitely differentiable and is zero outside a finite interval (ϕ is said to have compact support). C_0^∞ denotes the set of test functions.

Ex 1 $$\phi(x) = \begin{cases} e^{-\frac{x^2}{1-x^2}} & |x| < 1 \\ 0 & |x| \geq 1 \end{cases}$$

is in C_0^∞



would need to show

$\phi^{(n)}(1^-) = 0$ for all $n \geq 0$.

↑ n'th derivative

Let $A = \int_{-1}^{1} \phi(x)\,dx$ and

$$\psi(x) = \frac{1}{A}\phi(x).$$

(so that $\int_{-1}^{1} \psi(x)\,dx = \int_{-\infty}^{\infty} \psi(x)\,dx = 1$)

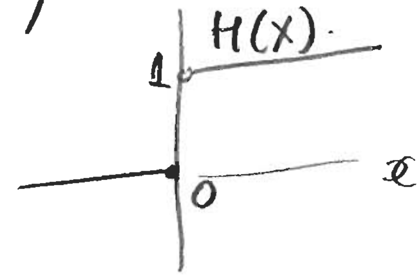
Consider $\psi_\varepsilon = \frac{1}{A\varepsilon}\psi\left(\frac{x}{\varepsilon}\right)$ ← Scaled ψ. (1)

$\varepsilon \to 0$, Ψ_ε narrower & thinner, preserves

$$\int_{-\infty}^{\infty} \Psi_\varepsilon(x)\,dx = 1$$

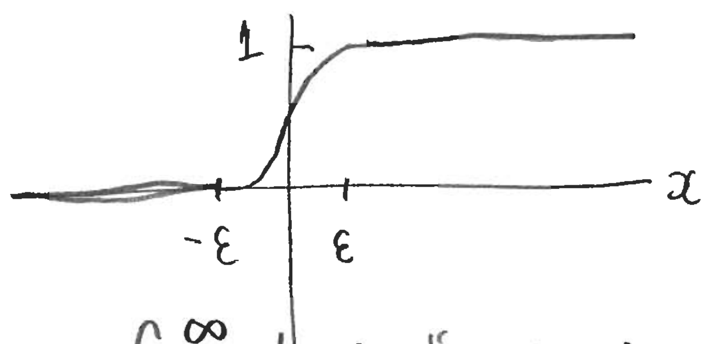
The functions $\Psi_\varepsilon(x)$ are examples of mollifiers (C_0^∞, positive, $\int \Psi_\varepsilon(x)dx = 1$)

Consider $H(x) = \begin{cases} 1 & x > 0 \\ 0 & x \leq 0. \end{cases}$



Consider $H_\varepsilon = H * \Psi_\varepsilon = \int_{-\infty}^{\infty} H(s)\, \Psi_\varepsilon(x-s)\,ds.$

($*$: convolution)



H_ε is in C^∞, but "approximates" $H(x)$, a useful tool for analysis.

<u>Defn</u> A generalized function is a linear functional on C_0^∞

<u>Ex</u> 2 Any integrable function $f(x)$ defines a linear functional T_f in the following way

$$T_f(\underset{\text{test fn}}{\phi}) = \int_{-\infty}^{\infty} f(x)\,\phi(x)\,dx \qquad (2)$$

Linearity $T_F(a\phi + b\psi) = a\, T_F(\phi) + b\, T_F(\psi)$

scalars

follows from the linearity of the integral.

But, not every generalized function can be written in this way.

Ex 3 $T_\delta(\phi) := \phi(0)$ is also linear.

defined

Even though this linear functional cannot be written in the form (2), we write formally that this T corresponds to the function

$\delta(x)$ ← Dirac δ-function

Defn The sequence of generalized functions $\{T_\varepsilon\}$ is said to converge weakly to T if

$$\lim_{\varepsilon \to \infty} T_\varepsilon(\phi) = T(\phi) \text{ for every } \phi \in C_0^\infty.$$

Ex 4 T_{ψ_ε} converge weakly to T_δ as $\varepsilon \to 0$,

So, even though T_δ can't be represented by integration with an actual function, it can be thought of as a limit of actual functions.

Ex 5 if $T_M(\phi) = \int_{-\infty}^{\infty} \cos(Mx)\, \phi(x)\, ds$

then $\lim_{M \to \infty} T_M(\phi) = 0$ for every $\phi \in C_0^\infty$,

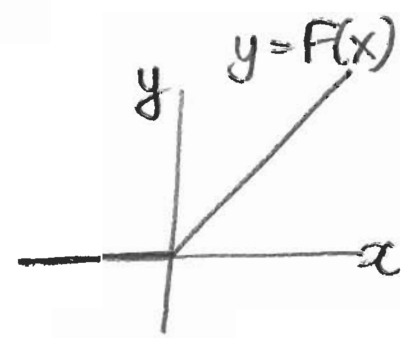
so weak convergence doesn't mean pointwise convergence, even for actual functions.

Differentiation of generalized functions can be defined using integration by parts.

$$\int_{-\infty}^{\infty} f'(x)\, \phi(x)\, dx = \cancel{f\phi\Big|_{-\infty}^{\infty}} - \int_{-\infty}^{\infty} f(x)\, \phi'(x)\, dx$$

ϕ has compact support

Thus, $T_{f'}(\phi) = -T_f(\phi')$ defines the derivative f' of the generalized function f.



Ex 6 If $F(x) = \begin{cases} x & x > 0 \\ 0 & x \le 0 \end{cases}$

then $F'(x) = H(x)$ [as a generalized function, the undefined value of $F'(0)$ does not matter].

Consider H' as a generalized function

$$T_{H'}(\phi) = -\int_{-\infty}^{\infty} H(x)\, \phi'(x)\, dx$$
$$= -\int_0^{\infty} \phi'(x)\, dx = -\cancel{\phi(\infty)}^{\,0} + \phi(0) = \phi(0).$$

This shows that $H' = \delta$, as generalized functions.

§ 2.2 Adjoint operators.

Consider the following problem for $u(x)$, $x \in [0,1]$ given $f(x)$:

$$\left.\begin{aligned}
&a(x)\, u'' + b(x) u' + c(x) u = f(x) \\
&\alpha_0 u(0) + \beta_0 u'(0) = 0 \\
&\alpha_1 u(1) + \beta_1 u'(1) = 0 \\
&a(x) \neq 0,\ b(x), c(x),\ \alpha\text{'s},\ \beta\text{'s given} \\
&\ (\alpha_j, \beta_j \text{ not both zero for } j = 0,1)
\end{aligned}\right\} \begin{array}{c}\mathbb{L} \\ \text{problem.}\end{array}$$

Summarize the problem as

$\mathbb{L} u = f$ "function" which has a fn as an argument & returns a fn.

linear boundary value operator - it includes the boundary conditions. If this problem is ! (uniquely) solvable for every f, we would write

$$u = \mathbb{L}^{-1} f.$$

Reminder: We want to find the Green's fn representation of $\mathbb{L}^{-1}$.

The adjoint of $\mathbb{L}$ ($\mathbb{L}^*$) is like the transpose of $\mathbb{L}$. It is also a differential operator with BC's.

Analogy: matrix $\mathbb{A}$, transpose $\mathbb{A}^T$ satisfies

$$(\underline{v}, \mathbb{A}\underline{u}) = (\mathbb{A}^T\underline{v}, \underline{u}) \quad \text{for all vectors } \underline{u}, \underline{v} \qquad (3)$$

Using notation $(\underline{x}, \underline{y}) = \underline{x} \cdot \underline{y} = \underline{x}^T \underline{y}$

Functions on $[0,1]$ have an inner product (L_2):

$$(u,v) = \int_0^1 u(x)v(x)dx$$

In analogy to (3), the adjoint $\mathbb{L}^*$ will satisfy

$$(v, Lu) = (L^* v, u) \qquad \text{for all } u, v.$$

$$\int_0^1 vLu\,dx = \int_0^1 v\left[au'' + bu' + cu\right]dx$$

integrate 1st & 2nd terms by parts.

$$= \int_0^1 \left[(-av)'u' - (bv)'u + cuv\right]dx + (avu' + buv)\Big|_0^1$$

integrate 1st term by parts.

$$= \int_0^1 \left[(av)'' - (bv)' + cv\right] u\,dx + \left(avu' + buv - (av)'u\right)\Big|_0^1$$

so the differential equation part of $\mathbb{L}^*$ is

$$\mathbb{L}^* v = (av)'' - (bv)' + cv$$
$$= av'' + (2a' - b)v' + (a'' - b' + c)v \qquad (4)$$

What about the BC's?

$$\left(a v u' + b u v - a' v u - a v' u\right)\Big|_0^1 = 0 \qquad (5)$$

Note: it can be shown that the $x=1$ & $x=0$ parts must both be zero. Consider $x=0$:

$$\alpha_0 u(0) + \beta_0 u'(0) = 0$$

Other than satisfying this condition, $u(0)$ and $u'(0)$ are arbitrary, i.e.

$$u(0) = -k\beta_0, \quad u'(0) = k\alpha_0, \quad \text{with } k \text{ arbitrary.}$$

Put this info in (5), $x=0$:

$$k\left(a(0)v(0)\alpha_0 - b(0)\beta_0 v(0) + a'(0)v(0)\beta_0 + a(0)v'(0)\beta_0\right) = 0$$

or $\hat{\alpha}_0 v(0) + \hat{\beta}_0 v'(0) = 0 \qquad (6)$

with

$$\hat{\alpha}_0 = a(0)\alpha_0 + (a'(0) - b(0))\beta_0$$

$$\hat{\beta}_0 = a(0)\beta_0$$

and similarly, $\hat{\alpha}_1 v(1) + \hat{\beta}_1 v'(1) = 0 \qquad (7)$

with

$$\hat{\alpha}_1 = a(1)\alpha_1 + (a'(1) - b(1))\beta_1$$

$$\hat{\beta}_1 = a(1)\beta_1.$$

Note: The problem for $\mathbb{L}^*$ (4, 6, 7) has the same form as that of the $\mathbb{L}$ problem.

§2.3 Green's functions for ODE BVP's.

Want $u(x) = \int_0^1 G(s,x) f(s)\,ds$ to be the solution of the $\mathbb{L}$ problem.

$$u(x) = \int_0^1 G(s,x)\, \mathbb{L} u \, ds$$

$$= \int_0^1 \mathbb{L}^* [G(\cdot,x)]\, u \, ds.$$

↑ adjoint, in s variables.

so $\mathbb{L}^* [G(\cdot,x)] = \delta(s-x)$. (8).

§2.4 Self adjoint operators

Let's concentrate on an important class of problems where $\mathbb{L} = \mathbb{L}^*$ (including BC's), $\mathbb{L}$ is said to be self-adjoint.

$$\mathbb{L}(u) = a u'' + b u' + c u$$

(4) $\Rightarrow$ $\mathbb{L}^*(v) = a v'' + (2a' - b) v' + (a'' - b' + c) v$

$\mathbb{L} = \mathbb{L}^*$ iff $\boxed{a' = b}$ for all x, then

$$\mathbb{L} u = (a u')' + c u$$

(like the form of the heat conduction problem with variable area).

In the case of the BC's of the $\mathbb{L}$ problem of §2.2, the boundary conditions of $\mathbb{L}^*$ are also the same as those for $\mathbb{L}$ if $\boxed{a' = b}$.

§2.5 Green's function for problem ⊛

$$u'' = f \qquad u(0) = u(1) = 0.$$

Self adjoint, want (8) ⇒

$$\frac{d^2}{ds^2} G(s,x) = \delta(s-x)$$

$$G(0,x) = G(1,x) = 0.$$

For $s<x$ and $s>x$, $\frac{d^2}{ds^2} G(s,x) = 0$, so it must be straight line segments. It must be continuous at $s=x$ (to make sense of the integration by parts in Eq (8)), but with a jump in slope of 1.

$$G(s,x) = \begin{cases} A\,s & s < x \\ (A+1)(s-1) & s > x. \end{cases}$$

satisfies BC's and slope jump. Solve for A to make G continuous at $s=x$:

$$Ax = (A+1)(x-1) \Rightarrow A = x-1.$$

so $G(s,x) = \begin{cases} (x-1)s & s<x \\ x(s-1) & s>x \end{cases}$

as stated in Ch I. You prove this works in Assignment I.

Note: $G(s,x) = G(x,s)$ for self-adjoint problems.

§ 2.6 A second example of a Green's fn is for the self adjoint problem

$$-u'' + u = f \qquad u(0) = u(1) = 0.$$

This describes heat conduction in a rod where heat transfers to an ambient temperature along the length of the rod.

Green's function:

$$-G_{ss} + G = \delta(x-s) \qquad (9)$$

$$G(0,x) = G(1,x) = 0$$

away from $s = x$, $-G_{ss} + G = 0$. At $s=x$,

$\underbrace{G(x+,x) - G(x-,x)}_{\text{Notation } [G(s=x,x)]} = 0$ ie. G is continuous

denotes "jump"

$$[G_s(s=x,x)] = -1$$ to give the δ RHS of (9).

On either side of $s = x$,

$$G = Ae^{s} + Be^{-s}$$

Actually, it's better to use hyperbolic trig functions

$$G = \begin{cases} A \sinh s & s < x \\ B \sinh(s-1) & s > x. \end{cases}$$

satisfies the equations and the BC's, A & B are chosen to match the 2 jump conditions (continuity & derivative jump at the bottom of the previous page)

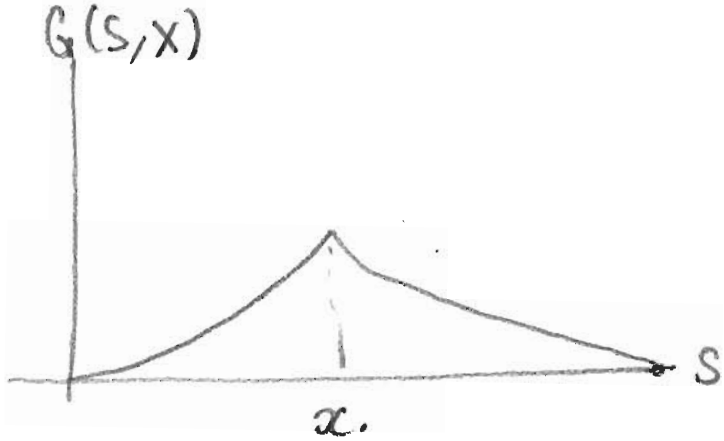
$$A \sinh x = B \sinh(x-1) \qquad \text{continuity}$$

$$A \cosh x - 1 = B \cosh(x-1). \qquad \text{derivative jump.}$$

solving for A & B (using hyperbolic trig identities) gives.

$$A = \frac{\sinh(1-x)}{\sinh(1)}, \quad B = -\frac{\sinh x}{\sinh 1}$$

or
$$G = \begin{cases} \dfrac{\sinh(1-x)\sinh s}{\sinh 1} & s < x \\ \dfrac{\sinh x \sinh(1-s)}{\sinh 1} & s > x. \end{cases}$$



Makes physical sense.

§2.7 Some Theory.

If the $\mathbb{L}$ problem is self adjoint then

(i) The normalized eigenfunctions $\{\mu_j\}_{j=1}^{\infty}$, $\|\mu_j\|^2 := \int_0^1 [\mu_j(x)]^2 dx = 1$, correspond to real eigenvalues and form an o.n. basis for functions on $[0,1]$, i.e. every function $f(x)$ (for which $\int_0^1 |f(x)|^2 dx$ exists) can be written as

$$f(x) = \sum_{j=1}^{\infty} f_j \, \mu_j(x)$$

where $f_j = \int_0^1 f(x)\, \mu_j(x)\, dx$.

If in addition the only solution of $\mathbb{L} u = 0$ is $u \equiv 0$, the following hold:

(ii) $\mathbb{L}$ is invertible

(iii) $\mathbb{L}^{-1}$ can be represented by a Green's function.

§2.8 Eigenanalysis of $\mathbb{L}$.

If $\mathbb{L}\,\mu(x) = \lambda\,\mu(x)$ [including BC's].

and we exclude $\lambda = 0$ so that $\mathbb{L}$ is invertible as above, then

$$\mu(x) = \int_0^1 G(s,x)\,\lambda\,\mu(s)\,ds$$

or $\frac{1}{\lambda}\mu = \mathbb{L}^{-1}\mu$, i.e. μ is also an eigenfunction of $\mathbb{L}^{-1}$ with eigenvalue $\frac{1}{\lambda}$.

Ex 7 Prob ⊛ $[\,u'' = \lambda u\,,\ u(0) = 0,\ u(1) = 0\,]$

has ~~eigenvectors~~ $\lambda_k = -(k\pi)^2$

eigenfunctions $\mu_k(x) = \frac{1}{\sqrt{2}}\sin(k\pi x)$

↑ normalizing factor.

So the theory §2.7 (i) proves that a sine series can be used to represent functions. This theory justifies all the separation of variables done in earlier courses.

Q 2.9 Consider the self-adjoint IL problem.
Assume it is invertible.

$$(a(x)u')' + c(x)u = f(x) \quad , \quad a(x) \neq 0$$

$$\alpha_0 u(0) + \beta_0 u'(0) = b_1$$

$$\alpha_1 u(1) + \beta_1 u'(1) = b_2$$

α_j & β_j not both zero, $j=0,1$.

$u \equiv 0$ if $f \equiv 0$.

Q: what if these were not zero?

Let $u\{f, b_1, b_2\}$ denote the solution above.
By linearity,

$$u\{f, b_1, b_2\} = u\{f, 0, 0\} + \boxed{u\{0, b_1, 0\}} + u\{0, 0, b_2\}$$

problem we have G's fn's for.

Solves $(au')' + cu = 0$ — general homog soln

$\alpha_0 u(0) + \beta_0 u'(0) = b_1$ $u(x) = Au_1(x) + Bu_2(x)$

$\alpha_1 u(1) + \beta_1 u'(1) = 0.$

plug in, get a linear system for A, B that is ! solvable by invertibility condition.

Let $V(x)$ be the solution of the problem above with $b_1 = 1$, then

$$u\{0, b_1, 0\} = b_1 \underbrace{V(x)}$$

Like the Green's function for the left boundary condition.

In 2-dimensions (2D) a boundary is a curve, 3D a surface, and the boundary Green's functions are more interesting.

§2.10 An example in an unbounded domain.

$$-u'' + u = f(x) \qquad u(x), \quad x \in (-\infty, \infty).$$

$$\lim_{x\to\infty} u(x) = 0, \quad \lim_{x\to -\infty} u(x) = 0. \qquad \text{(BC's)}.$$

(infinite rod with heat loss to ambient)

Formally, $u = \int_{-\infty}^{\infty} G(s,x) f(s)\,ds. \qquad (10).$

$$-G_{ss} + G = \delta(s-x). \qquad \text{[self adjoint]}.$$

$G = Ae^{s} + Be^{-s} \qquad s < x$

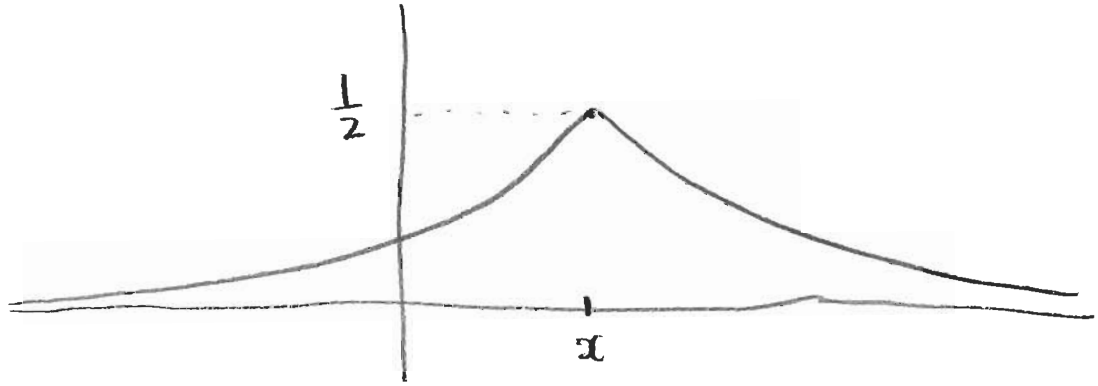
(also $s > x$, different constants).

$$G = \begin{cases} A\, e^{(s-x)} & s < x \\ A\, e^{-(s-x)} & s > x. \end{cases}$$

Satisfies the BC's & Continuity.

$$G_s(x_+) - G_s(x_-) = -1 \Rightarrow$$

$$A = \frac{1}{2}.$$



Note: $G(s,x) = \hat{G}(s-x)$ and (10) has the form of a convolution.

$$\hat{G}(y) = \begin{cases} \frac{1}{2} e^{y} & y < 0 \\ \frac{1}{2} e^{-y} & y > 0 \end{cases}$$

§2.11 Green's Function representation in terms of eigenfunctions.

$$u(x) = \sum_{j=1}^{\infty} u_j \mu_j(x), \quad f(x) = \sum_{j=1}^{\infty} f_j \mu_j(x)$$

by completeness of eigenfunctions.

$$f_j = \int_0^1 f(s) \mu_j(s) \, ds.$$

$$\mathbb{L}u = \sum_{j=1}^{\infty} u_j \lambda_j \mu_j(x) \qquad \mathbb{L}u = f \Rightarrow$$

$$u_j = \frac{f_j}{\lambda_j}.$$

So now $u(x) = \sum_{j=1}^{\infty} \frac{\left[\int_0^1 f(s)\,\mu_j(s)\,ds\right]}{\lambda_j}\,\mu_j(x)$.

exchange sum and integral,

$$G(s,x) = \sum_{j=1}^{\infty} \frac{\mu_j(s)\,\mu_j(x)}{\lambda_j}.$$

Note: For constant coefficient problems with no boundaries, the fourier transform plays a similar role and it is often possible to determine $\tilde{G}(s-x) = G(s,x)$ from its transform.

Math 401 Notes - III

Classification of 2D, 2nd order PDE's

General, linear constant coefficient PDE in 2 space variables $\underline{x} = (x_1, x_2)$:

$$a\frac{\partial^2 u}{\partial x_1^2} + 2b\frac{\partial^2 u}{\partial x_1 \partial x_2} + c\frac{\partial^2 u}{\partial x_2 \partial x_2} + d\frac{\partial u}{\partial x_1} + e\frac{\partial u}{\partial x_2} = f(\underline{x})$$

or
$$\sum_{i,j=1}^{2} A_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{2} L_i \frac{\partial u}{\partial x_i} = f(\underline{x}) \qquad (1)$$

assume not invertible

where $A_{11} = a$, $A_{12} = A_{21} = b$, $A_{22} = c$

$L_1 = d$, $L_2 = e$.

orthonormal change of coordinates

$$\underline{y} = \mathbb{P}\underline{x} \qquad \mathbb{P}^T = \mathbb{P}^{-1}$$

$$\frac{\partial u}{\partial x_i} = \sum_{k=1}^{2} \frac{\partial u}{\partial y_k}\frac{\partial y_k}{\partial x_i} P_{ki}$$

$$\frac{\partial^2 u}{\partial x_i \partial x_j} = \sum_{k=1}^{2}\sum_{l=1}^{2} \frac{\partial^2 u}{\partial y_k \partial y_l} P_{ki} P_{lk}$$

So now (1) reads (2)

$$\sum_{i,j}\sum_{k,l} A_{ij} P_{ki} P_{lj} \frac{\partial^2 u}{\partial y_k \partial y_l} + \sum_{i=1}^{2}\sum_{k=1}^{2} P_{ki} L_i \frac{\partial u}{\partial y_k} = f(\underline{x})$$

Note that if $\mathbb{C} = \mathbb{A}\mathbb{B}$,

$$C_{ik} = \sum_j A_{ij} B_{jk}.$$

so

$$\sum_{i,j} A_{ij} P_{ki} P_{\ell j} = C_{k\ell} \quad \text{where } \mathbb{C} = \mathbb{P}\mathbb{A}\mathbb{P}^T = \mathbb{P}\mathbb{A}\mathbb{P}^{-1}.$$

$\mathbb{A}$ is symmetric and so can be diagonalized by an orthogonal matrix:

$\mathbb{P}$ = matrix of normalized eigenvectors in columns, then

$$\mathbb{P}\mathbb{A}\mathbb{P}^{-1} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \qquad \lambda\text{'s eigenvalues of } \mathbb{A}.$$

Thus (2) becomes (3)

$$\lambda_1 \frac{\partial^2 u}{\partial y_1^2} + \lambda_2 \frac{\partial^2 u}{\partial y_2^2} + \hat{a} \frac{\partial u}{\partial y_1} + \hat{b} \frac{\partial u}{\partial y_2} = \underline{f}(\mathbb{P}\underline{x}).$$

$$\hat{a} = \sum_{i=1}^{2} P_{1i} L_i \qquad \hat{b} = \sum_{i=1}^{2} P_{2i} L_i.$$

Three cases:

(i) λ_1 & λ_2 have the same sign, (1) is called elliptic. Can further scale y_1 & y_2 to x & y so that

$$u_{xx} + u_{yy} + a u_x + b u_y = g(x,y).$$

When $a = b = 0$ this is called Poisson's equation.

(ii) λ_1 & λ_2 are opposite signs, (1) is called hyperbolic. Again scaling,

$$u_{xx} - u_{yy} + a u_x + b u_y = g(x,y)$$

x time $\quad$ y space.

When $a = b = 0$ (and $g \equiv 0$) this is called the (unforced) wave equation. Further change of variables $\eta = x + y$, $\xi = x - y$ $\Rightarrow$

$$u_{\eta\xi} = 0 \qquad \text{(unforced)}$$

This form leads to D'Alembert's solution.

(iii) If $\lambda_2 = 0$ (exclude also $\lambda_1 = 0$ since then (1) is not second order) assume $\hat{b} \neq 0$ (otherwise there is no differentiation in y_2 and the problem is a family of ODEs). Then (1) is called parabolic. Again scaling

$$-u_{xx} + u_y + a u_x = g(x,y).$$

When $a = 0$ this is called the heat equation.

Note: $\lambda_1 \lambda_2 = \det \mathbb{A} = \begin{vmatrix} a & b \\ b & c \end{vmatrix} = ac - b^2$

remember, b is half the coefficient of the $\frac{\partial^2 u}{\partial x_1 \partial x_2}$ term

so it's easy to determine the class from the original equation (1).

Note: The sign of $\det \mathbb{A}$ determines the local behaviour even when $a(\underline{x}), b(\underline{x}), c(\underline{x})$. Equations that change type in $\underline{x}$ are tricky (potential in transonic flow).

Ex Classify the PDE

$$\frac{\partial^2 u}{\partial x^2} + 4 \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = f.$$

$2b$, so $b = 2$.

$\det \mathbb{A} = 1 - 4 = -3$ so hyperbolic.

Let's look at Green's functions for these 3 important equations.

Math 401 Notes - IV

Green's Functions for PDE's.

§4.1 Heat equation

$$u_t = u_{xx} + f(x,t) \quad (1) \qquad \lim_{|x|\to\infty} u(x,t) = 0$$
$$u(x,0) = g(x). \qquad \text{for all } t.$$

u is a scaled temperature, and the equation describes heat transfer in an infinite rod with conduction (u_{xx}), external heating (f), and initial temperature (g).

As at the end of Ch 2, we can write

$$u\{f,g\} = \underbrace{u\{f,0\}}_{①} + \underbrace{u\{0,g\}}_{②}.$$

Let's look at ② first. For each t, write u in terms of its Fourier transform

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{u}(\alpha,t)\, e^{i\alpha x}\, d\alpha$$

$$\text{where } \hat{u}(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,t)\, e^{-i\alpha x}\, dx$$

Formally plug into the PDE (1), equate Fourier coefficients:

$$\frac{\partial}{\partial t}\hat{u}(\alpha,t) = -\alpha^2\,\hat{u}(\alpha,t)$$

$$\hat{u}(\alpha,0) = \hat{g}(\alpha).$$

Note that above there is no differentiation in α, so this is a family of ODE's, easy to solve.

$$\hat{u}(\alpha,t) = e^{-\alpha^2 t}\,\hat{g}(\alpha) \qquad (2).$$

Note: high wavenumber components of the initial data decay rapidly - physically reasonable.

Note: The backward heat equation $u_t = -u_{xx}$ has solutions $\hat{u}(\alpha,t) = e^{\alpha^2 t}\,\hat{g}(\alpha)$.

$\uparrow$

unbounded growth rate, solutions do not exist in general for $t>0$.

this problem is called ill-posed.

From (2), we see that u is obtained by convolving g with $C(x,t)$ where

$$\hat{C}(\alpha) = \frac{1}{\sqrt{2\pi}}\,e^{-\alpha^2 t}$$

Note: In the online Fourier Transform notes, there is a scaling factor missing in the convolution section 2.1.2

Standard tables give

$$C(x,t) = \frac{1}{2\sqrt{\pi t}}\, e^{-x^2/4t}.$$

and so

$$u(x,t) = \int_{-\infty}^{\infty} \underbrace{\frac{1}{2\sqrt{\pi t}}\, e^{-(s-x)^2/4t}} g(s)\, ds. \qquad (3).$$

This is the Green's fn for problem ② $G_2(t, s-x)$.

Note: $u(x,t)$ is C^∞ for all $t>0$ even if g is discontinuous.

Note: We would say that

$$\lim_{t\to 0} \frac{1}{2\sqrt{\pi t}}\, e^{-x^2/4t} = \delta(x) \qquad \text{(weakly)}.$$

(3) does not make sense at $t=0$, but has the right limiting behaviour as $t \to 0^+$.

There is a trick (Duhamel's principle) to find the Green's function for the interior forcing (f) from the one for problem ② given in (3).

$$\left.\begin{aligned} u_t &= u_{xx} + f(x,t) \\ u(x,0) &\equiv 0 \end{aligned}\right\} \quad (4).$$

Let $\phi(x,t,\tau)$ solve the following problem

$$\phi(x,\tau,\tau) = f(x,\tau)$$

$$\phi_t = \phi_{xx} \qquad t > \tau$$

This is just a version of (2) shifted in time and so

$$\phi(x,t,\tau) = \int_{-\infty}^{\infty} G_2(t-\tau, s-x) f(s,\tau)\,ds.$$

We will show that $u(x,t) = \int_0^t \phi(x,t,\tau)\,d\tau$ (5).
solves (1), written explicitly in (4).

$u(x,0) = 0$ ✓.

$$u_t = \phi(x,t,t) + \int_0^t \phi_t(x,t,\tau)\,d\tau$$

$$u_{xx} = \int_0^t \phi_{xx}(x,t,\tau)\,dx$$

$$u_t - u_{xx} = \underbrace{\phi(x,t,t)}_{f(x,t)} + \int_0^t \overbrace{\cancel{(\phi_t - \phi_{xx})}}^{0 \text{ by construction}}\,d\tau = f(x,t) ✓$$

From (5) we can write the solution of (1) as

$$u(x,t) = \int_{-\infty}^{\infty}\int_0^{\infty} G_1(s,x,t,\tau)\, f(s,\tau)\,d\tau\,ds$$

where $G_1 = \tilde{G}_1(s-x, t-\tau)$ and

$$\tilde{G}_1(\eta, \xi) = \begin{cases} \dfrac{1}{2\sqrt{\pi\xi}} e^{-\eta^2/4\xi} & \text{if } \xi > 0 \\ 0. & \text{if } \xi \leq 0. \end{cases}$$

Note: $\tilde{G}_1 = 0$ for $\xi \leq 0$ $(\tau > t)$ is an expression of causality, that the future data cannot affect the past temperature.

Note: The idea above in (4) [Duhamel's principle] is the same basic idea as Assignment I, #1.

§4.2 Wave Equation

$$\begin{aligned} u_{tt} - u_{xx} &= f(x,t) \\ u(x,0) &= g(x,t) \\ u_t(x,0) &= h(x,t) \end{aligned} \qquad \begin{aligned} &u, f, g, h \to 0 \\ &\text{as } |x| \to \infty. \end{aligned}$$

This describes the vertical displacement u of a wire under tension, with external forces (f), initial displacement (g), initial velocity (h), after linearization & scaling.

As above, write

$$u\{f,g,h\} = \underset{①}{u\{f,0,0\}} + \underset{\text{② plucked string}}{u\{0,g,0\}} + \underset{\text{③ struck string}}{u\{0,0,h\}}$$

We'll do ② & ③ first, then use Duhamel's principle to find ① from ③. We could proceed as before, using the Fourier transform, identifying the Green's function as a convolution in space for each time. But, let's use a different technique here.

Recall: $u_{tt} - u_{xx} = u_{\eta\xi}$ where $\eta = x-t$

$\xi = x+t.$

For ② & ③ above, $u_{\eta\xi} = 0$ so

$$(6) \begin{cases} u = F(\eta) + G(\xi) \\ \quad = \underbrace{F(x-t) + G(x+t)}_{\text{waves to the right and left with unit speed}} \end{cases}$$

$F(\cdot)$, $G(\cdot)$ are functions of one variable to be determined by matching initial conditions

This is known as D'Alembert's formula.

For problem ②,

$u(x,0) = f(x)$ $\qquad F(x) + G(x) = f(x) \qquad (7)$

$u_t(x,0) = 0$ $\qquad -F'(x) + G'(x) = 0$

integrate $\qquad -F(x) + G(x) = C \qquad (8)$

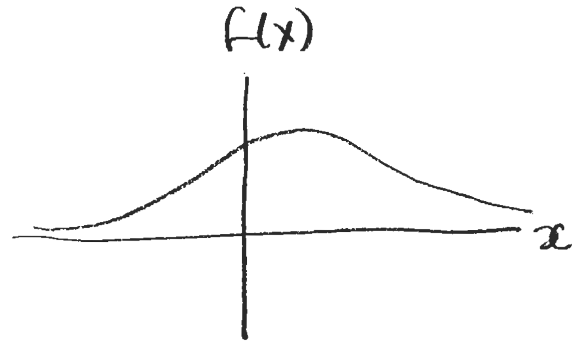
Solve (2) & (3) for each x:

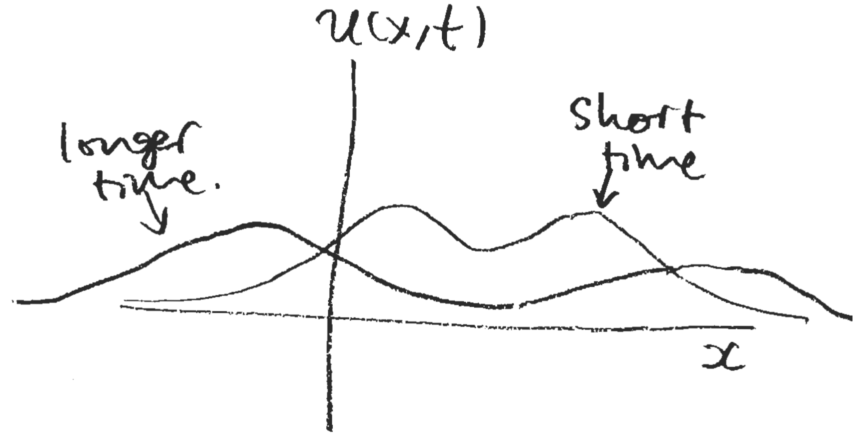
$$F(x) = \tfrac{1}{2} f(x) - C/2$$
$$G(x) = \tfrac{1}{2} f(x) + C/2$$

} putting back into (6), C cancels out, neglect it

$$u(x,t) = \tfrac{1}{2} f(x-t) + \tfrac{1}{2} f(x+t) \qquad (9)$$

The initial displacement divides into 2 halves, one travelling to the right, the other to the left.





We can just read off the Green's function from (9)

$$u(x,t) = \int_{-\infty}^{\infty} G_2(t, s-x)\, f(s)\, ds$$

$$G_2(t,y) = \tfrac{1}{2}\,\delta(y+t) + \tfrac{1}{2}\,\delta(y-t)$$

<u>Note</u>: This is very different from the behaviour of the heat equation in §4.1, where the solution at (x,t) depended on <u>all</u> initial data.

<u>Note</u>: Form (7) makes sense even if f is discontinuous, although then $u(x,t)$ cannot

actually solve the WE. You can make sense of this by taking the limit of mollified f's.

Move on to 3 using form (6)

$$u(x,0)=0 \qquad F(x)+G(x)=0 \qquad (10)$$

$$u_t(x,0)=h(x) \qquad -F'(x)+G'(x)=h(x)$$

Integrate $\qquad -F(x)+G(x) = \int_{-\infty}^{x} h(s)\,ds := \mathcal{H}(x). \quad (11)$

of (10, 11)

solution, $F(x) = -\frac{1}{2}\mathcal{H}(x)$, $G(x) = \frac{1}{2}\mathcal{H}(x)$

$$u(x,t) = -\frac{1}{2}\mathcal{H}(x-t) + \frac{1}{2}\mathcal{H}(x+t)$$

$$= \frac{1}{2}\int_{x-t}^{x+t} h(s)\,ds$$

Now this has the form

$$u(x,t) = \int_{-\infty}^{\infty} G_3(t, s-x)\, h(s)\,ds \qquad (12)$$

with $G_3(t,y) = \begin{cases} 1/2 & \text{if } -t \le y \le t \\ 0 & |y| > t. \end{cases}$

Proceed to 1 using Duhamel's principle.

Let $\phi(x,t,\tau)$ satisfy

$$\phi_{tt} - \phi_{xx} = 0 \quad t > \tau, \quad \phi(x,\tau,\tau) = 0, \quad \phi_t(x,\tau,\tau) = f(x,t).$$

This is just a version of ③ shifted in time, whose solution can be written in the form (12):

$$\phi(x,t,\tau) = \int_{-\infty}^{\infty} G_3(t-\tau, s-x)\, f(s,\tau)\, ds.$$

We will show that $u(x,t) = \int_0^t \phi(t,\tau,x)\, d\tau$ solves ①:

$u(x,0) = 0$ ✓.

$$u_t = \overbrace{\phi(x,t,t)}^{0} + \int_0^t \phi_t(x,t,\tau)\, d\tau$$

$u_t(x,0) = 0$ ✓.

$$u_{tt} = \underbrace{\phi_t(x,t,t)}_{f(x,t)} + \int_0^t \phi_{tt}(x,t,\tau)\, d\tau.$$

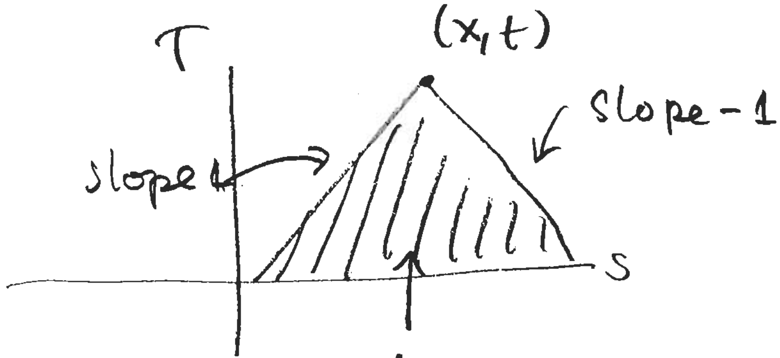
$$u_{xx} = \int_0^t \phi_{xx}(x,t,\tau)\, d\tau.$$

since $\phi_{tt} = \phi_{xx}$ for each τ, t, x we see that $u_{tt} = u_{xx} + f(x,t)$ ✓

The $u(x,t)$ that solves ① can be written as

$$u(x,t) = \int_0^{\infty}\int_{-\infty}^{\infty} G_1(s,x,\tau,t)\, f(s,\tau)\, ds\, d\tau$$

where $G(s,x,\tau,t)$ is $\frac{1}{2}$ in the (s,τ) domain below, and zero otherwise:



called a domain of dependence.

One final observation:

Consider problem ③: $u_{tt} = u_{xx}$, $u(x,0)=0$,
$u_t = h(x)$

Consider $v = u_t$, also satisfies $v_{tt} = v_{xx}$ (take $\frac{\partial}{\partial t}$ of $u_{tt} = u_{xx}$). Also,

$$v(x,0) = h(x)$$

$$v_t(x,0) = u_{tt}(x,0) = u_{xx}(x,0) = 0.$$

So if u solves ③ then u_t solves ②. That means, $G_2 = \frac{\partial}{\partial t} G_3$, which is true (in the generalized sense) as seen above.

§4.3 Poisson's Equation

$$u_{xx} + u_{yy} = f(x,y).$$

u describes temperature in a plate due to external heating (f) and heat conduction (all scaled).

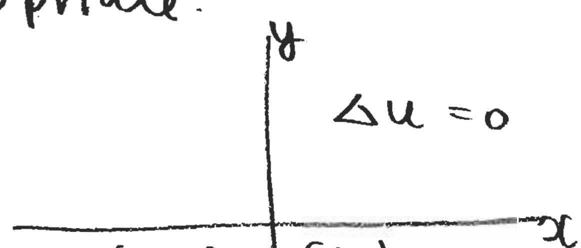
Notation: $\Delta u = u_{xx} + u_{yy}$

or $\nabla^2 u = u_{xx} + u_{yy}$

because $\nabla u = (u_x, u_y)$, $\nabla \cdot (\nabla u) = u_{xx} + u_{yy}$.

Δ is called the Laplacian. If $\Delta u = 0$ in a domain, u is said to be harmonic on that domain.

It will be shown below that neither x nor y (they appear symmetrically) can be considered to be time like variables. Consider the case where y is taken to be time-like and the following Initial Value Problem (IVP) would be appropriate:



$u(x,0) = f(x)$
$u_y(x,0) = h(x)$.

Fourier transform in x ...

$$\hat{u}(\alpha, y) = \hat{f}(\alpha) \cosh(\alpha y) + \hat{g}(\alpha) \sinh(\alpha y)$$

unbounded growth for $y > 0$, also $y < 0$.

This is an ill-posed problem.

§ 4.4 3D Well-posed self-adjoint Poisson's Problem.

For some technical reasons, involving how solutions behave for large $|\underline{x}|$, let's look at a problem in 3D first.

$$u(\underline{x}) \qquad \underline{x} = (x_1, x_2, x_3)$$

$$\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2}$$

$$= u_{11} + u_{22} + u_{33} \qquad \text{(Notation).}$$

<u>Theorem:</u> If $f \in C_0^\infty$ (or goes to zero sufficiently quickly as $|\underline{x}| \to \infty$ and is sufficiently smooth) then

$$\Delta u = f$$

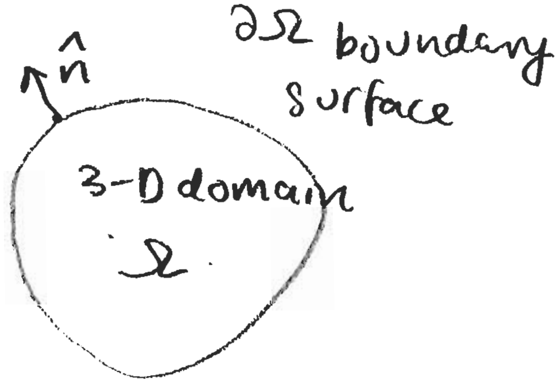
has a unique solution with $\lim_{|\underline{x}| \to \infty} u(\underline{x}) = 0$. (P3)

More specifically,

$|\underline{x}|\,|u(\underline{x})|$ and $|\underline{x}|^2\,|\nabla u(\underline{x})|$ are bounded.

In this section, we will show that (P3) is self adjoint, in the next section we will find the Green's Fn for it.

Divergence Theorem



$$\int_{\Omega} \nabla \cdot \underset{\sim}{g} \, dV = \int_{\partial\Omega} \hat{n} \cdot \underset{\sim}{g} \, dA$$

↑
volume integral

for any smooth vector function $\underset{\sim}{g}(\underset{\sim}{x})$. Also true in 2D, where the left side is an integral over a 2D domain (dV replaced by dA) and the right side is an integral over the boundary curve (dA replaced by dl). In 1D, it's the Fundamental Theorem of Calculus.

Note: $\nabla \cdot (u \nabla v) = \nabla u \cdot \nabla v + u \Delta v$.

So $$\int_{\Omega} u \Delta v \, dV = \int_{\partial\Omega} u \frac{\partial v}{\partial n} \, dA - \int_{\Omega} \nabla u \cdot \nabla v \, dV.$$

↑
using Divergence theorem

similarly $$\int_{\Omega} v \Delta u \, dV = \int_{\partial\Omega} v \frac{\partial u}{\partial n} - \int_{\Omega} \nabla u \cdot \nabla v \, dV.$$

Combining the two equations above we obtain Green's second identity:

$$\int_{\Omega} v \Delta u \, dv = \int_{\Omega} u \Delta v \, dv + \int_{\partial\Omega} \left(v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} \right) dA. \qquad (13).$$

Take $\Omega = B_R$, a ball of radius R.

$\partial\Omega = S_R$, surface of B_R.

Consider (13) as $R \to \infty$. If in this limit, the boundary terms disappear under the conditions of our problem for u & v:

$$|\underline{x}|\,|u|,\ |\underline{x}|^2\,|\nabla u|,\ |\underline{x}|\,|v|,\ |\underline{x}|^2\,|\nabla v| \text{ bounded},$$

then we could say (P3) is self adjoint.

Introduce spherical coordinates

$x_1 = r \cos\varphi \sin\theta$ $\qquad$ φ longitude $0 \le \varphi < 2\pi$

$x_2 = r \sin\varphi \sin\theta$ $\qquad$ θ latitude $0 \le \theta < \pi$

$x_3 = r \cos\theta$

In these coordinates, $\nabla f = \frac{\partial f}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial f}{\partial \theta} \hat{\theta} + \frac{1}{r \sin\theta} \frac{\partial f}{\partial \varphi} \hat{\varphi}.$

So $\left| \frac{\partial f}{\partial n} \right| = \left| \frac{\partial f}{\partial r} \right| \le |\nabla f|.$

$\uparrow$

on S_R

On S_R, $dA = R^2 \sin\theta \, d\theta \, d\varphi.$

So the boundary terms in (13) are

$$\int_{S_R} \left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right) dA =$$

$$R^2 \int_0^{2\pi}\int_0^{\pi} \left(\underbrace{v}_{O(\frac{1}{R})} \underbrace{\frac{\partial u}{\partial r}}_{O(\frac{1}{R^2})} - \underbrace{u}_{O(\frac{1}{R})} \underbrace{\frac{\partial v}{\partial r}}_{O(\frac{1}{R^2})} \right) \sin\theta \, d\theta \, d\varphi.$$

size $O(\frac{1}{R})$ $O(\frac{1}{R^2})$ $O(\frac{1}{R})$ $O(\frac{1}{R^2})$

$$= O\left(\frac{1}{R}\right) \to 0 \quad \text{as} \quad R\to\infty \text{ as required.}$$

Notation: $O(f(x))$ denotes a quantity bounded by a constant times $|f(x)|$ in a limit usually clear from the problem.

§ 4.5 Green's Function for (P3).

$$u(\underline{x}) = \iiint_{\underline{s}} G(\underline{s},\underline{x})\, f(\underline{s})\, d\underline{s}$$

where for each $\underline{x}$, $\Delta_{\underline{s}} G = 0$ for $\underline{s} \neq \underline{x}$

in $\underline{s}$ variables

and $|\underline{s}||G|$ and $|\underline{s}|^2|\nabla G|$ bounded.

Expect $G(\underline{s},\underline{x}) = \hat{G}(|\underline{s}-\underline{x}|) = \hat{G}(r)$,

r radial coordinate from origin $\underline{x}$

In spherical coordinates,

$$\Delta G = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial G}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial G}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 G}{\partial \varphi^2}$$

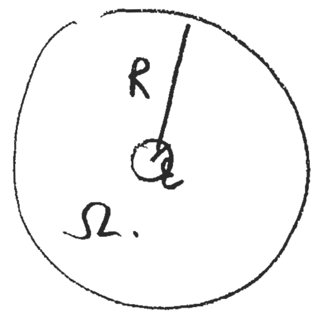
If $G = \tilde{G}(r)$ and $\Delta \tilde{G} = 0$ then

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \tilde{G}}{\partial r}\right) = 0.$$

$$\tilde{G} = A + \frac{C}{r}$$

$\tilde{G} = \frac{C}{r}$ is the only candidate. Let's see if this works and determine the appropriate C.

Consider (13) for $\Omega = B_R \setminus B_\varepsilon$



$V = \frac{C}{r}$ so $\Delta V = 0$ in Ω

$$\int_\Omega \frac{C}{r}\Delta u = \int_{S_R}\left(\frac{C}{R}\frac{\partial u}{\partial r} + u\frac{C}{R^2}\right) dA$$

$$- \int_{S_\varepsilon}\left(\frac{C}{\varepsilon}\frac{\partial u}{\partial r} + u\frac{C}{\varepsilon^2}\right) dA \qquad (14)$$

$\hat{n}$ inwards

Take $R \to \infty$, $\varepsilon \to 0$, want (14) to be $u(0)$ in the limit.

As $R \to \infty$, the S_R integral $\to 0$, similar to the integral limit on p.15.

$$\int_{S_\varepsilon} \left(\frac{c}{\varepsilon} \frac{\partial u}{\partial r} + u \frac{c}{\varepsilon^2} \right) dA$$

$$= \int_0^{2\pi} \int_0^{\pi} \left(\cancel{c\,\varepsilon \frac{\partial u}{\partial r}} + u\,c \right) \sin\theta \, d\theta \, d\varphi$$

0 as $\varepsilon \to 0$

$u(\varepsilon \cos\varphi \sin\theta, \varepsilon \sin\varphi \sin\theta, \varepsilon \cos\theta)$
$\to u(0)$ uniformly as $\varepsilon \to 0$.

$$= 4\pi C\, u(0).$$

So to obtain $u(0)$ in (14) in the limit $\varepsilon \to 0$, $R \to \infty$ is possible with $C = -\frac{1}{4\pi}$, i.e.

$$u(\underline{x}) = -\frac{1}{4\pi} \iiint_{\underline{S}} \frac{F(\underline{S})}{|\underline{S} - \underline{x}|} \, d\underline{S}$$

solves (P3)

§4.6 2D Poisson problem.

In polar coordinates (r, θ)

$$\Delta u = \frac{1}{r} (r u_r)_r + \frac{1}{r^2} u_{\theta\theta}$$

Radially symmetric solutions $u(r)$

$$u = A\ln r + \cancel{B}^{\,0}$$ can always adjust $u(\underline{x})$ by a constant

Following as above,

$$\Delta u = f$$

is solved in 2D by

$$u(\underline{x}) = \frac{1}{2\pi}\iint_{\underline{s}} \log|\underline{s}-\underline{x}|\, f(\underline{s})\, d\underline{s}. \qquad (15)$$

↑ natural log.

Why I didn't do this one first is that here,

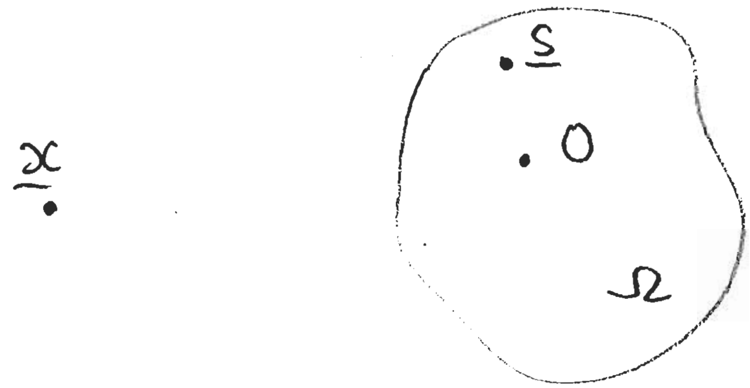
$$\lim_{|\underline{x}|\to\infty} u(\underline{x}) \neq 0.$$

There are technical difficulties in specifying the problem that (15) solves uniquely.

<u>Note</u>: (15) solving $\Delta u = f$ gives u a potential (gravitational / electrical) when f is a scaled density (mass / charge). $\pm\nabla u$ gives the resulting gravitational / electrical force field. Even though (15) gives u unbounded, the resulting fields $\nabla u \to 0$ as $|\underline{x}| \to \infty$.

§4.7 Multipole expansions.

Consider again (15). Suppose that f has compact support Ω (i.e, $f(x) \equiv 0$ outside Ω) centered around the origin. You are interested in evaluating $u(\underline{x})$ (also $\nabla u(\underline{x})$) at $\underline{x}$ in the far field ($\underline{x}$ large).



There are different ways you can do this:

1. Discretize a large domain D, including all points $\underline{x}$ where you want to know u. Numerically solve $\Delta u = f$ in D with appropriate conditions on ∂D to approximate far field conditions (like $\frac{\partial u}{\partial n} = 0$ on ∂D but there are better things you can do).

   Advantages: straight forward
   computes $u(\underline{x})$ at all grid points $\underline{x}$.

   Disadvantages: very large computational domain D might be required, loss of accuracy in small far field values of ∇u.

(2) Use numerical integration of (15):

Advantages: numerical integration on small domain Ω, easy to get high accuracy.

Disadvantages: New computation for every evaluation point $\underline{x}$.

(3) Approximate (3) using a multipole expansion

$$\log|\underline{s}-\underline{x}| = \frac{1}{2}\log|\underline{s}-\underline{x}|^2$$

$$= \frac{1}{2}\log\left\{(\underline{s}-\underline{x})\cdot(\underline{s}-\underline{x})\right\}$$

$$= \frac{1}{2}\log\left\{|s|^2 - 2\underline{s}\cdot\underline{x} + |x|^2\right\}.$$

cosine law.

multiply $\{\ \}$ by $\dfrac{|\underline{x}|^2}{|\underline{x}|^2}$,

$$= \log|\underline{x}| + \frac{1}{2}\log\left\{1 - \frac{2\underline{s}\cdot\underline{x}}{|\underline{x}|^2} + \frac{|\underline{s}|^2}{|\underline{x}|^2}\right\}. \qquad (16).$$

Assumption: for all target $\underline{x}$ and all sources $\underline{s}\in\Omega$,

$$\frac{|\underline{s}|}{|\underline{x}|} < \varepsilon, \text{ small.}$$

So in (16), in $\{\ \}$ the second term is $O(\varepsilon)$, the third $O(\varepsilon^2)$.

To highest order, $\log|\underline{s}-\underline{x}| \approx \log|\underline{x}|$ and (15) reads

$$u(\underline{x}) \approx \frac{1}{2\pi}\log|\underline{x}| \underbrace{\iint_\Omega f(\underline{s})\,d\underline{s}}_{\text{total heatsources in } \Omega} \qquad (17).$$

Note: We just need to evaluate one integral an we can evaluate $u(\underline{x})$ approximately (for $\underline{x}$ in the far field) easily for all $\underline{x}$.

Physically, (17) says that far away, the heat sources distributed through Ω look like they could all be concentrated at the origin.

Next order correction, using (16).

$$\frac{1}{2}\log\left\{1-\frac{2\underline{s}\cdot\underline{x}}{|\underline{x}|^2}+\frac{|\underline{s}|^2}{|\underline{x}|^2}\right\} \approx \frac{1}{2}\log\left\{1-\frac{2\underline{s}\cdot\underline{x}}{|\underline{x}|^2}\right\}.$$

$$\approx -\frac{\underline{s}\cdot\underline{x}}{|\underline{x}|^2}.$$

$\log(1+\varepsilon)\approx\varepsilon$.

2 integrals, moments of f.

So (15) reads

$$u(\underline{x}) \approx \frac{1}{2\pi}\log|\underline{x}|\iint_\Omega f(\underline{s})\,d\underline{s} - \frac{1}{2\pi}\frac{\underline{x}}{|\underline{x}|^2}\cdot\iint_\Omega \underline{s}\, f(\underline{s})\,d\underline{s} \qquad (18)$$

more accurate than (17)

As before, (18) is easy to compute when the three integrals are known, for any x in the far field.

Higher order terms can be added in the same way (using complex notation makes the algebra much more straight-forward). (17), (18) and higher order versions are called multipole expansions.

If $\Omega \subset B_R$ and $|\underline{x}| > 2R$ $(\varepsilon < \frac{1}{2})$, machine precision can be obtained with 20 terms.

The only disadvantage of using multipole expansions is that they can only give accurate values in the far field.

§4.8 Poisson's problem in the upper half plane.

$$\left.\begin{aligned} \Delta u &= f(x,y) \quad y \geq 0. \\ u(x,0) &= g(x). \end{aligned}\right\} \quad (PU)$$

As usual, we can write $u\{f,g\} = \underset{①}{u\{f,0\}} + \underset{②}{u\{0,g\}}$.

① We'll use a trick, called the method of images.

$\cdot\ (S,T)$ Free space $\frac{1}{4\pi}\log\{(x-S)^2+(y-T)^2\}$.

$\Delta(\ \ ) = \delta(\underline{s}-\underline{x})$.

$\cdot\ (S,-T)$ Free space $\frac{1}{4\pi}\log\{(x-S)+(y+T)^2\}$.

$\Delta(\ \ ) = 0$ for $y>0$.

<u>Note</u>: if these are subtracted,

$$\frac{1}{4\pi}\log\left\{\frac{(x-S)^2+(y-T)^2}{(x-S)^2+(y+T)^2}\right\}.$$

the result satisfies $\Delta(\ \ ) = \delta(\underline{s}-\underline{x})$ and equals zero when $T=0$. Must be the Green's function for (PU):

$$u(x,y) = \frac{1}{4\pi}\int_0^\infty\int_{-\infty}^\infty \underbrace{\log\left\{\frac{(x-S)^2+(y-T)^2}{(x-S)^2+(y+T)^2}\right\}}_{G_1(x-S,\,y,\,T)} f(S,T)\,dS\,dT \quad (19).$$

<u>Note</u>: For f with compact support, I believe you can show that the resulting u has the property $\lim_{y\to\infty} u(x,y) = 0$.

For ②, there are 2 ways to proceed:

Ⓐ Fourier transforms

$$\hat{u}(\alpha, y) = e^{-|\alpha| y}\, \hat{g}(\alpha).$$

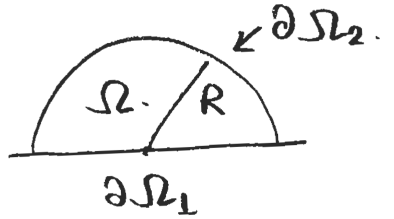
I dentify $\frac{1}{\sqrt{2\pi}} e^{-|\alpha| y}$ as the transform of $\hat{G}(x,y)$ and then.

$$u(x,y) = \int_{-\infty}^{\infty} \hat{G}(s-x, y)\, g(s)\, ds.$$

Ⓑ Go back to (13) using u the solution of ②, $\Delta u = 0$, $u(x,0) = g(x)$ and $V = G_2(x-s, y, \tau)$ from (19) for fixed x, y.

$\Delta V = \delta((s,\tau) - (x,y))$ and $V(x-s, y, 0) = 0.$

Take $\Omega = B_R \cap \{(x,y) : y \geq 0\}$.



boundary integral on $\partial\Omega_2$ $\to 0$ as $R \to \infty$.

as $R \to \infty$

$$0 = u(x,y) + \int_{-\infty}^{\infty} g(s) \left.\frac{\partial V}{\partial \tau}\right|_{\tau=0} ds.$$

Think about this, $-\left.\frac{\partial G}{\partial \tau}\right|_{\tau=0}$ will be the Green's function for problem ②.

$$G(s,x,T,y) = \frac{1}{4\pi}\Big(\ln\{(s-x)^2+(T-y)^2\} - \ln\{(s-x)^2+(T+y)^2\}\Big).$$

$$\left.\frac{\partial G}{\partial T}\right|_{T=0} = \frac{1}{\pi}\,\frac{-y}{(s-x)^2+y^2}$$

so $$u(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} \underbrace{\frac{+y}{(s-x)^2+y^2}}_{G_2}\, g(s)\,ds \qquad (20).$$

solves ②

<u>Note</u>: $\lim_{\varepsilon\to 0} \frac{1}{\pi}\frac{\varepsilon}{x^2+\varepsilon^2} = \delta(x)$ weakly.

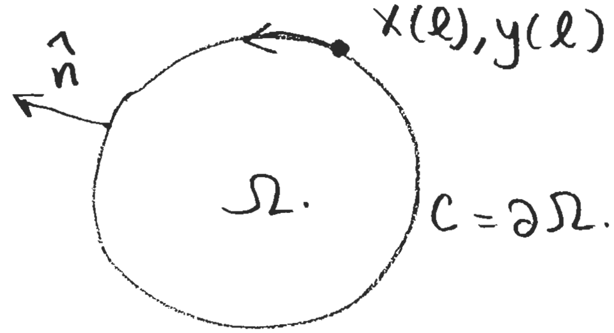
<u>Note</u>: that the formula (20) gives $u(x,0)\equiv 0$, but $u(x,0^+) = g(x)$. Also, $u(x,0^-) = -g(x)$. We will see something similar in the next section.

§4.9 Single and Double Layer Potentials, application to the 2D Dirichlet problem.

External heating $f(\underline{x})$ per unit area in Ω, $\Delta u = f$.

$$u(\underline{x}) = \frac{1}{2\pi}\iint_{\Omega} \log|\underline{x}-\underline{s}|\, f(\underline{s})\, d\underline{s}$$

Consider now a simple, closed curve C, enclosing a domain Ω:



Suppose C is parametrized by arc length l $x(l)$, $y(l)$, with $\dot{x}^2 + \dot{y}^2 = 1$ for all l.

Consider

$$u(\underline{x}) = \frac{1}{2\pi} \int_C \log |\underline{x} - (x(l), y(l))| \, g(l) \, dl. \quad (21)$$

$\uparrow$ physical interpretation of external heating per unit length on C.

u is harmonic ($\Delta u = 0$) off C.

If u is harmonic, then each component of ∇u is also harmonic and $\hat{d} \cdot \nabla u$ is also harmonic for any direction $\hat{d}$, so

$$\hat{n} \cdot \nabla_s \log |\underline{x} - \underline{s}|$$

evaluated at $\underline{s} = (x(l), y(l))$ is also harmonic away from $\underline{x} = (x(l), y(l))$.

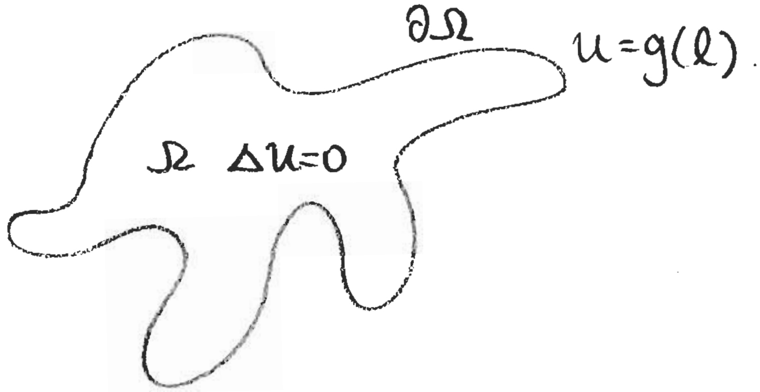
so

$$u(\underline{x}) = \frac{1}{2\pi} \int_C \left(\hat{n}_\ell \cdot \nabla_s \log|\underline{x} - \underline{s}| \right) \Big|_{\underline{s} = (x(\ell), y(\ell))} \mu(\ell)\, d\ell \qquad (22).$$

is also harmonic off C. Here, μ is called a dipole density.

(21) is called a single layer potential, (22) is called a double layer potential.

Consider now the Dirichlet problem in a bounded domain Ω:



(D)

We'd like to know the Green's function for (D),

$$u(x,y) = \int_\ell G_D(\ell, x, y)\, g(\ell)\, d\ell.$$

But except in simple cases (disk, conformal mapping) this is not known. Could discretize Ω and use Finite Element approximation (we'll discuss this in the weeks to follow).

There is another approach, representing the solution to Ⓓ as a double layer potential. This is not a perfect match ($\mu \neq g$).

<u>Motivation</u>: If we had the Green's function for the Poisson problem for the domain Ω,

i.e. $$u(\underline{x}) = \int_{\Omega} G_P(\underline{s}, \underline{x}) f(\underline{s}) \, d\underline{s}$$

solved
$$\Delta u = f \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega.$$

Then $$G_D = \hat{n}_\ell \cdot \left(\nabla_s G_P(\underline{s}, \underline{x}) \right) \Big|_{\underline{s} = (x(\ell), y(\ell))}.$$

Same reasoning as our last result in §2.8. But, we don't know G_P either. However, using the free space Green's function results in a double layer potential, should capture some of the behaviour of G_P.

Some technical computations show:

a) (22) can be evaluated at points $x(k), y(k)$ on the curve C

$$u(k) = u(x(k), y(k)) = \int_C \mu(\ell) K(k, \ell) \, d\ell \qquad (23).$$

where
$$K(k,l) = \begin{cases} \dfrac{1}{2\pi} \dfrac{[y(k)-y(l)]x'(l) - [x(k)-x(l)]y'(l)}{[x(s)-x(t)]^2 + [y(s)-y(t)]^2} & k \neq l \\ \dfrac{1}{4\pi}\left(y''(k)\,x'(k) - x''(k)\,y'(k)\right) & k = l \end{cases}$$

b) Taking the interior limit of (22) towards the boundary point $x(k), y(k)$ gives

$$u_i(k) = \underbrace{U(k)}_{\text{above in (23)}} + \frac{1}{2}\mu(k) \tag{24}$$

We want $u_i(k) = g(k)$, so $\mu(k)$ must solve the integral equation

$$g(k) = \frac{1}{2}\mu(k) + \int_C K(k,l)\,\mu(l)\,dl.$$

Numerical advantages of this approach (over FE discretization of the domain):

(i) Lower dimensional problem

(ii) Well conditioned problem.

(iii) Fast evaluation of discretized integral term using multipole expansions.

Properties (ii) & (iii) allow fast solution for discretized values of μ using Krylov subspace (conjugate gradient) methods of linear algebra.

This idea can be generalized to other problems and is called the Boundary Integral or Boundary Element method.

§5.10 Euler-Lagrange Equations with constraint.

Consider the minimization of

$$F[u] = \int_0^1 \underset{\substack{\uparrow \\ f(u,p,x)}}{f(u, u', x)}\, dx \qquad u(0) = u(1) = 0 \quad (16)$$

subject to

$$\int_0^1 g(u, u', x)\, dx = 0. \qquad (17).$$

$g(u,p,x)$.

Proceeding as in §5.2 & §5.3, we consider

$$u = \underset{\substack{\uparrow \\ \text{minimizer.}}}{u_0} + \varepsilon \eta$$

To highest order in ε, (16) reads.

$$F_{min} + \varepsilon \int_0^1 \left(f_u - (f_p)'\right) \eta \, dx. \qquad (18)$$

and (17) reads

$$\cancel{\int_0^1 g(u_0, u_0', x)\, dx}^{\,0} + \varepsilon \int_0^1 (g_u \eta + g_p \eta')\, dx$$

$$= \varepsilon \int_0^1 \left(g_u - (g_p)'\right) \eta \, dx. \qquad (19).$$

Following the argument of §5.2, (19) allows any $\eta \perp$ to $g_u - (g_p)'$. Thus, (18) requires $f_u - (f_p)'$ to be parallel to $g_u - (g_p)'$, i.e.

$$\left.\begin{aligned} &f_u - (f_p)' + \lambda(g_u - (g_p)') = 0 \\ \text{and } &\int_0^1 g(u, u', x)\, dx = 0 \end{aligned}\right\}$$

Math 401 – Notes V
Functional Minimization.

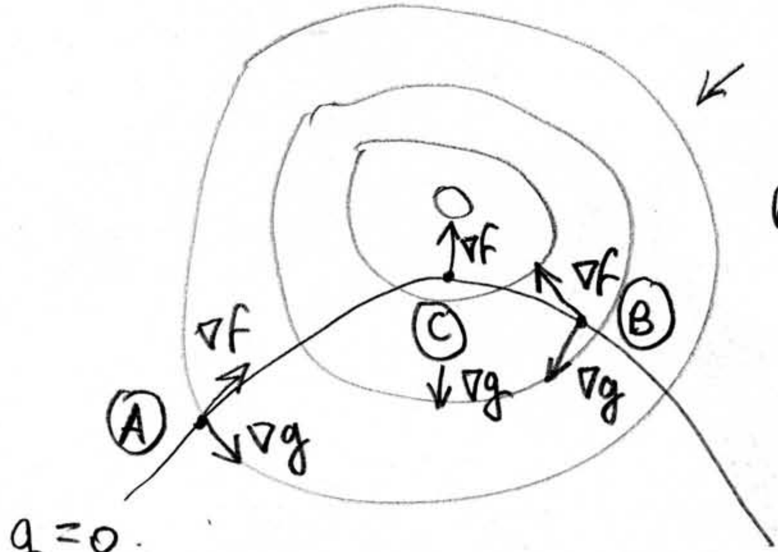
§5.1 Review of optimization in R^n.

If Ω is a closed, bounded domain in R^n, and $f(\underline{x})$ is continuous on Ω, differentiable in the interior of Ω, then the following applies:

<u>Theorem</u>: f has a maximum and a minimum value on Ω, attained either on the boundary ($\partial\Omega$) or in the interior at a point at which $\nabla f = 0$.

<u>Recipe</u>: Look at values on the boundary, look at values at interior points where $\nabla f = 0$.

To find extreme values of $f(\underline{x})$ subject to a constraint $g(\underline{x}) = 0$, these can only occur at values of $\underline{x}$ where ∇f points in the same direction as ∇g, as seen in the following diagram.



contours of f
(ie curves along which f is constant).

Note: ∇f is $\perp$ to contour lines.

At (A) & (B), where ∇g is not parallel to ∇f it is possible to move along the curve $g=0$ to increase f. The maximum occurs at (C) where ∇f is parallel to ∇g, i.e.

$$\nabla f + \lambda \nabla g = 0.$$

This can be combined with $g(\underline{x})=0$ compactly if we consider

$$F(\underline{x},\lambda) = f(\underline{x}) + \lambda g(\underline{x})$$

and look for values of $\underline{x}$ & λ such that

$$\tilde{\nabla} F = \underline{0}$$

$$\left(\nabla, \frac{\partial}{\partial x}\right).$$

We want to continue now with the optimization of functionals $F[u]$.

F does not have a discrete number of inputs but an infinite number, represented by the values of a (continuous) function $u(x)$. To proceed, we must put some restrictions on the form of F.

§5.2 Functionals involving integrals of scalar functions of one variable.

$u(x)$, $x \in [0,1]$ $\qquad$ $f(u,x)$ specifies F.

$$F[u] = \int_0^1 f(u(x), x)\, dx \tag{1}$$

If we want to minimize (maximize) F, use the idea we had in Chapter 1:

Consider $\underset{\substack{\uparrow \\ \text{minimizer}}}{u = u_0} + \varepsilon h(x)$, $\quad h$ arbitrary.

$$F(\varepsilon) = \int_0^1 f(u_0 + \varepsilon h, x)\, dx$$

$$= \underbrace{\int_0^1 f(u_0, x)\, dx}_{F_{min}} + \varepsilon \int_0^1 f_u(u_0, x)\, h\, dx + O(\varepsilon^2) \tag{2}$$

Considering (2) as $\varepsilon \to 0$, u_0 can only be a minimizer if $f_u(u_0(x), x) = 0$ for every x. For each x, this is an algebraic condition for $u(x)$.

Let's add an integral constraint

$$\int_0^1 g(u(x), x)\, dx = 0 \qquad (3)$$

to the optimization of (1).

$$\int_0^1 g(u_0 + \varepsilon\eta, x)\, dx = \cancel{\int_0^1 g(u_0, x)\, dx}^{\,0}$$

$$+ \varepsilon \int_0^1 g_u(u_0(x), x)\, \eta\, dx \; + \; O(\varepsilon^2)$$

Retaining the same order terms as in (2), we see that to preserve the constraint as $\varepsilon \to 0$, we must only allow η that satisfy

$$\int_0^1 g_u(u_0, x)\, \eta(x)\, dx.$$

That is, $\eta \in L^\perp$ where $L = \text{Span}\{g_u(u_0(x), x)\}$ in the space of functions on $[0,1]$. Considering $\eta \in L^\perp$ in (2) we see $u_0(x)$ can only be a minimizer if $f_u(u_0, x) \in (L^\perp)^\perp = L$, that is

$$f_u(u_0(x), x) + \lambda\, g_u(u_0(x), x) = 0 \qquad (4)$$

for some constant λ. (3) & (4) are in the same form as the Lagrange multiplier for the finite dimensional case in §5.1. For given λ, (4) is an algebraic solve for $u_0(x)$. Condition (3) with $u = u_0$ is the global constraint that determines the correct value of λ.

Ex 1 Minimize

$$F[u] = \int_0^{1/2} \underbrace{\frac{\frac{1}{2}-x}{u(x)}}_{f(u,x)} dx \quad \text{subject to} \quad \int_0^{1/2} \underbrace{u}_{g(u,x)}\, dx = \frac{1}{2}$$

Note: This is our heated rod example from Assignment #1, assuming the cross-sectional areas are symmetric about $x = \frac{1}{2}$. Here, $u(x)$ is the cross-sectional area, not the solution.

In this case $f_u = \dfrac{(x-\frac{1}{2})}{u^2}$ $\quad g_u = 1$ and (4) reads

$$\frac{x-\frac{1}{2}}{u^2} + \lambda = 0 \quad \text{with} \quad \int_0^{1/2} u\, dx = \frac{1}{2}.$$

$$\Rightarrow \quad u = \underset{\substack{\uparrow \\ \text{renamed } \sqrt{1/\lambda}.}}{C} \sqrt{1/2 - x}$$

$$C \int_0^{1/2} \sqrt{\tfrac{1}{2}-x}\, dx = \frac{1}{2} \quad \Rightarrow \quad C = \frac{3}{4}\, 2^{3/2}$$

so $u_0(x) = \dfrac{3}{4}\, 2^{3/2} \sqrt{\tfrac{1}{2}-x}$ is the minimizer.

The minimum value of F is obtained at this $u_0(x)$;

$$F_{min} = \int_0^{1/2} \frac{\frac{1}{2}-x}{\frac{3}{4}\, 2^{3/2} \sqrt{\frac{1}{2}-x}}\, dx = \frac{1}{9}.$$

Note: The optimal rod cross-sectional area has $u(\frac{1}{2}) = 0$, soln may need to be modified to be physically reasonable.

$u_0(x)$ is the only candidate for a minimizer. Does this mean that F really is minimized by u_0? [yes, in this case].

In general, F could have only one candidate u_0 for a minimizer, but have other u_n such that

$$\lim_{n\to\infty} F(u_n) = -\infty$$

We can think of F having a saddle-point structure at u_0. In this case, there is no minimum of F.

Finite dimensional analogy:

$$f(x,y) = x^2 - y^2$$

$\nabla f = 0$ only at $(0,0)$ but f is not minimized or maximized there.

To rule out such behaviour in functional minimization is difficult. For our example, we have some physical intuition, if not a proof.

§5.3 Euler-Lagrange Equations for functionals of scalar functions of one variable.

More general than (1):

$$F[u] = \int_0^1 \underbrace{f(u(x), u'(x), x)}_{f(u,p,x)} \, dx \qquad (5)$$

Additional conditions must be given, either

(A) $u(0)=0,\ u(1)=0$ (or other, specific values)

(B) no constraints on u.

To minimize (maximize) F, proceed as before,

$$\frac{d}{d\varepsilon} F[u_0+\varepsilon\eta]\Big|_{\varepsilon=0} = 0 \Rightarrow$$

$$\int_0^1 \left(f_u(u_0, u_0', x)\eta + f_p(u_0, u_0', x)\eta'\right) dx = 0$$

Integrate the second term above by parts.

$$\int_0^1 \left(f_u - \frac{d}{dx} f_p\right)\eta \, dx + \eta f_p(u_0, u_0', x)\Big|_0^1 = 0.$$

This must be true for every function $\eta(x)$. For (A), $\eta(0)=\eta(1)=0$, the boundary term cancels and we require

$$f_u - \frac{d}{dx}(f_p) = 0 \qquad (6)$$

This is called the Euler-Lagrange equation.

For (B) we still have (6) but also require

$$f_p(u_0, u_0', x) = 0 \quad \text{at } x=0,1$$

(natural boundary conditions).

Ex 2 Recall our linearized, scaled wire energy

$$F[u] = \int_0^1 \left\{\tfrac{1}{2}(u')^2 + m(x)\,u(x)\right\} dx$$

This is in the form (5) with $F(u,p,x) = \frac{1}{2}p^2 + m(x)u$.

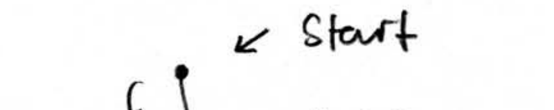
$F_u = m(x)$, $F_p = p$ so (6) reads.

$$m(x) - \frac{d}{dx}(u') = 0.$$

$$\text{or } u'' = m(x) \quad \checkmark.$$

Conditions (A) lead to a solvable problem. Conditions (B) lead to $u'(0) = u'(1) = 0$. In assignment I we showed this problem only had a solution when $\int_0^1 m(x)\,dx = 0$. Considering the functional of this example on constant functions, we see the functional will have no minimum unless this condition is satisfied.

Ex 3 Brachistochrone problem.



H, L given: find the curve $y = u(x)$ down which a bead would slide without friction starting at rest and accelerating due to gravity in the least time.

Energy Conservation $\frac{1}{2}mv^2 = mg(H-y)$

Speed $v = \sqrt{2g(H-y)}$.

Consider a segment of the curve between x and $x+\Delta x$

The length of the segment is approximately

$$\sqrt{1+(u'(x))^2}\ \Delta x$$

The time to traverse the segment is approximately

$$\frac{\sqrt{1+(u')^2}}{\sqrt{2g(H-y)}}\ \Delta x.$$

Sum over all segments, take limits

total time $\quad F[u] = \frac{1}{\sqrt{2g}} \int_0^L \underbrace{\sqrt{\frac{1+(u')^2}{H-u}}}\, dx.$

scales time, doesn't affect optimal curve.

$$f(u,p,x) = \sqrt{\frac{1+p^2}{H-u}}$$

$$f_u = \frac{1}{2}\frac{(1+p^2)^{1/2}}{(H-u)^{3/2}} \qquad f_p = \frac{p}{\sqrt{(1+p^2)(H-u)}}$$

E-L reads

$$\frac{1}{2}\frac{(1+(u')^2)^{1/2}}{(H-u)^{3/2}} - \frac{d}{dx}\left\{\frac{u'}{\sqrt{(1+(u')^2)(H-u)}}\right\} = 0. \qquad (7)$$

$u(0)=H$, $\ u(L)=0$. This is a nonlinear, 2nd order ODE boundary value problem. Could solve numerically - would be easy if you had a good initial guess. Maybe $H(1-x/L)$ would be good enough, but if not consider the following idea.

If the bead starts with speed V_0, then

$$\frac{1}{2} m v^2 = mg\,(H-u) + \frac{1}{2} m V_0^2$$

$$V = \sqrt{2g}\ \sqrt{(H-u) + V_0^2/2g}$$

To get the E-L equations for this case, just replace $H-u$ by $(H-u) + V_0^2/2g$ in (7) above. Consider optimal solutions $u(x; V_0)$. There will be continuous dependence on V_0 [not proven]. Physical intuition makes us suspect

$$\lim_{V_0 \to 0} u(x; V_0) = H(1 - x/L).$$

So for large enough V_0, $H(1-x/L)$ is a great initial guess for the nonlinear solver for $u(x; V_0)$. It would then be possible to gradually reduce V_0, using the solution at the previous V_0 as an initial guess (close to the solution by the continuity in V_0). This technique is called the method of continuation.

§5.4 Beltrami Identity.

Notice that in the case of the Brachistochrone problem, the functional came from an "f" that did not depend on x. In such a situation,

$$\left(u' f_p(u,u') - f(u,u')\right)'$$
$$= \cancel{u'' f_p} + u'\,(f_p)' - f_u u' - \cancel{f_p u''}$$
$$= u'\left(f_u - f_p'\right).$$

So solutions of the E-L equations $f_u - f_p' = 0$ have the property that

$$u' f_p - f = \text{constant}.$$

(The Beltrami identity). This is a first order equation. Analytic solutions of the brachristochron problem (as cycloids) follow from this approach.

§5.5 Inequality Constraints. $f(u,p,x)$.

Consider the Brachristochrone problem, minimize

$$F[u] = \int_0^L \sqrt{\frac{1+(u')^2}{H-u}}\,dx \quad u(0)=H, \; u(L)=0.$$

with the added constraint that $u(x) \geq \phi(x)$ for all x, ϕ a given function such that $\phi(x) < H$ for all x, $\phi(L) \leq 0$. We can introduce $S(x)$ ("slack" variable) and let

$u(x) = \phi(x) + S^2(x)$, constraint satisfied for any S.

Now $F[s] = \int_0^1 \underbrace{f(\phi + s^2,\ \phi' + 2sq,\ x)}_{g(s,q,x).}\, dx$

where $q = s'$.

E-L equations $\quad g_s - (g_q)' = 0.$

$g_s = 2s f_u + 2q f_p.$

$g_q = 2s f_p.$

$$g_s - (g_q)' = 2s f_u + 2s' f_p - (2s f_p)' = 0$$

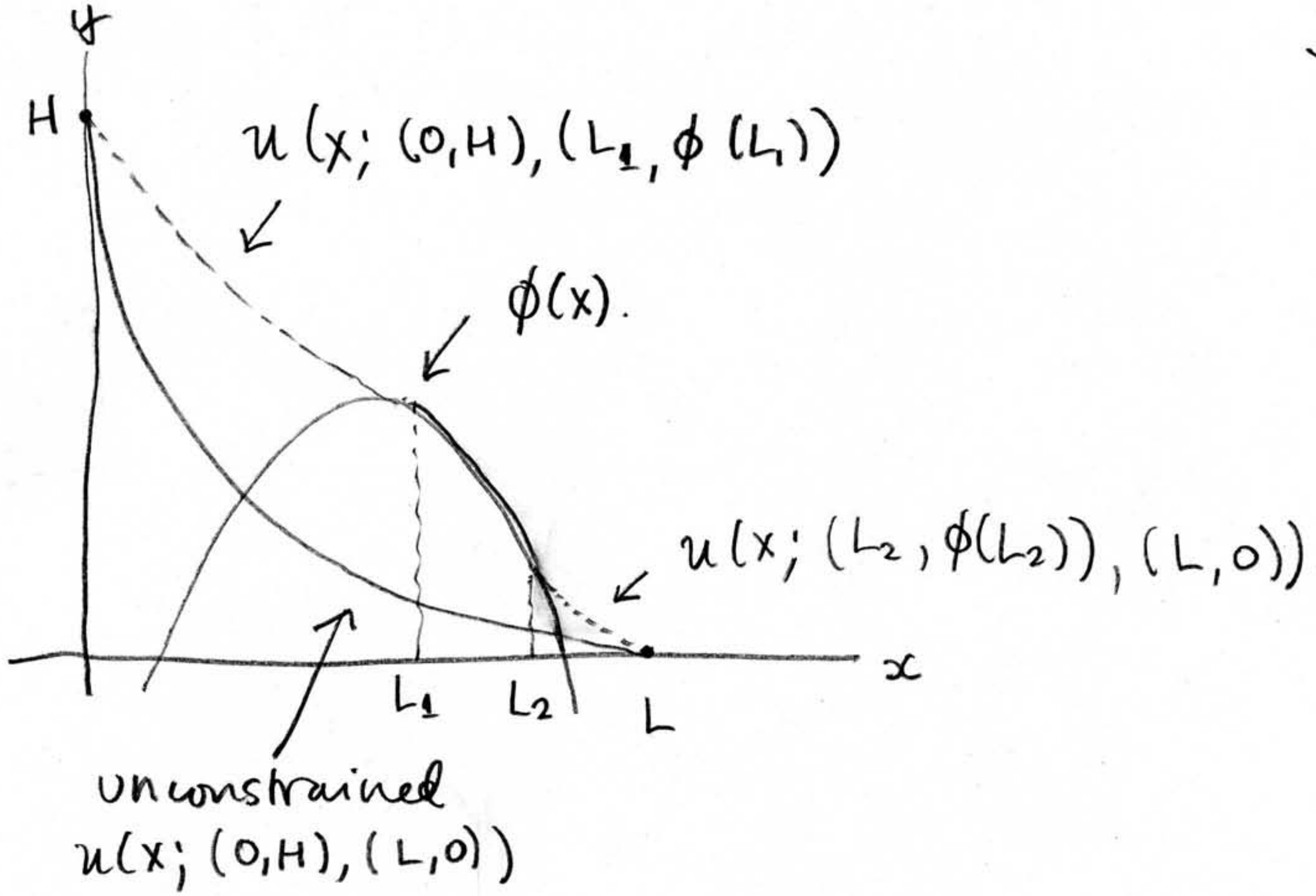
$$\Rightarrow 2s f_u + \cancel{2s' f_p} - 2s f_p' - \cancel{2s' f_p} = 0$$

$$2s\,(\underbrace{f_u - (f_p)'}_{\text{unconstrained E-L.}}) = 0.$$

So, the optimal solution is composed of two types of solutions (in intervals): Either $s = 0$ ($u(x) = \phi(x)$, solution lies on the constraint) or u solves the unconstrained equations.

Consider the example at the top of the next page. Using notation $u(x; (x_0, y_0), (x_1, y_1))$ for solutions of (7) with the given boundary conditions, we know that the optimal

$$u(x) = \begin{cases} u(x; (0, H), (L_1, \phi(L_1))) & x \in [0, L_1] \\ \phi(x) & x \in [L_1, L_2]. \\ u(x; (L_2, \phi(L_2)), (L, 0)). \end{cases}$$



L_1 and L_2 still need to be determined, but the optimal time is now only a function of 2 variables, L_1 & L_2.

Conjecture (from the class): The optimal curve and the constraint are tangent at L_1 & L_2, ie.

$$u'(L_1; (0,H), (L_1, \phi(L_1))) = \phi'(L_1).$$
$$u'(L_2, (L_2, \phi(L_2)), (L,0)) = \phi'(L_2).$$

These are equations that could be solved numerically for L_1 & L_2. The exact solutions for the unconstrained problems would be really useful here.

For general ϕ, there could be more than one interval where the optimal u lies on the constraint.

Another approach: Consider the speed

$$v = \sqrt{2g\,(H-u)}\,.$$

We could approximate the condition that $u(x) \geq \phi(x)$ by introducing a factor that would make the bead move very slowly if it were below $\phi(x)$. The optimal path would then avoid $u(x) \leq \phi(x)$. Possible implementation:

introduce $$V_M(u,x) = \begin{cases} 1 & \text{if } u \geq \phi(x) \\ \frac{1}{M} & \text{if } u < \phi(x). \end{cases}$$

replace v above by

$$\sqrt{2g\,(H-u)}\; V_{M,\varepsilon}(u,x)$$

mollified with radius ε.

Then minimize

$$F_{M,\varepsilon}[u] = \frac{1}{\sqrt{2g}} \int_0^L \sqrt{\frac{1+(u')^2}{(H-u)}} \cdot \frac{1}{V_{M,\varepsilon}(u,x)}\, dx.$$

Solve the E-L equations to find the optimal $u_{M,\varepsilon}(x)$

The solution to the original problem is recovered in the limit $M \to \infty$, $\varepsilon \to 0$.

Amenable to numerical approximation, known as a capturing method. It has the advantage that the number of transitions from equation to constraint does not have to be known before computations begin.

§5.6 Energy arguments for the Poisson problem.

Euclidean $\mathbb{R}^n$ $\quad (\underline{x}, \underline{y}) = \sum_{i=1}^{n} x_i y_i$

$$|\underline{x}| = \sqrt{(\underline{x}, \underline{x})}.$$

$|\underline{x} \cdot \underline{y}| \leq |\underline{x}|\,|\underline{y}|$ Cauchy-Schwartz Inequality

$|\underline{x} + \underline{y}| \leq |\underline{x}| + |\underline{y}|$ Triangle inequality

These results are also true for the inner product

$$(u,v) = \int_0^1 u(x)\, v(x)\, dx, \quad \|u\| = \sqrt{\int_0^1 |u|^2 dx}$$

defined for all functions u, v in L^2, space of functions for which $\int_0^1 |u|^2 dx$ exists.

Technical points: L^2 is formed as the completion of $C^\infty[0,1]$ in the L^2 norm. Functions in L^2 don't need to be continuous.

Consider now the space H_0^1 of functions u such that $u(0)=0$, $u(1)=0$ and

$$\|u\|_{H_0^1}^2 = \int_0^1 \left\{ (u')^2 + u^2 \right\} dx = \|u'\|_{L^2}^2 + \|u\|_{L^2}^2$$

is finite.

This is the natural space of functions to consider the energy that led to the problem $u'' = m$, $u(0) = u(1) = 0$.

$$F[u] = \int_0^1 \left\{ \frac{1}{2}(u')^2 + mu \right\} dx$$

since for any $u \in H_0^1$,

C-S.

$$|F[u]| \leq \frac{1}{2}\|u'\|^2_{L^2} + \|m\|_{L^2}\|u\|_{L^2}$$

$$\leq \frac{1}{2}\|u'\|^2_{H_0^1} + \|m\|_{L^2}\|u\|_{H_0^1}$$

$F[u]$ defined (finite) for all $u \in H_0^1$, Considering $u \in H_0^1$ allows more general solutions (ie. when m is in L^2, could be discontinuous).

<u>Note:</u> $|u(x)| = \left| \int_0^x u'(s)\,ds \right|$

$$= |(1, f)| \quad \text{where } f(s) = \begin{cases} u'(s), & s \leq x \\ 0, & s > x. \end{cases}$$

$$\leq \|u'\|_{L^2} \quad \text{C-S}$$

So if $u \in H_0^1$, u is bounded by $\|u'\|_{L^2}$ (9)

A similar argument shows that $u \in H_0^1$ is continuous (10).

(9) & (10) are only true in 1D.

If $|u(x)| \leq \|u'\|_{L^2}$ then

$$\|u\|_{L^2} \leq \|u'\|_{L^2} \tag{11}$$

Using a different argument, (11) is true in higher dimensions (Poincaré Lemma). Look back at

$$\|u\|_{H_0^1}^2 = \|u'\|_{L^2}^2 + \|u\|_{L^2}^2$$

Using (11) we can see that $\|u'\|_{L^2}$ is an equivalent norm to the H_0^1 norm, in that

$$\|u'\|_{L^2}^2 \le \|u\|_{H_0^1}^2 \le 2\,\|u'\|_{L^2}^2 \qquad (12).$$

Go back to $F[u] = \int_0^1 \left\{ \frac{1}{2}(u')^2 + \underset{\text{continuous.}}{m} u \right\} dx \quad u(0) = u(1) = 0$

E-L $\quad u'' = m.$

Consider $\underbrace{F[u}_{\text{minimizer}} + \underbrace{h]}_{\text{arbitrary fn in } H_0^1} = \int_0^1 \left\{ \frac{1}{2}(u')^2 + u'h' + \frac{1}{2}(h')^2 + mu + mh \right\} dx$

Integrate 2nd term by parts

$$F[u+h] = F_{min} + \frac{1}{2}\int_0^1 (h')^2 dx + \int_0^1 (\cancelto{0}{-u'' + m})\, h\, dx$$

so $F[u+h] = F_{min} + \frac{1}{2}\|h'\|_{L^2}^2$

$$F[u+h] - F_{min} \geqslant \frac{1}{4}\|u\|_{H_0^1}^2$$

We've shown for this problem, u really is the only minimum for all functions in H_0^1.

§5.7 Eigenvalue problems.

In §5.6 we showed (11), $\|u\|_{L^2} \le \|u'\|_{L^2}$ for $u \in H_0^1$. In fact,

$$\|u\|_{L^2} \le c\, \|u'\|_{L^2} \qquad \text{with } c < 1$$

for all $u \in H_0^1$. How would we find the smallest c that worked?

$$\frac{1}{c} = \min_{u \neq 0,\, \in H_0^1} \frac{\|u'\|_{L^2}}{\|u\|_{L^2}} \qquad \left(\text{then } \frac{\|u'\|_{L^2}}{\|u\|_{L^2}} \geq \frac{1}{c} \cdots\right).$$

$$\frac{1}{c^2} = \min_{u \in H_0^1,\, u \neq 0} \frac{\|u'\|_{L^2}^2}{\|u\|_{L^2}^2} \qquad (13).$$

$$\frac{1}{c^2} = \min \int_0^1 \underbrace{(u')^2}_{f(u,p,x) = p^2} dx \qquad \text{with } u(0) = u(1) = 0$$

subject to the constraint $\int_0^1 \underbrace{u^2}_{g(u,p,x) = u^2} dx = 1$

(just scale u in (13) by $1/\|u\|_{L^2}$).

E-L equations with constraint

$$f_u - (f_p)' + \lambda g_u = 0$$

$$-2u'' + 2\lambda u = 0$$

$$u'' = \lambda u \qquad u(0) = u(1) = 0 \qquad (14).$$

i.e. the minimizer must be an eigenfunction of an ODE boundary value problem. If u solves (14) with $\int_0^1 u^2\,dx = 1$, then

$$\int_0^1 (u')^2 dx = -\int_0^1 u u'' dx$$

$$= -\lambda \int_0^1 u^2 dx = -\lambda.$$

So the eigenvalues of (14) are all negative, and the minimum of the functional is $-\lambda_1$ where λ_1 is the eigenvalue of smallest absolute value.

For this problem, $\lambda = -n^2\pi^2$, $\lambda_1 = -\pi^2$, leading to the smallest $C = \frac{1}{\pi}$ in the Poincaré Lemma.

The eigenvalue λ_1 of smallest absolute value has great importance in applications:

(A) $u_t = (A(x) u_x)_x \qquad u(0) = u(1) = 0.$

Describes transient heat conduction in a rod of variable area. Separate variables:

$$u = \sum_{i=1}^{\infty} \underbrace{A_i}_{\text{determined by } u(x,0),\text{ initial data}} e^{\lambda_i t} \phi_i(x) \qquad (15)$$

where $\phi_i(x)$ are eigenfunctions

$$(A(x)\phi_i'(x))' = \lambda_i \phi_i(x) \qquad \phi_i(0) = \phi_i(1) = 0.$$

The dominant term in (15) for large time is the

$i=1$ term. λ_1 is found by

$$-\lambda_1 = \min_{u \in H_0^1} \int_0^1 A(x)\,(u')^2 dx \quad \text{subject to } \int_0^1 u^2 dx = 1.$$

The minimizing function is $\phi_1(x)$.

(B) $m(x)\, u_{tt} = u_{xx} \qquad u(0) = u(1) = 0.$

Wave motion in a wire of variable mass.
Separate variables

$$u = \sum_{i=1}^{\infty} \left(A_i \cos \mu_i t + B_i \sin \mu_i t \right) \phi_i(x).$$

Where $\phi_i(x)$ are generalized eigenfunctions

$$\phi_i'' = \lambda_i\, m(x)\, \phi_i \qquad \phi_i(0) = \phi_i(1) = 0.$$

and $\lambda_i = -\mu_i^2$. $\dfrac{\mu_1}{2\pi}$ is the lowest frequency of vibration, and μ_1 can be found by

$$\mu_1^2 = \min_{u \in H_0^1} \int_0^1 (u')^2 dx$$

subject to $\int_0^1 m(x)\, u^2 dx = 1.$

§5.8 Forshadowing Chapter 7

We have a number of minimization problems of interest for $u \in H_0^1$:

(i) $\int_0^1 \left\{ \frac{1}{2} A(x)\,(u')^2 + m(x)\, u(x) \right\} dx$ solves $(Au')' = m.$

(ii) $\int_0^1 A(x)(u')^2\,dx$ with constraint $\int_0^1 u^2 dx = 1$

solves $(A(x)u')' = \lambda_1 u$.

(iii) $\int_0^1 (u')^2 dx$ with constraint $\int_0^1 m(x)\,u^2 dx = 1$

Solves $u'' = \lambda_1 M(x)\,u$.

We could approximate these problems in the following way: Consider a finite set $\{\psi_i\}_{i=1}^n$ of l.i. functions in H_0^1. A numerical method results if you replace minimization over H_0^1 by minimization over span$\{\psi_i\}$. One choice of $\{\psi_i\}$ results in the finite element method, the subject of Chapter 7. For any choice of $\{\psi_i\}$, the discrete minimization provides upper bounds on $-\lambda_1$ in (ii) & (iii).

§5.9 Higher Dimensional functions.

$$u(x,y) \quad , (x,y) \in \Omega, \quad u\big|_{\partial\Omega} = 0.$$

$$F[u] = \int_\Omega \underset{\substack{\uparrow \\ f(u,\,p,\,q,\,x,\,y)}}{f(u, u_x, u_y, x, y)}\, dx\,dy$$

minimizing F:

$$0 = \frac{d}{d\varepsilon} F[u + \varepsilon \underset{\substack{\uparrow \\ 0 \text{ on } \partial\Omega}}{h}]\Big|_{\varepsilon=0} = \int_\Omega (f_u h + f_p h_x + f_q h_y)\,dx\,dy$$

Integrate by parts,

$$\Rightarrow \int_\Omega \left(f_u - (f_p)_x - (f_q)_y\right) \eta \, dx\, dy = 0.$$

$$\Rightarrow f_u - (f_p)_x - (f_q)_y = 0 \qquad \text{2D E-L.}$$

Ex 4 Minimize

$$F[u] = \int_\Omega \left(\tfrac{1}{2}(u_x^2 + u_y^2) + m(x,y)\, u \right) dx\, dy$$

$$\uparrow$$

$$f = \tfrac{1}{2} p^2 + \tfrac{1}{2} q^2 + m u.$$

$$\frac{\partial f}{\partial u} = m, \quad \frac{\partial f}{\partial p} = p = u_x, \quad \frac{\partial f}{\partial q} = q = u_y.$$

2D E-L $\qquad m - u_{xx} - u_{yy} = 0 \Rightarrow$

$\Delta u = m \qquad$ 2D Poisson problem in Ω.

Math 401 - Notes VI
Level set methods.

§6.1 Optimal travel paths.

velocity $V(x,y)$.

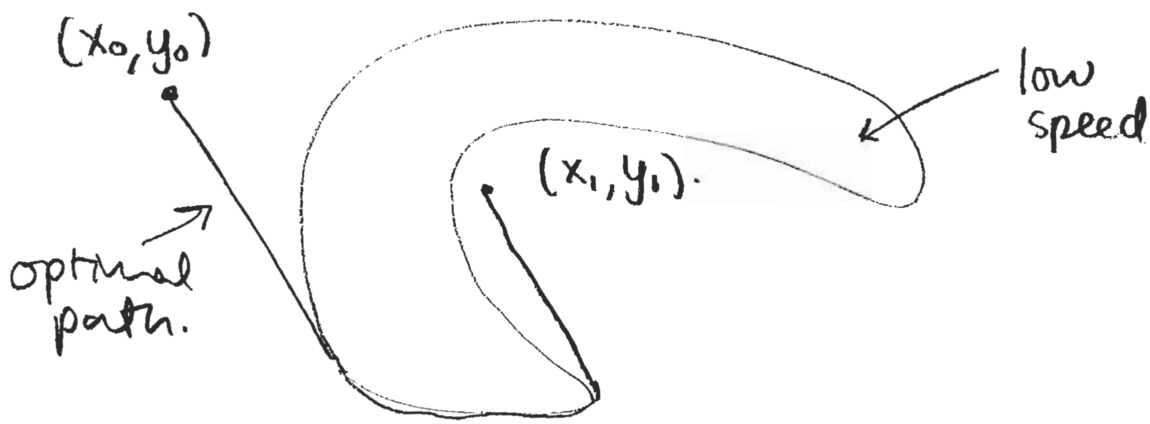
Find the curve between (x_0, y_0) and (x_1, y_1) that gives the fastest travel time.

If the assumption $y = u(x)$ is reasonable then

$$T[u] = \int_{x_0}^{x_1} \frac{\sqrt{1+(u')^2}}{V(x,y)}\,dx \qquad u(x_0) = y_0, \quad u(x_1) = y_1.$$

To minimize T, u solves the E-L equations.
The Brachistochrone problem was an example, where $V(x,y) = \sqrt{2g(y_0 - y_1)}$

But there are situations where the optimal curve is not a function $y(x)$ [or $x(y)$].



Could use a parametrized curve $u(s), v(s)$, parameter $s \in [0,1]$. For a parameter subinterval of length Δs, the curve length is

$$\approx \sqrt{(u')^2 + (v')^2}\, \Delta s.$$

Time to travel this distance is

$$\approx \frac{\sqrt{(u')^2 + (v')^2}}{V(u,v)}\, \Delta s.$$

sum, take limits, get travel time

$$T[u,v] = \int_0^1 \frac{\sqrt{(u')^2 + (v')^2}}{V(u,v)}\, ds. \qquad (1).$$

To minimize, could use vector versions of the E-L equations. However, there are an infinite number of minimizers. This is easy to see: there are many different parametrizations of the optimal curve. Could force a unique parametization, i.e. one where s is scaled arc length, that is

$$\sqrt{(u')^2 + (v')^2} = C, \quad \text{constant in } s$$

or write this as $$\left\{ (u')^2 + (v')^2 \right\}' = 0 \qquad (2).$$

I am not sure how to combine (1) & (2)... look for other approaches.

§6.2 Steady State level set method

Suppose $T(x,y)$ is known, the optimal travel time from (x_0, y_0) to (x,y) from the previous problem.

Then, the optimal path to (x_1, y_1), $(u(\sigma), v(\sigma))$ [σ is not the parametrization from the last section and goes from $\sigma = 0$ at (x_1, y_1) backward to (x_0, y_0)] solves

$$u(0) = x_1 \quad ; \quad v(0) = y_1.$$

$$\frac{d(u,v)}{d\sigma} = -\nabla T(u,v) \qquad \sigma \geq 0 \qquad (3)$$

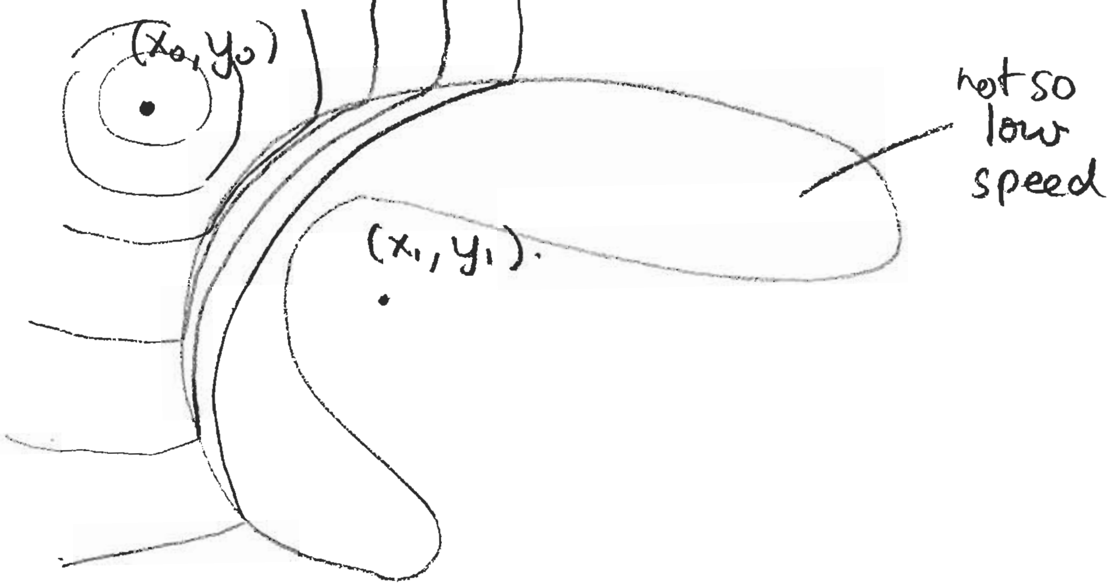
Leave aside the question of how to see this is the optimal path.

(3) is an example of gradient flow. Along the solutions $T(\sigma) = T(u(\sigma), v(\sigma))$

$$\frac{dT}{d\sigma} = \nabla T \cdot \frac{d(u,v)}{d\sigma} = -|\nabla T|^2.$$

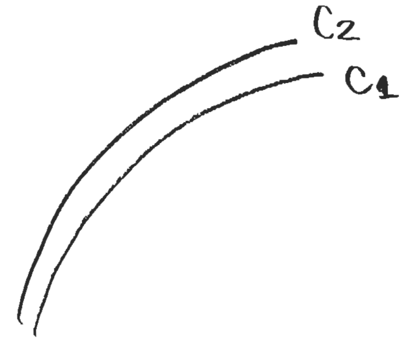
i.e. T decreases as σ increases and must tend to a (possibly) local minimum of T. Here, it is clear that T has only one minimum at (x_0, y_0), so $u(\sigma), v(\sigma)$ does tend to (x_0, y_0) and describes a path from (x_0, y_0) to (x_1, y_1) (considering σ in reverse).

Want to prove $(u(\sigma), v(\sigma))$ is the optimal path and determine a way to find $T(x,y)$.



Consider the level sets of T above. If $V(x,y)$ is continuous at (x_0, y_0), local level sets are almost circles, and the graph of $T(x,y)$ looks like a cone near that point. Recall that ∇T is $\perp$ to level sets of T.

Consider two level sets of T



$C_1: \; T(x,y) = t$

$C_2: \; T(x,y) = t + \Delta t.$

Remember, $V(x,y)$ so moving in any direction has the same speed starting from a point (x,y) on C_2. For $T(x,y)$ to be the optimal time, (x,y) must come from the point on C_1 that would reach it the quickest. That is, the point C_1 that leads to (x,y) must move in the direction ∇T. This proves the optimality of (3). The normal distance Δx between the two points must be

$$\Delta x \approx V(x,y)\,\Delta t. \qquad (4)$$

Now $\displaystyle T\Big((x,y) - \Delta x \underbrace{\frac{\nabla T}{|\nabla T|}}_{\hat{n}}\Big) \approx t.$

$$\underbrace{T(x,y)}_{t+\Delta t} - \nabla T\cdot\Big(\Delta x \frac{\nabla T}{|\nabla T|}\Big) \approx t. \quad \text{Using (4),}$$

$$V(x,y)\,|\nabla T|\,\Delta t \approx \Delta t$$

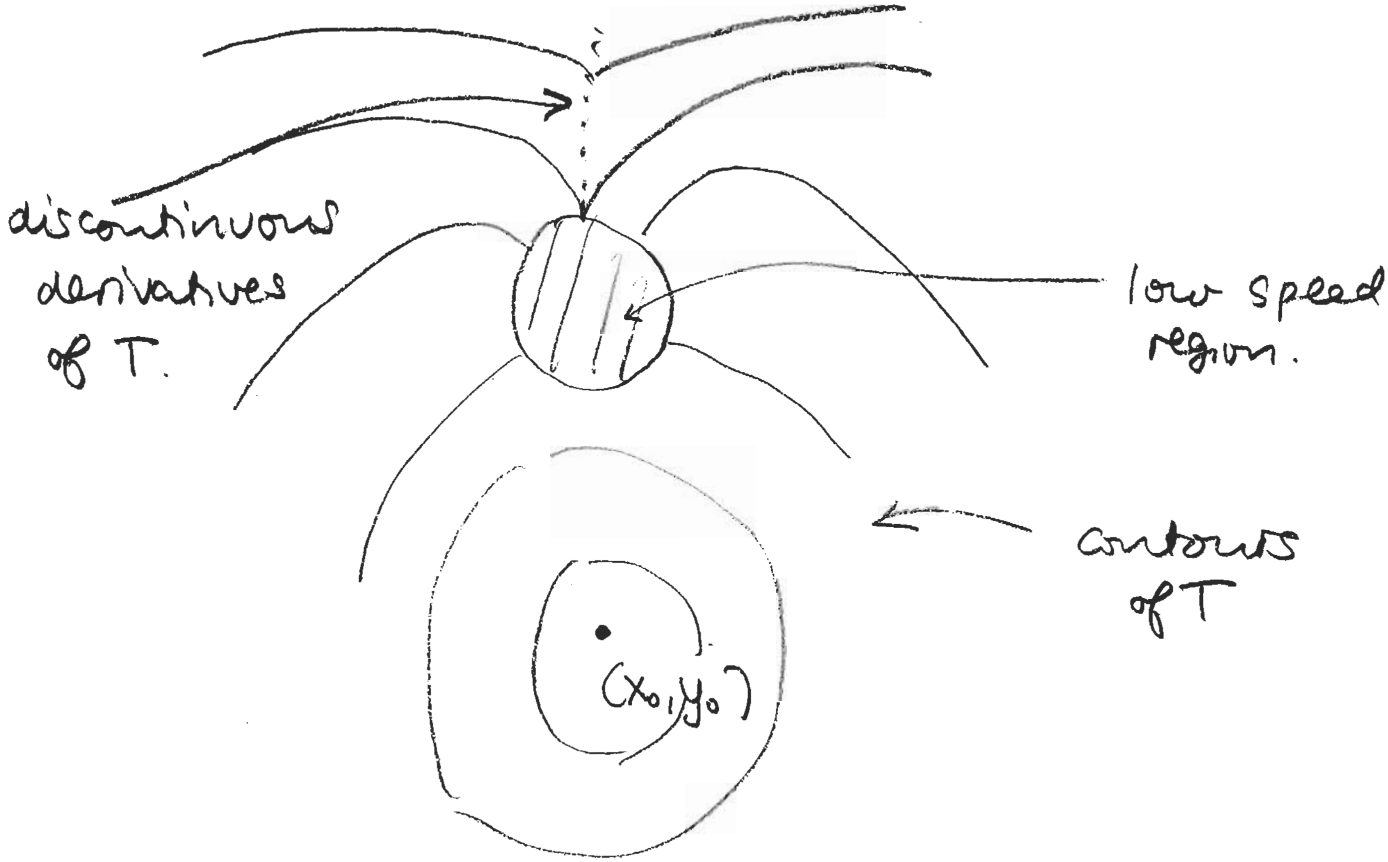
Take limits $\Delta t \to 0$, $\quad V(x,y)|\nabla T| = 1.$

$$\text{or} \quad T_x^2 + T_y^2 = \frac{1}{V^2} \qquad (5).$$

This is the eikonal equation, which can have multi-valued solutions. If you can solve (5), you find optimal times and optimal paths (by solving (3)) for all points (x,y).

$T(x,y)$ can have discontinuities derivatives as can be seen from the following example (next page).

Numerical methods that solve for the correct $T(x,y)$ are a bit tricky, can borrow ideas from shock capturing methods for hyperbolic conservation laws.



§6.3 Transient Level Set Method

Extend the ideas of the previous section in the following way. Suppose we had the situation where we have a curve C_t (parametrized by $(x(s,t), y(s,t))$) that move with normal speed given by

$$V(x,y) \quad \Big[\text{possibly} \quad V(x,y,\underset{\substack{\uparrow \\ \text{curve normal}}}{\hat{n}},\underset{\substack{\uparrow \\ \text{curve curvature.}}}{\kappa})\Big].$$

with F not necessarily always positive (or negative). We could use a tracking method:

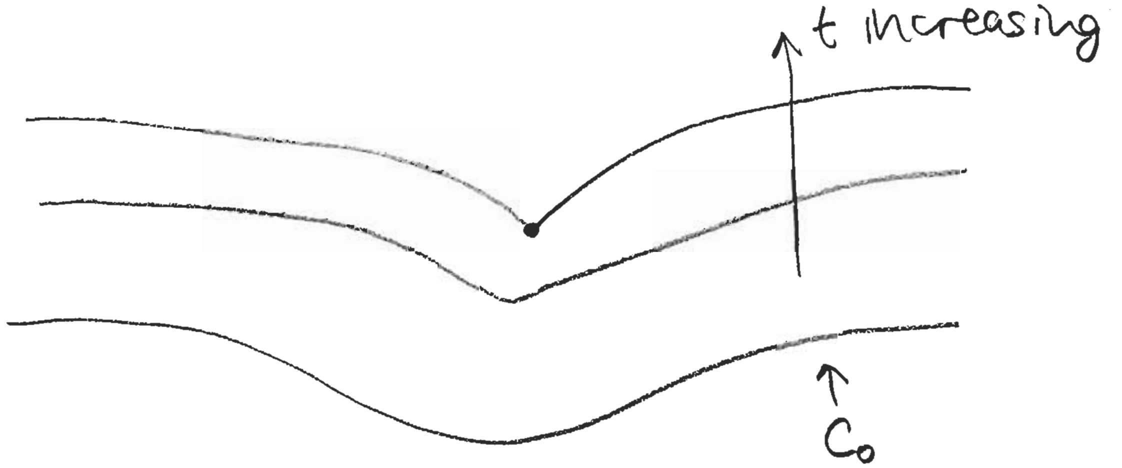
$$\hat{n} = \frac{(-y_s, x_s)}{\sqrt{(x_s)^2 + (y_s)^2}}$$

$$(x,y)_t = V(x,y)\hat{n}.$$

Note: There is no natural extension to surfaces moving in 3D.

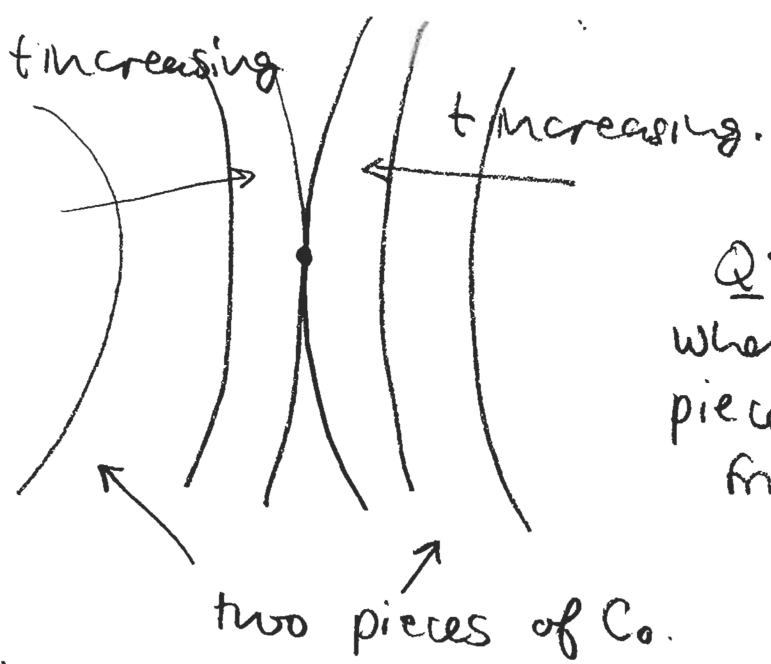
Even in 2D, there can be complications as shown in the next two examples with $V \equiv 1$.

Ex 2



Q: What to do after this cusp?

Ex 3



Q: What to do when the two pieces of the front meet?

The answer to the Q's depends on the physical situation being modelled. Say the curve describes a combustion front, with burnt material behind the front, unburnt ahead. [These are ideas about the entropy or irreversibility of the physics behind the front tracking methods].

A nice idea to get around the change of topology is to write the tracked curve $C_t = (X(s,t), y(s,t))$ as the level set of a function $\phi(x,y,t)$, i.e.

$$C_t = \{ (x,y) : \phi(x,y,t) = 0 \} \qquad (6)$$

The topological change of Ex 3 can happen naturally in this setting.

Can make $\phi(x,y,0)$ so that

$$C_0 = \{ (x,y) : \phi(x,y,0) = 0 \}.$$

for example by letting

$$\phi(x,y,0) = \underset{\substack{\uparrow \\ \text{signed, + on unburnt side, - on} \\ \text{burnt side.}}}{\text{distance}} ((x,y), C_0)$$

But, how to make ϕ have property (6) at later time? Consider

$$\phi(x(s,t), y(s,t), t) = 0.$$

Take $\partial/\partial t$

$$\phi_t + \phi_x x_t + \phi_y y_t = 0. \qquad (7)$$

But $(x,y)_t = V(x,y)\,\hat{n}$ and $\hat{n} = \dfrac{\nabla\phi}{|\nabla\phi|}$

(outward ✓).

So (7) reads

$$\phi_t + \nabla\phi \cdot \left\{ V(x,y) \frac{\nabla\phi}{|\nabla\phi|} \right\} = 0.$$

$$\phi_t + V(x,y)\,|\nabla\phi| = 0 \qquad (8).$$

So far, (8) only has to apply on the zero level set. Extend (8) to apply at all points (x,y). Now all level sets of ϕ move with normal velocity $V(x,y)$. (8) is in a class called Hamilton-Jacobi equations

$$\phi_t(\underline{x},t) + H(\nabla\phi, \underline{x}) = 0.$$

All this relates to §6.2 in the following way. Consider

$$\phi(x,y,t) = T(x,y) - t.$$

Level sets of ϕ move normally with speed V so ϕ satisfies (8), ie

$$-1 + V(x,y)\,|\nabla T| = 0$$

$$V(x,y)\,|\nabla T| = 1 \qquad \text{(the equation derived in §6.2).}$$

§6.4 Mathematical structure and approximation of level set equations.

Consider $V \equiv 1$, (8) reads

$$\phi_t + \sqrt{\phi_x^2 + \phi_y^2} = 0$$

Let $u = \phi_x$, $v = \phi_y$, take derivatives of the equation above w.r.t. x & y:

$$\left.\begin{aligned} u_t + \left(\sqrt{u^2+v^2}\right)_x &= 0 \\ v_t + \left(\sqrt{u^2+v^2}\right)_y &= 0 \end{aligned}\right\} \quad (9).$$

<u>Note</u>: This formulation is not the way to actually <u>solve</u> for ϕ, just to see its structure.

(9) is a system of nonlinear conservation laws in 2D.

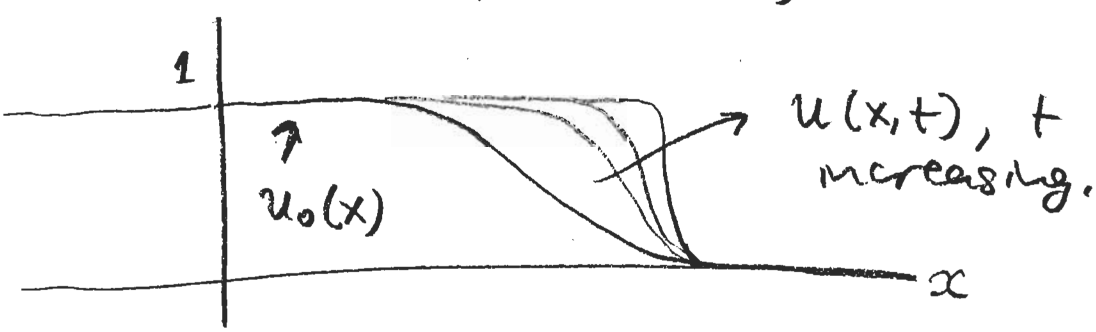
Recall scalar conservation laws in 1D.

<u>Ex</u> 4 Burger's equation $u(x,t)$

$$u_t + \left(\tfrac{1}{2}u^2\right)_x = 0 \qquad u(x,0) = u_0(x) \text{ given}$$

Solutions are constant along characteristics

$$u(s + u_0(s)t, t) = u_0(s).$$



$u(x,t)$ can develop shocks (discontinuities)
Expect (9) to have discontinuities, then ϕ to have discontinuous derivatives - This matches our expectations for $T(x,y)$ from §6.2.

Numerical methods to approximate (8) & (4) must handle these discontinuities correctly:

(i) discrete conservation

(ii) discrete entropy conditions.

To solve (4), $T_x^2 + T_y^2 = \frac{1}{[V(x,y)]^2}$ on a discrete grid $T_{ij} \approx T(ih, jh)$ can be approximated by solving

$$\max(T_{i,j} - T_{i-1,j}, 0)^2 + \min(T_{i+1,j} - T_{i,j}, 0)^2 + \max(T_{i,j} - T_{i,j-1}, 0)^2 + \min(T_{i,j+1} - T_{i,j}, 0)^2 = h^2 / [V(ih, jh)]^2 \qquad (10).$$

Note that $T_{i,j}$ will only be influenced above by neighboring, smaller T values. As an equation for $T_{i,j}$, (10) is a quadratic. The largest root must be taken.

To solve (10), sweep outward from the point where $T=0$, adding at each step the point that is adjacent to the region of known T values that has the smallest time computed by (10). For details and applications, see

math.berkeley.edu/~sethian/

Math 401 - Notes VII

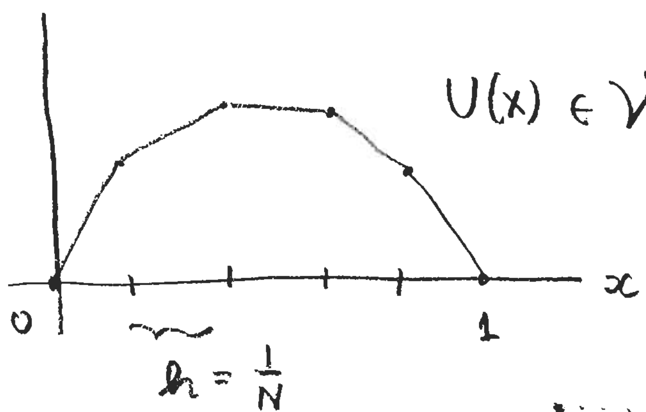
Finite Element Method

§7.1 Rayleigh Ritz Finite Element Method

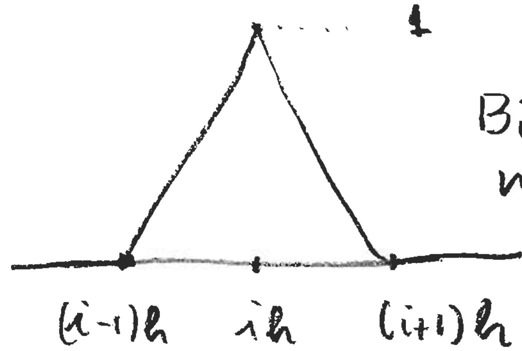
$$u'' = f(x) \qquad u(0)=0,\ u(1)=0$$

solves
$$\min_{u \in H_0^1} F[u] = \int_0^1 \left\{ \frac{1}{2}(u')^2 + f u \right\} dx \qquad (1).$$

We could instead minimize over a subset V_h of H_0^1. For example, V_h piecewise linear functions on a grid (zero on boundary):



Basis of V_h:



$B_i(x)$, value 1 at node i, zero at other nodes.

$U \in V_h$ can be written $U(x) = \sum_{i=1}^{N-1} U_i B_i(x)$.

U_i are the coefficients of the basis vectors, with the nice feature that they are the values of $U(x)$ at the node points.

Minimize (1) over V_h rather than over H_0^1.

$$F[U] = \int_0^1 \left\{ \frac{1}{2} \left(\sum_{i=1}^{N-1} U_i B_i'(x) \right) \left(\sum_{j=1}^{N-1} U_j B_j'(x) \right) + \sum_{i=1}^{N-1} U_i B_i(x) f(x) \right\} dx.$$

2nd term above is $\underline{F} \cdot \underline{U}$ where $F_i = \int_0^1 N_i(x) f(x)\, dx$ and $\underline{U}$ is the N-1 vector of values U_i.

1st term $\frac{1}{2} \underline{U} \cdot \mathbb{A} \underline{U}$ where $A_{ij} = \int_0^1 N_i' N_j' \, dx$.

$$\min_{V \in Y_h} F[V] = \min_{\underline{U} \in \mathbb{R}^{N-1}} \left(\frac{1}{2} \underline{U} \cdot \mathbb{A} \underline{U} + \underline{F} \cdot \underline{U} \right)$$

This is a quadratic minimization, elementary linear algebra $\Rightarrow$ minimum attained when

$$-\mathbb{A} \underline{U} = \underline{F}.$$

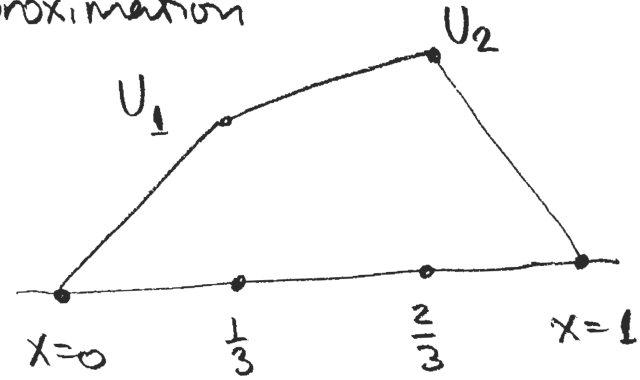
We can solve for $\underline{U}$, get the approximation $V(x)$ to the original problem.

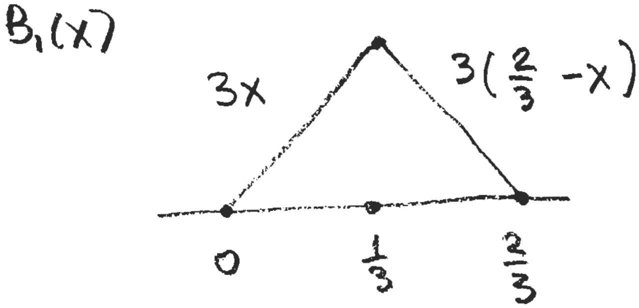
<u>Q1</u>: What do $\mathbb{A}$ & $\underline{F}$ look like?

<u>Q2</u>: How close is $V(x)$ to $u(x)$, the exact solution?

<u>Ex</u> $u'' = e^x$, $u(0) = 0$, $u(1) = 0$, $N = 3$.

approximation





$$A_{11} = \int_0^1 (B_1')^2\,dx = \int_0^{1/3} [(3)^2]\,dx + \int_{\frac{1}{3}}^{2/3} [(-3)^2]\,dx = 6.$$

Note: A_{22} has the same form $= 6$.

$$A_{12} = A_{21} = \int_0^1 B_1' B_2'\,dx = \int_{1/3}^{2/3} (-3)(3)\,dx = -3.$$

$$F_1 = \int_0^1 B_1(x) e^x\,dx = \int_0^{1/3} 3x\, e^x\,dx + \int_{1/3}^{2/3} 3(\tfrac{2}{3}-x) e^x\,dx \quad (2)$$

Can be done analytically. But (proofs later) the FE soln is second order accurate:

$$U_h(x) = u(x) + O(h^2).$$

Thus, it is reasonable to evaluate (2) with a second order quadrature method and not change the order of accuracy of the method. Use the trapezoidal rule to approximate (2).

$$\int_0^{1/3} \underbrace{3x e^x}_{g(x)}\,dx \approx \underbrace{\frac{1}{3}}_{h} \cdot \frac{1}{2}\left(g(0) + g(\tfrac{1}{3})\right) = \frac{1}{6} e^{1/3}.$$

and $\displaystyle\int_{1/3}^{2/3} 3(\tfrac{2}{3}-x) e^x\,dx \approx \frac{1}{6} e^{1/3}.$

So $F_1 \approx \frac{1}{3} e^{1/3}$. Similarly, $F_2 \approx \frac{1}{3} e^{2/3}$.

So $-AU = F$ with trapezoidal rule approximation of F:

$$\begin{pmatrix} -6 & 3 \\ 3 & -6 \end{pmatrix}\begin{pmatrix} U_1 \\ U_2 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} e^{1/3} \\ e^{2/3} \end{pmatrix}$$

$$\underbrace{9}_{\frac{1}{h^2} \text{ with } h = \frac{1}{3}.}\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} U_1 \\ U_2 \end{pmatrix} = \begin{pmatrix} e^{1/3} \\ e^{2/3} \end{pmatrix}.$$

So the FEM is the same as a finite difference when the grid is uniform, but the FEM allows an easy extension to higher dimensional problems in domains with complex geometry.

§7.2 Galerkin FEM.

Consider (1) again. Clearly, for all $v \in H_0^1$

$$\int_0^1 (u'')\, v \, dx = \int_0^1 f(x)\, v(x)\, dx \qquad \text{integrate by parts...}$$

$$-\int_0^1 (u')(v')\, dx = \int_0^1 f(x)\, v(x)\, dx. \qquad (3).$$

If $u \in H_0^1$ satisfies (3) for every $v \in H_0^1$, u is called a weak solution of (1). If u is a weak solution and is twice differentiable, then u solves (1) and is said to be a strong solution.

Can consider an approximate weak solution $U \in V_h$ so that (3) was satisfied for all $v \in V_h$.

<u>Note</u>: U solves the same problem as the last section.

If u solves a minimization problem, we can use Rayleigh Ritz to find an approximation.
If the problem for u has a weak formulation, we can use the Galerkin FE approach. Our simple problem (1) has both structures.

§7.3 Galerkin FEM in 2D.

$u(x,y)$ solves

$\leftarrow$ $u=0$ on $\partial\Omega$

Weak formulation, for all $v \in H_0^1(\Omega)$

$$\int_\Omega (\Delta u) v \, dA = \int_\Omega f v \, dA$$

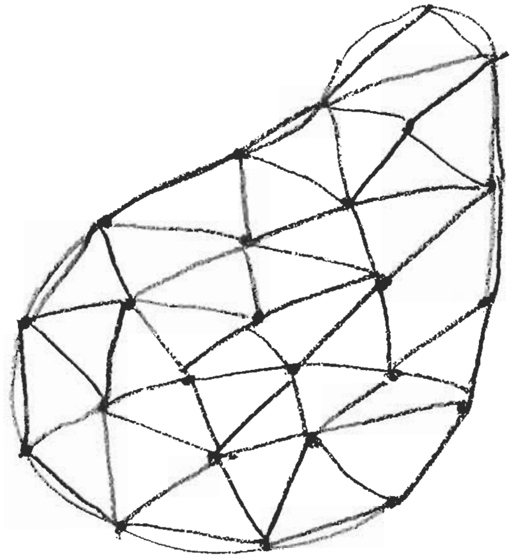
$$-\int_\Omega (\nabla u \cdot \nabla v) \, dA + \underbrace{\int_{\partial\Omega} v \frac{\partial u}{\partial n} \, ds}_{\text{zero since } v \in H_0^1} = \int_\Omega f v \, dA.$$

so $$-\int_\Omega (\nabla u \cdot \nabla v) \, dA = \int_\Omega f v \quad \forall \, v \in H_0^1 \qquad (4).$$

Consider $U(x) \in V_h$, the space of piecewise linear functions on a triangular mesh.

Write $U(x,y) = \sum_i U_i B_i(x,y)$ $\leftarrow$ basis functions piecewise linear, value 1 at node i, 0 at others.

sum over internal nodes



(4) is approximated by

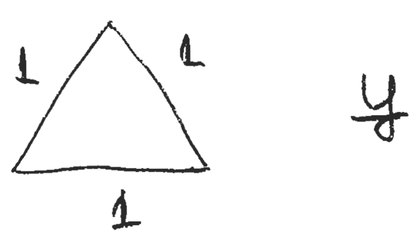
$$-\mathbb{A}\underline{U} = \underline{F}$$

where $A_{ij} = \int_\Omega (\nabla B_i)\cdot(\nabla B_j)\, dA.$

Note that the integrand is piecewise constant on triangles and that $A_{ij} \neq 0$ only if i & j are adjacent, so $\mathbb{A}$ is sparse (mostly zeros).

$$F_i = \int_\Omega B_i(\underline{x}) f(\underline{x})\, d\underline{x}.$$

This last term is a sum of integrals on triangles around the i'th node. Each triangle can be mapped to a reference triangle



The map $\underline{x} \to \underline{y}$ on a triangle is linear, so the determinant of its Jacobian $|J|$ is constant.

So $\int_{\Delta} B_i(\underline{x}) f(\underline{x})\, d\underline{x} = \frac{1}{|J|} \int_{\Delta.} B_{ref}(\underline{y}) f(\underline{y})\, d\underline{y}.$

original $\qquad\qquad$ reference

These integrals (all over the same reference triangle) can then be approximated by Gaussian quadrature methods.

The construction of $\mathbb{A}$ and the approximation of $\underline{F}$ can be done automatically in a code. Good (but heuristic) methods for grid generation for a given domain Ω are also available.

§7.4 1D convergence analysis.

Consider the problem

$$-u'' = f \qquad u(0) = u(1) = 0$$

(just (1) with a convenient sign shift).

Exact solution u, $\quad (u', v') = (f, v) \quad \forall\, v \in H_0^1.$

FE solution U, $\quad (U', V') = (f, V) \quad \forall\, V \in V_h$ (5)

Since $V_h \subset H_0^1$,

$$(u', V') = (f, V)$$

Subtract from (5) denoting the error $e = U - u$.

$$(e', V') = 0 \qquad \forall\, V \in V_h \qquad (6)$$

Note that $V = U - \psi = U - u + (u - \psi)$

$$= e + (u - \psi) \quad \in V_h.$$

Insert this V in (6).

$$(e', e' + (u-\psi)') = 0$$

$$(e', e') = |(e', u - \psi')|$$

$$\|e'\|^2 \le \|e'\| \, \|(u-\psi)'\| \qquad \text{C-S.}$$

So $\|e'\| \le \|(u-\psi)'\|$ for all $\psi \in V_h$ (6).

<u>Note:</u> We have written the error in the solution with a bound from an approximation problem!

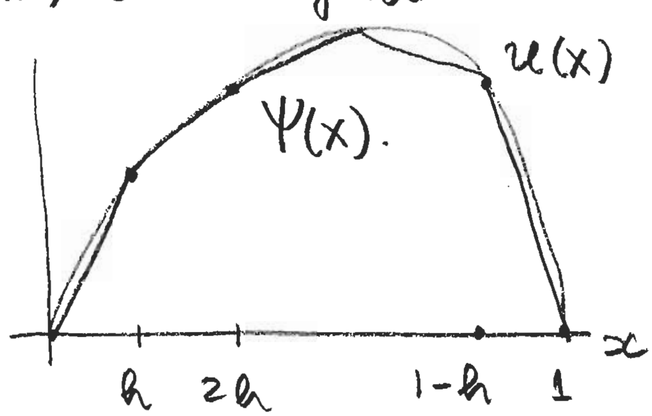
If $f \in C^0$ (continuous), $u \in C^2$.

If $f \in L_2$, $u \in H^2$ (functions for which $\underbrace{\|u\|^2 + \|u'\|^2 + \|u''\|^2}_{\|u\|^2_{H_2}}$ is finite) and

$$\|u\|_{H_2} \le C \|f\|$$

for a constant C independent of f. These results can be shown from the Green's function representation.

For 1D functions $u \in H^2$ implies that u is continuous. Take $\psi(x)$ to be the linear interpolant of $u(x)$ on the grid:



On each subinterval $I = [ih, (i+1)h]$ consider

$W = u - \psi$

$w(ih)=0, \quad w((i+1)h)=0.$

so (Rollé) $w'(\xi)=0$ for some $\xi \in I$.

$$w'(x) = \int_{\xi}^{x} w''(s)\,ds = \int_{\xi}^{x} u''(s)\,ds$$

since $\psi''=0$ (ψ linear on I).

$$\text{so } |w'(x)| \le \int_I |u''(s)|\,ds$$
$$\le \sqrt{h}\sqrt{\int_I |u''|^2 ds} \qquad \text{CS}$$

$$\text{so } \int_I [w'(x)]^2 dx \le h^2 \int_I [u'']^2 dx.$$

Now sum over all subintervals I,

$$\|w'\|_{L^2}^2 \le h^2 \|u''\|_{L^2}^2 \le h^2 \|u\|_{H_2}^2$$
$$\le h^2 C^2 \|f\|_{L^2}^2$$

Recall $w = u-\psi$, so returning to (6) we have

$$\|e'\|_{L^2} \le \|(u-\psi)'\|_{L^2} \le h\,C\,\|f\|_{L^2} \qquad (7).$$

Recall $\|e\|_{H_0^1} \le 2\|e'\|_{L^2}$ (Poincaré)

So we have proven convergence of the scheme in H_0^1. First order convergence in $(u-U)'$ is the best that can be attained for general grids and piecewise linear approximation.

It is observed computationally that $u-U$ converges in L_2 with second order.

Let W (different w from above) solve the problem

$$-W'' = e \qquad W(0)=0,\ W(1)=0.$$

In weak form,

$$(W', v') = (e, v) \qquad \forall\, v \in H_0^1$$

Pick $v = e$

$$(W', e') = (e, e).$$

Use (5), $(e', \psi') = 0 \quad \forall\, \psi \in H_0^1.$

$$((W-\psi)', e') = (e, e).$$

$$\|e\|_{L^2}^2 = (e', (W-\psi)')$$

$$\le \|e'\|_{L^2} \|(W-\psi)'\|_{L^2}$$ interpolate W as before

$$\le \underbrace{\|e'\|_{L^2}}\, h\, C \|e\|_{L^2}$$ e is the "f" for the problem for W

$$\le hC\|f\|_{L^2}$$

So $\|e\|_{L^2} \le C^2 h^2 \|f\|_{L^2}$, second order convergence as expected.

Proving convergence in higher dimensions is very similar, but with two additional technical difficulties:

1. H_2 functions are not necessarily continuous in high enough dimensions – other interpolants must be used.
2. Ω_h is not equal to Ω when $\partial\Omega$ has curved boundaries.

Proving maximum norm convergence is technically difficult.

Chapter 8 - Perturbation Methods

§8.1 Convergence of Taylor series.

Ex 1 Recall $\sin x = x - \frac{x^3}{6} + \frac{\varepsilon^5}{5!} - \cdots = \sum_{i=1}^{\infty} \frac{(-1)^{i+1} x^{2i-1}}{(2i-1)!}$ (1)

McLaurin series.

Finite truncation of the McLaurin series gives the Taylor polynomial approximation $P_n(\varepsilon)$. This is the "best" polynomial approximation in the sense that

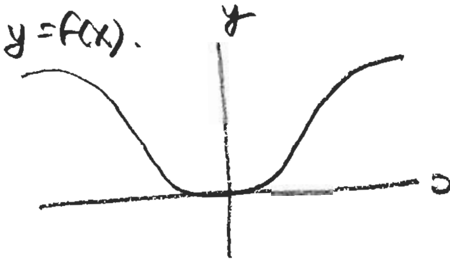
$$\max_{x \in [-\varepsilon, \varepsilon]} |P_n(x) - \sin x|$$

is as small as possible over the choice of n'th order polynomials for ε small enough. In this example, the series (1) converges for all x. Think of this as a nice bonus, not always true in what we will do in this Chapter.

Ex 2 Recall $\frac{1}{1-x} = 1 + x + x^2 + \cdots = \sum_{i=0}^{\infty} x^n$ (2).

(geometric series). Finite truncation again gives $P_n(x)$, best polynomial approximation for x small enough. The series (2) converges for all $|x|<1$. The radius of convergence is the distance (from the base point $x=0$) to the nearest singularity in the complex analytic extension of the function.

Ex 3 Consider $f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$



$f \in C^\infty$. $f^{(n)}(0) = 0$ for all n. Now consider

$$g(x) = f(x) + e^x, \quad g^{(n)}(0) = 1 \text{ for all } n.$$

$$P_n(x) = \sum_{i=0}^{n} \frac{g^{(n)}(0)}{n!} x^n = \sum_{i=0}^{n} \frac{x^n}{n!}$$

↑ for $g(x)$

For fixed x, $\lim_{n\to\infty} P_n(x) = e^x$, not $g(x)$. That is, the Mclaurin series of g does not converge to g except at $x = 0$. But... for x small enough, $P_n(x)$ is still the best n'th order polynomial approximation of $g(x)$.

Some of the perturbation series we'll develop will have this spirit, that for fixed ε (even if it is small) adding another term to the expansion may not improve the accuracy. Fixed order series may only be good approximations for ε "small enough".

In real applications of perturbation (or asymptotic) analysis, it is often difficult to determine when ε is "small enough". Numerical results on simple, related problems can help.

§8.3 Motivational Examples.

Ex 4 Approximate the root of

$$x^4 + \underbrace{0.01}_{\varepsilon}\, x - 1 = 0 \qquad (3).$$

near $x = 1$. Assume $x \approx 1 + \varepsilon x_1$ and plug into (3).

↑ x_0, base soln when $\varepsilon = 0$.

$$(1+\varepsilon x_1)^4 + \varepsilon(1+\varepsilon x_1) - 1 \approx 0.$$

Expand $(1+\varepsilon x_1)^4$ and retain terms of $O(1)$ and $O(\varepsilon)$, equating the coefficients of this size:

$1:$ $\quad 1 - 1 = 0 \quad \checkmark$ (we got x_0 right).

$\varepsilon:$ $\quad 4x_1 + 1 = 0, \quad x_1 = -\frac{1}{4}$ $\quad (4)$

So $x \approx 1 - \varepsilon/4$ and when $\varepsilon = 0.01$, $x \approx 0.9975$. The exact x (found by Newton's method) is $x \approx 0.997497$.

Alternative method for Ex 4: Consider $x(\varepsilon)$, $x(0) = 1$.
Differentiate (3) w.r.t. ε,

$$4x^3 \frac{dx}{d\varepsilon} + x + \varepsilon \frac{dx}{d\varepsilon} = 0.$$

Evaluate at $\varepsilon = 0$ $(x=1)$, $\frac{dx}{d\varepsilon} = -\frac{1}{4}$. Linear approximation $x(\varepsilon) \approx 1 - \varepsilon/4$.

This alternative method seems quite nice, but can break down as shown in the next example.

Ex 5 Approximate the roots near $x=1$ of

$$(x-1)^4 + \underbrace{0.01}_{\varepsilon}\, x = 0 \qquad (5).$$

The alternative approach above leads to

$$4(x-1)^3 \frac{dx}{d\varepsilon} + x + \varepsilon \frac{dx}{d\varepsilon} = 0$$

and $\frac{dx}{d\varepsilon}$ is indeterminate at $\varepsilon = 0$, $x = 1$.

Try instead writing a perturbation series

$$x \approx 1 + \delta x_1, \quad \delta \text{ small}.$$

In the previous example, we could have started with this, then discovered at (4) that

$$4x_1\delta + \varepsilon = 0$$

and it would have made sense to take $\delta = \varepsilon$. In this example, something different will happen. Plug the form $x \approx 1 + \delta x_1$ into (5)

$$\delta^4 x_1^4 + \varepsilon + \varepsilon\delta x_1 \approx 0.$$

We want to match the ε term. The $O(\varepsilon\delta)$ term can't work since δ is small, so the δ^4 term must be $O(\varepsilon)$. Take $\delta = \varepsilon^{1/4}$ or equivalently $\varepsilon = \delta^4$.

$$\delta^4 x_1^4 + \delta^4 + \delta^5 x_1 \approx 0.$$

δ^4: $\quad x_1^4 = -1$

$$x_1^2 = \pm i, \quad x_1 = \pm\left(\frac{1\pm i}{\sqrt{2}}\right).$$

So the roots of (5) are given approximately by

$$x \approx 1 \pm \varepsilon^{1/4}\left(\frac{1\pm i}{\sqrt{2}}\right).$$

Expect the next term to be $O(\delta^2) = O(\sqrt{\varepsilon})$ so we should not expect this to be very accurate unless ε is <u>very</u> small.

Return to Ex 4. There we had a small parameter ε, and a perturbation series in integer powers of ε. Such an expansion is called a <u>regular</u> expansion. This wasn't true in Ex 5, and there are various ways problems can lead to perturbation expansions that are not regular.

§8.3 Examples of regular expansions in ODE Boundary Value Problems.

Ex 6 Let's revisit quickly the example from § 1.6 on the uniformly heated rod with slightly varying cross-sectional area.

$$\left(\underbrace{\left\{ \frac{1+\varepsilon \sin \pi x}{1+2\varepsilon/\pi} \right\}}_{A(x),\ \int_0^1 A(x)\,dx = 1.} u' \right)' = -1 \qquad (6)$$

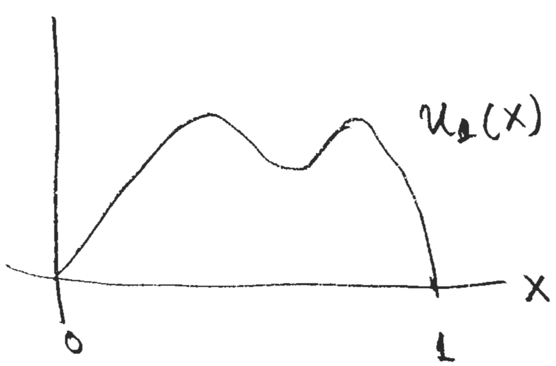
Assume $u(x) \approx \underbrace{-\frac{1}{2}x^2 + \frac{1}{2}x}_{u_0(x)} + \varepsilon u_1(x)$. Plug into (6), equate powers of ε:

$$\varepsilon: \quad u_1'' = \sin \pi x - \frac{2}{\pi} + \pi \cos \pi x \left(x - \frac{1}{2}\right).$$

$$u_1(0) = 0, \quad u_1(1) = 0.$$

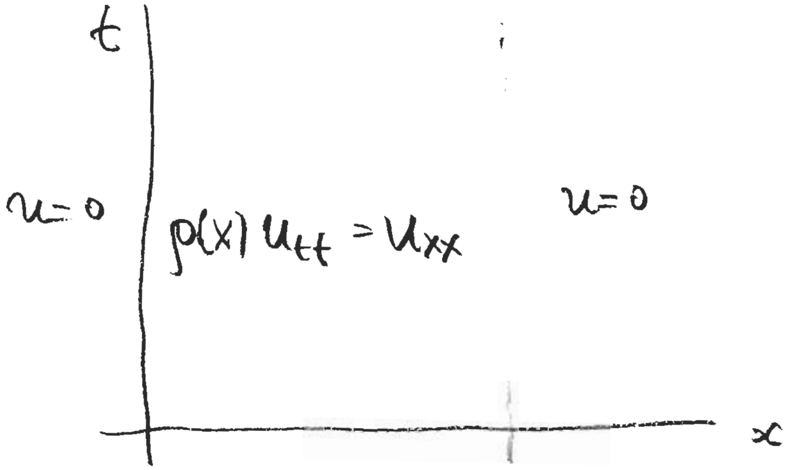
Note: u_1 solves the simple $\varepsilon = 0$ version of (6) with forcing terms from $A(x)$, $u_0(x)$ and their derivatives.

$$u_1(x) = \frac{1}{\pi^2} \sin \pi x + \frac{1}{2\pi} \cos \pi x - \frac{1}{\pi} x \cos(\pi x) - \frac{1}{\pi^2} x^2 + \frac{1}{\pi} x - \frac{1}{2\pi}.$$



Insight: negative ε (rod thinner in the middle) would reduce the maximum temperature.

Ex 7 Consider a string of length 1, fixed at both ends ($x=0$ & $x=1$) under tension $T=1$ with density $\rho(x) = 1 + 0.01x$ in suitable units.



Find the dominant angular frequency μ of this string ($\lambda_1 = -\mu^2$).

Separate variables $u(x,t) = X(x)T(t)$.

$$\frac{T''}{T} = \frac{X''}{\rho(x) X} = \lambda \qquad \text{expect } \lambda < 0, \ \lambda = -\mu^2.$$

$$T'' = -\mu^2 T, \quad T = A\cos\mu t + B\sin\mu t.$$

(confirms the idea that μ is an angular frequency).

$$X'' = -\mu^2 (1 + \underbrace{0.01}_{\varepsilon} x) X \qquad X(0) = 0, \ X(1) = 0 \quad (7).$$

Note: It is possible to determine X & μ exactly in terms of Airy's functions, but let's use the asymptotic approach to get approximate solutions in more familiar form.

Expand $\mu(\varepsilon) \approx \mu_0 + \varepsilon \mu_1$

$$X(\varepsilon, x) \approx X_0(x) + \varepsilon X_1(x)$$

Plug these expressions into (7), equate powers of ε:

$$1: \quad X_0'' = -\mu_0^2 X_0 \qquad X_0(0) = X_0(1) = 0 \qquad (8a)$$

$$\varepsilon: \quad X_1'' = -2\mu_0\mu_1 X_0 - \mu_0^2 X_1 - \mu_0^2 x X_0 \qquad (8b).$$
$$X_1(0) = X_1(1) = 0$$

(8a) is the $\varepsilon = 0$ eigenvalue problem. We are interested in the fundamental frequency, so take

$$\mu_0 = \pi, \quad X_0(x) = \sin \pi x$$

Could have chosen other eigenvalues ($\mu_0 = n\pi$, $X_0 = \sin n\pi x$) and perturbed those. Use $y(x)$ to denote $X_1(x)$ in (8b):

$$\left.\begin{aligned} y'' + \pi^2 y &= -2\pi\mu_1 \sin \pi x - \pi^2 x \sin \pi x \\ y(0) &= y(1) = 0 \end{aligned}\right\} \quad (9).$$

This problem has to determine both μ_1 and $y(x)$. Use the method of undetermined coefficients to solve (9). It is an exceptional case, since the RHS has both homogeneous solutions and x times homogeneous solutions. Look for $y(x)$ in the form

$$\begin{aligned} y(x) = {} & A \sin \pi x + B \cos \pi x \leftarrow \text{homogeneous.} \\ & \left.\begin{aligned} &+ a x \sin \pi x + b x \cos \pi x \\ &+ c x^2 \sin \pi x + d x^2 \cos \pi x \end{aligned}\right\} \text{particular.} \end{aligned}$$

Plug this form into (9), equate function coefficients to determine a, b, c and d.

$$\Rightarrow \quad y(x) = A\sin \pi x + B\cos \pi x - \frac{1}{4}x\sin \pi x + \mu_1 x \cos \pi x + \frac{\pi}{4}x^2 \cos \pi x.$$

Match to $y(0)=0, \ y(1)=0$

$y(0)=0 \Rightarrow B=0.$

$y(1)=0 \Rightarrow -\mu_1 - \pi/4 = 0 \Rightarrow \mu_1 = -\pi/4.$

So $\mu \approx \pi - \varepsilon\pi/4$. With $\varepsilon = 0.01$, $\mu \approx 3.1337$ and this is correct to the exact μ to all digits shown.

The corresponding eigenvector

$$X(x) = \sin \pi x + \varepsilon\left\{ -\frac{1}{4}x\sin \pi x - \frac{\pi}{4}x\cos \pi x + \frac{\pi}{4}x^2\cos \pi x + A\sin \pi x \right\}$$

A has not yet been determined. Notice above that A just scales the $O(1)$ eigenfn by $1 + O(\varepsilon)$, and the size of that eigenfn is arbitrary. It is reasonable to take $A = 0$.

<u>Discussion</u>: We could pretend that <u>Ex</u> 7 is a problem coming from quality control concerns at a guitar factory. The question is how large ε can be ($\varepsilon=0$ corresponds to string cross-sectional area variations from manufacturing processes) before the string makes a sound that is noticeably wrong? This is the kind of problem for which asymptotics is well

suited, since very small changes in sound quality are detectable by expert ears. The small changes are difficult to resolve using computational approximation of the full problem (but computing the asymptotic problem for μ_1, X_1, which are not small can be done in a straight-forward way in cases where they cannot be solved analytically).

More realistically, μ can always be adjusted (by tuning, adjusting the tension T) to be a specific value. The ratio of μ's is fixed, however, by the string construction. For a perfect string ($\varepsilon = 0$), $\mu = n\pi$, so the frequencies are in integer ratios to the fundamental frequency. To approach this issue, we'd compute asymptotic expressions for a number of frequencies

$$\mu^{(n)} \approx n\pi + \varepsilon \mu_1^{(n)} \qquad \left[\mu_1^{(1)} \approx -\pi/4 \ \text{computed above} \right].$$

Then compute

$$\frac{\mu^{(n)}}{\mu^{(1)}} \approx \frac{n\pi + \varepsilon \mu_1^{(n)}}{\pi - \varepsilon \pi/4} \approx n + \varepsilon \left(\frac{n}{4} + \frac{\mu_1^{(n)}}{\pi} \right).$$

and determine how small ε needed to be to make this "acceptable" for a given range of n. The results would help the manufacturer assess the trade-off between quality and the cost of more control on the manufacturing process.

<u>Note</u>: "Real" strings are composite materials and nonlinear effects must be considered, I believe.

§8.4 Regular perturbations for 2D Elliptic problems.

Ex 8 Consider the following problem for $\psi(r,\theta)$

$$\Delta\psi + \psi^2 = 0 \quad \text{in disk } r \le 1.$$

with $\psi(1,\theta) = \varepsilon\cos\theta$.

Let $u = \varepsilon\psi$ then

$$\left.\begin{aligned} &\Delta u + \varepsilon u^2 = 0 \\ &u(1,\theta) = \cos\theta \end{aligned}\right\} \quad (10).$$

Note: The small boundary terms (data) showed us that the quadratic terms in the equation were very small. Often, the small parameter ε appears in a problem after scaling.

Consider a regular perturbation expansion for (10):

$$u \approx u_0(r,\theta) + \varepsilon u_1(r,\theta) + \cdots$$

Plug into (10), equate coefficients of powers of ε:

$$1: \quad \Delta u_0 = 0 \; ; \quad u_0(1,\theta) = \cos\theta \qquad (11a).$$

$$\varepsilon: \quad \Delta u_1 = -u_0^2 \; ; \quad u_1(1,\theta) = 0 \qquad (11b).$$

In polar coordinates,

$$\Delta\psi = \psi_{rr} + \frac{1}{r}\psi_r + \frac{1}{r^2}\psi_{\theta\theta}$$

Solving the Dirichlet problem (11a) can be done using an explicit Green's function representation, but a series soln from separation of variables can also be used:

$$\psi(r,\theta) = \sum_{n=-\infty}^{\infty} A_n r^{|n|} e^{in\theta}$$

where $\{A_n\}$ are the complex Fourier coefficients of the Dirichlet data $\psi(1,\theta)$. (11a) clearly has a simple solution of this form:

$$u_0(r,\theta) = r\cos\theta.$$

<u>Note</u>: I did not notice when I prepared this example that $u_0 \equiv y$. This might have made the algebra that follows trivial...

Now (11b) reads

$$u_{1,rr} + \frac{1}{r}u_{1,r} + \frac{1}{r^2}u_{1,\theta\theta} = -r^2\cos^2\theta$$
$$= -\frac{r^2}{2}(1+\cos 2\theta).$$

In general, $u_1(r,\theta) = \sum_{n=-\infty}^{\infty} B_n(r)e^{in\theta}$. From the form of the RHS above we can see that the $n \neq 0, 2$ terms will have $B_n(r) \equiv 0$, so

$$u_1(r,\theta) = B_0(r) + B_2(r)\cos(2\theta).$$

plug in

$$B_0'' + \frac{1}{r}B_0' = -\frac{r^2}{2}$$
$$(rB_0') = -r^3/2$$
$$rB_0' = -r^4/8 + C$$
$$B_0' = -r^3/8 + C/r.$$ B_0 bounded at $r=0$

$$B_0 = -r^4/32 + \cancel{C\ln r} + D$$

matching $B_0(1) = 0$,

$$B_0(r) = \frac{1}{32}(1-r^4)$$

$$B_2'' + \frac{1}{r}B_2' - \frac{4}{r^2}B_2 = \frac{r^2}{2}$$

Euler equation, homogeneous soln's of the form r^α, $\alpha = \pm 2$.

particular soln $-\frac{1}{2}r^2\ln r$.

$$B_2(r) = -\frac{1}{2}r^2\ln r.$$

So $u_1(r,\theta) = \frac{1}{32}(1-r^4) - \frac{1}{2} r^2 \ln r \,(\cos 2\theta)$.

$$u(r,\theta) \approx r\cos\theta + \varepsilon \left\{ \frac{1}{32}(1-r^4) - \frac{1}{2} r^2 \ln r \cos 2\theta \right\}.$$

Ex 9 Consider the problem for $u(r,\theta)$, $\Delta u = 0$ in the perturbed disk $\{(r,\theta) : r \leq 1+\varepsilon\cos\theta\}$, with $u(1+\varepsilon\cos\theta, \theta) = \cos\theta$.

Try $u(r,\theta) \approx \overbrace{r\cos\theta}^{u_0} + \varepsilon\, u_1(r,\theta)$ again.

$$\Delta u = 0, \quad \Delta u_1 = 0.$$

On boundary

$$u(1+\varepsilon\cos\theta, \theta) \approx u(1,\theta) + \varepsilon\, u_r(1,\theta)\cos\theta$$

$$\approx \underset{\substack{\uparrow \\ u_0(1,\theta)}}{\cos\theta} + \varepsilon\, u_1(1,\theta) + \varepsilon\, u_{0,r}(1,\theta)\cos\theta.$$

This should equal $\cos\theta$, so the ε terms must cancel:

$$u_1(1,\theta) = -\frac{\partial}{\partial r} u_0(1,\theta)\cos\theta = -\cos^2\theta$$

$$= -\frac{1}{2}(1+\cos 2\theta).$$

This boundary condition leads to two terms in the series for harmonic functions at the bottom of p.10:

$$u_1(r,\theta) = -\frac{1}{2} - \frac{r^2}{2}\cos 2\theta.$$

So $u(r,\theta) \approx r\cos\theta + \varepsilon\left\{-\frac{1}{2} - \frac{r^2}{2}\cos 2\theta\right\}$.

§8.5 Free Boundary Value Problems.

Ex 10 Minimize $F[u] = \int_0^{x_0} \{(u')^2 + 1\}\, dx$
where $u(0) = 1$ and $u(x_0) = 0$ (x_0 is the first positive zero of u). Determining x_0 is part of the problem, it is a "free boundary".

Soln A: Suppose x_0 were given. We could minimize

$$F[u] = \int_0^{x_0} \{(u')^2 + 1\}\, dx \quad , u(0) = 1, \ u(x_0) = 0$$

using standard techniques

E-L $\qquad -u'' = 0 \Rightarrow u(x) = 1 - x/x_0.$

$$(u')^2 = \frac{1}{x_0^2}$$

So $\quad \underbrace{F_{min}}_{\text{at given } x_0} = \int_0^{x_0} \left(\left(\frac{1}{x_0}\right)^2 + 1 \right) dx = \frac{1}{x_0} + x_0.$

Now, we can minimize over all x_0 (1D optimization) giving $x_0 = 1$

$$\underbrace{F_{min}}_{\text{over all } x_0} = 2.$$

Note: in 2D, it would not be possible to check all possible free boundaries (curves) so we find an alternate approach below.

Soln B: Derive (then apply) the E-L equations for free boundary value problems.

More generally, we seek the

$$\min \quad F[u] = \int_0^{x_0} f(x, u, u')\, dx \qquad u(0) \text{ given } > 0$$

with x_0 the first positive zero of $u(x)$.

Consider $F[u+\varepsilon\eta] = \int_0^{\hat{x}_0} f(x, u+\varepsilon\eta, u'+\varepsilon\eta')\, dx$ (13).
where $\eta(0)=0$, $\eta(x_0)$ is arbitrary and $\hat{x}_0$ solves

$$u(\hat{x}_0) + \varepsilon\eta(\hat{x}_0) = 0 \qquad (14).$$

Break (13) into 2 parts: I_1 from 0 to x_0 and I_2 from x_0 to $\hat{x}_0$. The first integral is handled in the usual way, giving (15).

$$I_1 = F_{min} + \varepsilon\left\{\int_0^{x_0} (f_u - (f_p)')\,\eta\, dx + f_p\,\eta\Big|_{x=x_0}\right\} + O(\varepsilon^2)$$

Considering (14) we can see that

$$\hat{x}_0 - x_0 = -\varepsilon\frac{\eta(x_0)}{u'(x_0)} + O(\varepsilon^2).$$

So $I_2 \approx -\varepsilon \dfrac{f}{u'}\eta\Big|_{x=x_0} + O(\varepsilon^2)$.

Considering I_1 in (15) and I_2 above summing to $F[u+\varepsilon\eta]$ in (13), we see that for u to be a minimizer

$$f_u - f_p' = 0 \qquad x \in [0, x_0].$$

and $f_p - \dfrac{f}{u'} = 0$ at $x = x_0$

$$\text{or } \quad u' f_p - f = 0 \quad \text{at } x = x_0. \qquad (16).$$

Let's apply this to Ex 10, where $F[u] = (u')^2 + 1$.

$$-u'' = 0, \quad u(0) = 1, \qquad u(x_0) = 0$$

$$\Rightarrow \quad u(x) = 1 - x/x_0 \qquad \text{as before.}$$

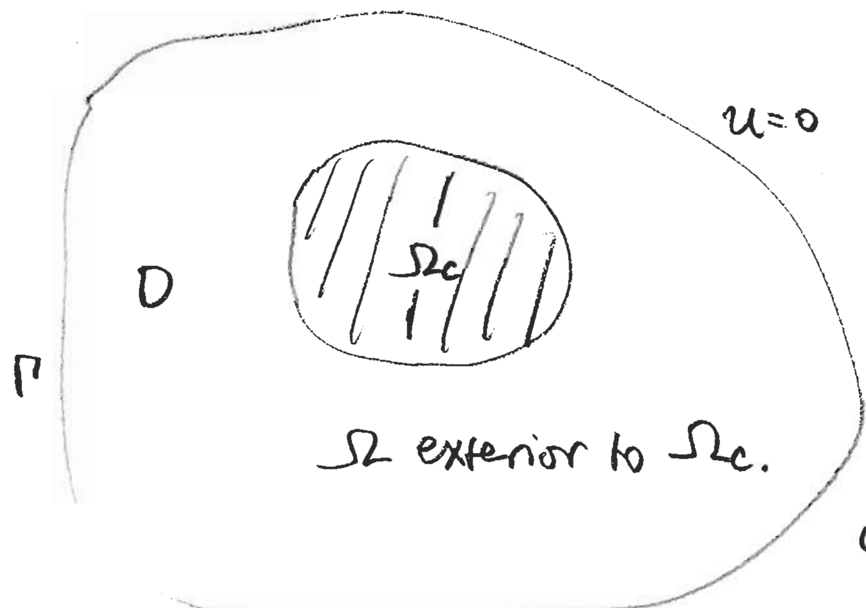
(16) reads $(u')(2u') - [(u')^2 + 1] = 0$ at $x = x_0$,

$$(u')^2 = 1 \Rightarrow \frac{1}{x_0^2} = 1 \Rightarrow x_0 = 1.$$

The extension to free boundary value problems in 2D is given below.

minimize $F[u(x,y)] = \int_D f(x, u, \overbrace{u_x}^{p}, \overbrace{u_y}^{q})\, dA$

Where D is the domain shown below:



Where Ω_c is a given domain and values $u\big|_{\partial\Omega_c}$ are given, >0

D can be described as the part of $\Omega \cap \{u \geq 0\}$ connected to $\partial\Omega_c$.

The E-L equations here are the usual

$$f_u - (f_p)_x - (f_q)_y = 0.$$

with boundary conditions on the free boundary Γ

$$u = 0$$

and $\quad f_p u_x + f_q u_y - f = 0$

Note: 2 BC's are specified on the free boundary. Consider that one allows the solution of the second order equation for u, the other determines the free boundary location Γ.

Ex 11 Consider the free boundary problem

$$\text{minimize} \quad F[u] = \int_D (u_x^2 + u_y^2 + 1)\, dA$$

where Ω_c is the disk of radius 1 and $u\big|_{\partial\Omega_c} = 1$.

Apply the equations above

$$\Delta u = 0, \quad u = 1 \quad \text{on } \partial\Omega_c.$$

Look for radially symmetric solutions. As found in Chapter 4,

$$u = A + B \ln r$$

$$u(1) = 1 \Rightarrow A = 1, \qquad u = 1 + B \ln r.$$

On the free surface (hopefully $r = R_0$),

$$|\nabla u|^2 = \left(\frac{\partial u}{\partial r}\right)^2 + \cancel{\left(\frac{1}{r}\frac{\partial u}{\partial \theta}\right)^2}$$

$$u = 0 \text{ on } r = R_0$$

The condition at the bottom of p. 15 in this case reads

$$2u_x(u_x) + (2u_y)u_y - (u_x^2 + u_y^2 + 1) = 0$$

or $|\nabla u|^2 = 1$. Using the result above

$$\left(\frac{\partial u}{\partial r}\right)^2 = 1 \quad \Rightarrow \quad \frac{\partial u}{\partial r} = -1 \quad (u \text{ must be decreasing}).$$

With $u = 1 + B\ln r$ we have at $r = R_0$:

$$u(R_0) = 0 \Rightarrow 1 + B\ln R_0 = 0.$$

and $\frac{\partial u}{\partial r} = -1 \Rightarrow \frac{B}{R_0} = -1.$

Combining we have $1 - R_0 \ln R_0 = 0$, which has a unique root R_0, with $R_0 > 1$. $u = 1 - R_0 \ln r$.

<u>Ex</u> 12 Derive an asymptotic correction to the solution above when

$$\partial\Omega_c = \{(r,\theta) : r = 1 + \varepsilon\cos\theta\}.$$

Expect solutions of the form

$$u(r,\theta) \approx u_0(r) + \varepsilon u_1(r)\cos\theta, \qquad u_0(r) = 1 - R_0 \ln r,$$

free boundary $\quad r = R_0 + \varepsilon R_1 \cos\theta.$

$$\Delta u = 0 \Rightarrow \Delta u_0 = 0 \quad \checkmark$$

$$\Delta u_1 = 0 \Rightarrow u_1(r) = Ar + B/r.$$

On $\partial\Omega_c$,

$$1 = u(1 + \varepsilon\cos\theta, \theta) \approx \underbrace{1}_{u_0(1)} + \varepsilon u_0'(1)\cos\theta + \varepsilon u_1(1)\cos\theta$$

so $u_1(1) = -u_0'(1) = R_0$

$$\Rightarrow \quad A + B = R_0 \qquad (17).$$

The free boundary conditions must give two more conditions for A, B, and R_1.

$$u(R_0 + \varepsilon R_1 \cos\theta) \approx 0$$

$$\Rightarrow u_0'(R_0) R_1 \cos\theta + u_1(R_0)\cos\theta = 0 \qquad \text{at } O(\varepsilon).$$

so $\quad -R_1 + AR_0 + \dfrac{B}{R_0} = 0 \qquad (18).$

Also, $|\nabla u|^2 = 1$ on the free boundary.

$$|\nabla u|^2 = \left(\frac{\partial u}{\partial r}\right)^2 + \underbrace{\left(\frac{1}{r}\frac{\partial u}{\partial \theta}\right)^2}_{O(\varepsilon^2).} = 1$$

so $\quad \dfrac{\partial u}{\partial r}(R_0 + \varepsilon R_1 \cos\theta, \theta) \approx -1$

$$\underbrace{u_0'(R_0)}_{=-1} + \varepsilon\left(u_0''(R_0) R_1 \cos\theta + u_1'(r)\cos\theta\right) \approx -1.$$

At $O(\varepsilon)$ above we have

$$\underbrace{u_0''(R_0)}_{1/R_0} R_1 + u_1'(R_0) = 0.$$

so $\quad \dfrac{R_1}{R_0} - \dfrac{B}{R_0^2} = 0 \qquad (19).$

(17), (18), and (19) are a linear system for A, B, R_1:

$$A=0,\ B=R_0,\ R_1 = 1.$$

So $\quad R \approx R_0 + \varepsilon \cos\theta$

$$u \approx 1 - R_0 \ln r + \varepsilon R_0 \cos\theta / r.$$

§8.6 Regular Perturbations of Wave Equations.

$$u_{tt} = c^2 u_{xx} \qquad (20)$$

c is the wave speed, take $c \equiv 1$ for now, let $c(x)$ later.

Consider a wave travelling to the right of the form $u(x,t) = \cos(x-t)$. It is easier if we take

$$u(x,t) = \mathrm{Re}\left\{ e^{i(x-t)} \right\} \qquad (21).$$

Suppose $c(x) \equiv 1$ for $x<0$ but can vary for $x>0$. We could consider a wave of the form (21) incident at $x=0$ for a "long time". We could then consider solutions of the form

$$u(x,t) = \mathrm{Re}\left\{ \underbrace{A(x)}_{\text{complex amplitude}} e^{-it} \right\} \qquad (22).$$

for all x. We will apply this idea to the following example:

<u>Ex</u> 13. Suppose $c(x) = \begin{cases} 1 & x<0 \\ W & x>0 \end{cases}$, $W>0$ constant.

For $x<0$, solutions of the form (22) put into (20) give

$$-A = A'' \Rightarrow A(x) = a e^{ix} + b e^{-ix} \qquad (23).$$

<u>Note</u>: Since the problem is linear, we can compute the complex solution, then take the real part at the end.

We considered the wave (21) incident, so this leads to $a=1$ in (23). The second term in (23) leads to a solution u of the form

$be^{-i(x+t)}$, a wave travelling to the left. This represents a reflected wave and b must be determined.

For $x > 0$,

$$-A = W^2 A'' \Rightarrow A(x) = c\,e^{ix/W} + d\,e^{-ix/W},$$

leading to a solution

$$u(x,t) = c\,e^{i(x/W - t)} + d\,e^{-i(x/W + t)}.$$

transmitted wave, c to be determined

d=0, no incoming wave from the left.

At $x=0$, we can apply the physical conditions that u and u_x be condinuous.

$$[u] = 1 + b - c = 0.$$

$$[u_x] = i - ib - i\,c/W = 0.$$

This is a linear system for b & c with solution

$$b = \frac{W-1}{W+1} \quad \text{(reflection coefficient).}$$

$$c = \frac{2W}{W+1} \quad \text{(transmission coefficient).}$$

<u>Note</u>: if $W = 1 + O(\varepsilon)$, reflection is $O(\varepsilon)$ and transmission is $1 + O(\varepsilon)$.

Ex 14. Consider a situation where

$$c(x) = 1 + \varepsilon \phi(x)$$

where $\phi(x)$ has compact support in $[0,1]$. Consider $\varepsilon\phi$ to be a defect in a transmission line. Determine the $O(\varepsilon)$ reflection and $O(\varepsilon)$ transmission correction to an incident wave of the form (21). Take the form (22) as before

$$A(x) = \begin{cases} e^{ix} + \varepsilon R e^{-ix} & x \le 0, \text{ R to be determined} \\ e^{ix} + \varepsilon B_1(x) & x \in [0,1], \; B_1(x) \text{ to be det.} \\ e^{ix} + \varepsilon T e^{i(x-1)} & x \ge 1, \text{ T to be determined.} \end{cases}$$

Note: $[A]$ and $[A_x] = 0$ at both $x = 0, 1$.

At $x = 1$, $\quad B_1(0) = R$

$\quad\quad\quad\quad B_1'(0) = -iR$

That is, $\quad B_1(0) - i B_1'(0) = 0$

Similarly, $\quad B_1(1) + i B_1'(1) = 0$ $\quad\quad$ (24).

Putting u of the form (22) with A as above into the wave equation (20) gives at $O(\varepsilon)$ in $[0,1]$:

$$-B_1 = B_1'' - 2\phi(x) e^{ix}$$

or $$B_1'' + B_1 = 2\phi(x) e^{ix} \quad\quad (25)$$

Homogeneous solutions of (23) are

$$Ce^{ix} + De^{-i(x-1)}$$

Note that BC's (24) force $C=0$, $D=0$ so solutions to (24, 25) can always be found and are unique.

Take a specific example, $\phi(x) = \sin \pi x$,

$\phi(x) = \frac{1}{2i}\left(e^{i\pi x} - e^{-i\pi x}\right)$. (25) reads

$$B_1'' + B_1 = \frac{1}{2i} e^{i(\pi+1)x} - \frac{1}{2i} e^{i(1-\pi)x}$$

This has solution

$$B_1(x) = Ce^{ix} + De^{-i(x-1)} + \frac{1}{2i(1-(\pi+1)^2)} e^{i(\pi+1)x} - \frac{1}{2i(1-(1-\pi)^2)} e^{i(1-\pi)x}.$$

Could apply (24) to determine C & D. Then one can determine

$$R = B_1(0), \quad T = B_1(1).$$

§ 8.7 A problem with multiple time scales.

Ex 15 Consider the following problem for $u(x,t)$, $x \in [0,1]$, $t \geq 0$:

$$\left.\begin{aligned} u_t &= -\varepsilon V(x) u + u_{xx} \\ u_x(0,t) &= u_x(1,t) = 0, \quad u(x,0) \equiv 1 \end{aligned}\right\} \quad (26).$$

where $V(x) > 0$ for all x. We'll take

$$V(x) = 2 + \cos 2\pi x$$

Note: This could describe heat conduction in a rod with temperature u. The rod is insulated at the ends, but not perfectly insulated along its length. The term $-\varepsilon V(x) u$ describes heat loss to ambient (zero temperature). The heat loss coefficient $V(x)$ varies along the length of the rod.

Let's do a regular expansion of the solution to (26), although it will prove to give unsatisfactory results. Clearly, $u_0(x,t) \equiv 1$, and if

$$u(x,t) \approx 1 + \varepsilon u_1(x,t)$$

we can insert into (26) and collect terms of $O(\varepsilon)$:

$$u_{1,t} = -(2 + \cos 2\pi x) + u_{1,xx}. \qquad u_1(x,0) \equiv 0$$

$$u_{1,x}(0,t) = 0, \quad u_{1,x}(1,t) = 0.$$

It is natural to consider solutions $u_1(x,t)$ in the form of a cosine series:

$$u_1(x,t) = \sum_{n=0} A_n(t) \cos n\pi x.$$

Only $A_0(t)$ and $A_2(t)$ will be nonzero,

$$A_0' = -2 \qquad A_0 = -2t$$

$$A_2' = -1 - 4\pi^2 A_2 \qquad A_2 = -\frac{1}{4\pi^2}\left(1 - e^{-4\pi^2 t}\right)$$

so $$u(x,t) \approx 1 + \varepsilon\left(-2t - \frac{1}{4\pi^2}\left(1 - e^{-4\pi^2 t}\right)\right). \qquad (27)$$

As predicted, this does not look right. At $t = O(\frac{1}{\varepsilon})$, the approximation u goes negative, which is unphysical. We would like to have an approximation that is uniformly valid for all x and t.

To get insight, consider the simpler case $V(x) \equiv 2$ (the average value of the previous V). In this case,

$$u(x,t) = e^{-2\varepsilon t}$$

is the solution. The term $1 - 2\varepsilon t$ in (27) came as a Taylor approximation to this exponential, not valid for long times of $O(\frac{1}{\varepsilon})$.

We can derive a better asymptotic description of this problem by explicitly adding dependence of the solution on these long time scales. Introduce

$$\tau = \varepsilon t \qquad \leftarrow \text{slow time scale.}$$

Consider $$u \approx u_0(\tau, x) + \varepsilon u_1(t, \tau, x).$$

over a long time, the $O(\varepsilon)$ terms can have an $O(1)$ effect.

With this expression,

$$u_t \approx \varepsilon u_{0,\tau} + \varepsilon u_{1,t}$$

$$u_{xx} = u_{0,xx} + \varepsilon u_{1,xx}.$$

(26) Now has an $O(1)$ term

$$u_{0,xx} = 0 \qquad u_{0,x}(0,\tau) = u_{0,x}(1,\tau) = 0$$
$$u_0(x,0) = 1.$$

This shows that at each τ, u_0 is constant in x, i.e. $u_0(x,\tau) = f_0(\tau)$, with $f_0(0) = 1$.

Consider now the $O(\varepsilon)$ terms in (26):

$$f_0' + u_{1,t} = -\underbrace{\left(2 + \cos(2\pi x)\right)}_{V(x)} f_0(\tau) + u_{1,xx}. \qquad (28).$$

There is some question of what terms go together in (28). It seems reasonable to match

$$f_0' = -2f \quad \Rightarrow \quad f_0(\tau) = e^{-2\tau}$$

Consider the remaining terms in (28)

$$\left.\begin{aligned} &u_{1,t} = -\cos(2\pi x)\, e^{-2\tau} + u_{1,xx} \\ &u_{1,x}(0,t) = 0, \quad u_{1,x}(1,t) = 0, \quad u_1(x,0) \equiv 0. \end{aligned}\right\} (29).$$

Don't forget that $u_1(x,t,\tau)$. Consider an additive term in u_1 of the form

$$g_1(\tau)\, W(x) + f_1(\tau).$$

Inserting this into (29) shows that $f_1(\tau)$ is arbitrary, $g_1(\tau) = e^{-2\tau}$ and $W(x)$ solves

$$W'' = \cos(2\pi x) \qquad W'(0) = W'(1) = 0. \qquad (30)$$

This is the problem we looked at in Assignment I (#4). Solutions are only obtained if the right hand side has average value 0 (✓). Now it is clear that following (28) we had to take

$$f_0' = -V_{ave} f$$

to be able to handle the problem above, so that match is not arbitrary. The solution to (30) is

$$W = -\frac{1}{4\pi^2}\cos(2\pi x).$$

This solution is arbitrary up to a constant, but that constant can be absorbed into $f_1(\tau)$. Now consider

$$u_1(x,t,\tau) = e^{-2\tau}\left\{-\frac{1}{4\pi^2}\cos(2\pi x)\right\} + f_1(\tau) + \hat{u}_1(x,t).$$

Inserting this expression into (29) gives

$$\hat{u}_1(t) = \tilde{u}_{1,xx} \qquad \hat{u}_1(x,0) = \frac{1}{4\pi^2}\cos(2\pi x)$$

$$\hat{u}_{1,x}(0,t) = 0, \quad \hat{u}_{1,x}(1,t) = 0$$

which has solution

$$\hat{u}_1(x,t) = \frac{1}{4\pi^2}\cos(2\pi x)\, e^{-4\pi^2 t}.$$

The $O(\varepsilon)$ term $f_1(T)$ is still to be determined. This is done by considering the $O(\varepsilon^2)$ terms in (26):

$$f_1' + u_{2,t} = u_{2,xx} - (2+\cos 2\pi x)\Big\{\underbrace{f_1 + \frac{1}{4\pi^2}\cos 2\pi x\left(e^{-4\pi^2 t} - e^{-2T}\right)}_{u_1}\Big\}. \qquad (31)$$

As above, a term of the form $g_2(T)W_2(x)$ in u_2 can match the slow terms in the second line of (31) that have average value zero. Thus, f_1 must match the average value of these terms, i.e.

$$f_1' = -2f_1 + \frac{1}{8\pi^2}e^{-2T} \qquad f_1(0) = 0.$$

since $\cos^2(2\pi x) = \frac{1}{2} - \underbrace{\frac{1}{2}\cos(4\pi x)}_{\text{average value zero.}}$

so $f_1(T) = +\frac{1}{8\pi^2}Te^{-2T}$.

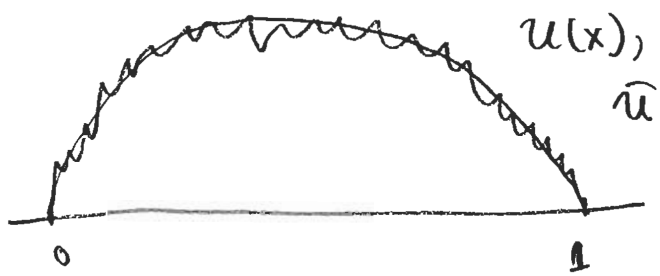
Summing up,

$$u \approx e^{-2\varepsilon t} + \varepsilon\left\{\frac{1}{8\pi^2}\varepsilon t e^{-2\varepsilon t} + \frac{1}{4\pi^2}\cos 2\pi x\left(e^{-4\pi t} - e^{-2\varepsilon t}\right)\right\}.$$

I believe this can be shown to be a uniformly $O(\varepsilon^2)$ approximation over all (x,t).

§8.8 Homogenization.

Consider $c(x)\, u'' = f(x)$, $u(0)=0$, $u(1)=0$, where $c(x)$ varies over very short spatial scales, ie. $c(x) = w(x/\varepsilon)$ where w is 1-periodic. We assume $w(x) > 0$ for all x. In this situation, u will also vary over the small spatial scale ($O(\varepsilon)$) but we are only interested in the average of the solution over these scales.



$u(x)$, with fine scale detail,
$\bar{u}(x)$ average.

This is another multiple scale problem, assume

$$u(x) \approx u_0\left(x, \tfrac{x}{\varepsilon}\right) + \varepsilon\, u_1\left(x, \tfrac{x}{\varepsilon}\right) + \varepsilon^2 u_2\left(x, \tfrac{x}{\varepsilon}\right) + \cdots$$

$\uparrow$ "y" where u_j is 1-periodic in y.

$$\frac{du}{dx} \approx u_{0,x} + \frac{1}{\varepsilon} u_{0,y} + \varepsilon\, u_{1,x} + u_{1,y} + \varepsilon^2 u_{2,x} + \varepsilon u_{2,y}$$

$$\left.\begin{aligned}\frac{d^2u}{dx^2} \approx\ & u_{0,xx} + \frac{2}{\varepsilon} u_{0,xy} + \frac{1}{\varepsilon^2} u_{0,yy} + \\ & \varepsilon\, u_{1,xx} + 2 u_{1,xy} + \frac{1}{\varepsilon} u_{1,yy} + \\ & \varepsilon^2 u_{2,xx} + 2\varepsilon\, u_{2,xy} + u_{2,yy}\end{aligned}\right\} \quad (32)$$

put this expression into $w(y)\, u'' = f(x)$, match powers of ε:

$O(\frac{1}{\varepsilon^2})$ $u_{0,yy} = 0$ $\Rightarrow$ $u_0(x,y) = A(x) + \cancel{B(x)y}$ not periodic.

Thus $u_0(x)$ does not have fine scale structure, this is the averaged solution we're interested in, to first order.

$O(\frac{1}{\varepsilon})$ $c(y)\left(2\, \underbrace{\cancel{u_{0,xy}}}_{0 \text{ since } u_0(x)} + u_{1,yy}\right) = 0$. As above, $u_1(x)$ only.

$O(1)$: $c(y)\left(u_0'' + 2\, \underbrace{\cancel{u_{1,xy}}}_{0 \text{ since } u_1(x)} + u_{2,yy}\right) = f(x)$. (33).

$$u_0'' + u_{2,yy} = \frac{1}{c(y)} f(x)$$

To proceed, integrate this equation over $y \in [0,1]$ at each x. Note that $\int_0^1 u_{2,yy}\, dy = u_{2,y}\Big|_0^1 = 0$ since u_2 (and its derivatives) are periodic in y.

$$u_0'' = \left[\int_0^1 \frac{dy}{c(y)}\right] f(x).$$

Thus u_0 solves an averaged version of the original equation

$$\bar{c}\, u_0'' = f(x)$$

but with $\bar{c} = \dfrac{1}{\left[\int_0^1 \frac{dy}{c(y)}\right]}$ (the harmonic average of c).

As a check, let's consider (33), which we have only satisfied in an average way. We have

$$u_{2,yy} = \frac{1}{c(y)} \cdot f(x) - u_0'' = \left(\frac{1}{c(y)} - \frac{1}{\bar{c}} \right) f(x).$$

At each x, this can be solved with y periodic because the RHS has average value in y zero. Let $\phi_2(y)$ solve

$$\phi_2'' = \frac{1}{c(y)} - \frac{1}{\bar{c}} \qquad \phi_2 \text{ 1-periodic and } \phi_{2,ave} = 0.$$

Then $u_2(x,y) = A_2(x) + f(x)\,\phi_2(y)$, where $A_2(x)$ is determined when $O(\varepsilon^2)$ are matched.

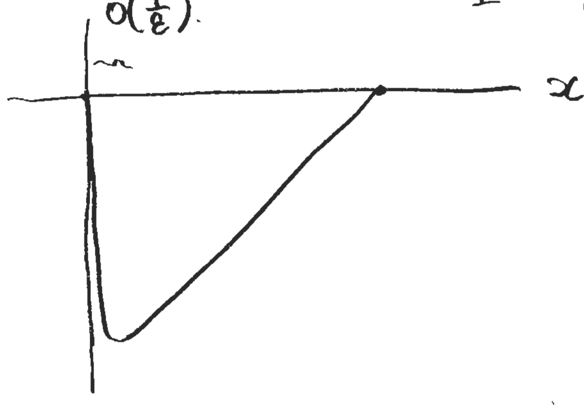
§8.9 An Example of a Singular Perturbation

Ex 16 Consider the ODE BVP

$$\varepsilon w'' + w' = 1, \qquad w(0) = w(1) = 0.$$

An explicit solution can be constructed:

$$w = (x-1) + \frac{e^{-x/\varepsilon} - e^{-1/\varepsilon}}{1 - e^{-1/\varepsilon}} = \alpha(x) + \beta(x/\varepsilon). \tag{34}$$



Note that when $\varepsilon=0$, $w = x + c$, only one of the boundary conditions can be matched. Let's see why the problem wants to match $w(1)=0$ smoothly and how we could consider the behaviour of the solution at $x=0$ in a systematic way.

Consider $w(x) \approx W_0(x) + W_1\left(\frac{x}{\varepsilon}\right) + W_2\left(\frac{1-x}{\varepsilon}\right)$

Note: we should consider $W_1\left(\frac{x}{\delta}\right)$ and work out later that $\delta=\varepsilon$, but let's take this as given.

Plug into the equation,

$$\varepsilon W_0'' + W_0' + \frac{1}{\varepsilon}\left(W_1'' + W_1'\right) + \frac{1}{\varepsilon}\left(W_2'' - W_2'\right) = 1.$$

We have at $O\left(\frac{1}{\varepsilon}\right)$ separately considering $x\approx 0$, and $x \approx 1$ respectively,

$$W_1'' + W_1' = 0, \quad W_2'' - W_2' = 0$$

so $W_1 = A_1 + B_1 e^{-x/\varepsilon}$, $W_2 = A_2 + B_2 e^{(1-x)/\varepsilon}$.

The terms A_1, A_2 can be absorbed into W_0. If W_2 is to have a significant effect at $x \approx 1$, then $B_2 = O(1)$. However, then this term is exponentially large away from $x=1$, hence $B_2 = 0$.

Now at $O(1)$ away from $x=0,1$, we have

$W_0' = 1$, so $W_0 = x + C$.

Now $w(x) \approx x + C + B_1 e^{-x/\varepsilon}$. Match BC'S,

$w(1) \approx 1 + C = 0 \Rightarrow C = -1$ ($e^{-1/\varepsilon}$ is an exponentially small correction).

$w(0) = -1 + B_1 \Rightarrow B_1 = 1$,

so $w(x) \approx (x-1) + e^{-x/\varepsilon}$

Agrees with the exact solution (34) except for exponentialy small terms.

1

Math 401, Spring 2007
Assignment #1.
due Monday, January 22.

#1. Solve the following problem for $u(x)$, $x \geq 0$:

$$u' + au = f(x) \qquad [f(x) \text{ given}].$$

$$u(0) = 0.$$

Recall: the integrating factor method, multiply the DE by e^{ax}.

From the form of the solution, determine the Green's function for this problem.

#2. With $f(x)$ given and continuous at all x, consider $u(x)$ given by

$$u(x) = \int_0^1 G(s,x) f(s)\, ds.$$

with $G(s,x) = \begin{cases} s\,(x-1) & 0 \leq s \leq x \\ x\,(s-1) & x \leq s \leq 1 . \end{cases}$

Show that $u''(x)$ exists for all x in $(0,1)$ and that $u'' = f$. Also show that $u(0) = u(1) = 0$.

#3. Consider the following discrete approximation of $u'' = f$, $u(0) = u(1) = 0$:

N subintervals, length $h = \frac{1}{N}$.

$U_i \approx u(ih) \qquad i = 1, \dots N-1$.

$$\frac{U_{i-1} - 2U_i + U_{i+1}}{h^2} = f(ih). \qquad i = 1, \dots N-1. \quad (*).$$

Note: At $i=1$, the term $U_0 = 0$ above and at $i = N-1$, $U_N = 0$.

Write the equations $(*)$ as an $(N-1)\times(N-1)$ linear system

$$\mathbb{A}\,\underline{U} = \underline{F}.$$

Determine $G = \mathbb{A}^{-1}$ explicitly.

Note: Do either #4 or #5 (both for bonus credit).

#4 Recall from the notes the nonlinear string problem energy

$$E[V] = k\left[\int_0^L \sqrt{1+[V'(x)]^2} - L\right]^2 + g\int_0^L \mu(x)\,V(x)\,dx.$$

with $\mu(x)$ the given suspended mass distribution and $V(0) = V(L) = 0$.

Using Euler's approach, determine the DE for $v(x)$.

#5 Approximate the solutions to the variable cross-sectional area heat conduction problem

$$((1+\varepsilon \sin \pi x)\, u')' = -1$$

$$u(0) = u(1) = 0$$

numerically, and compare to the first order perturbation approximation for $\varepsilon = 0.01$ and 0.1.

Optional problem: How would you construct a metal rod of given length that is uniformly heated but held at a reference temperature at both ends to minimize the maximum temperature of the rod. The mass of the rod is fixed.

That is, $\min \left(\max_{x \in [0,1]} u(x) \right)$ where u solves

$\min$ over all functions $A(x)$ with $A(x) > 0$, $\int_0^1 A(x) = 1$.

$$(Au')' = -1$$

$$u(0) = u(1) = 0.$$

Note: I'm not sure how to solve this problem myself

Notes on Assignment I:

Re #2: The following theorem will be useful:

If $f(s,x)$ is differentiable (milder conditions on s dependence are possible) then

$$\frac{d}{dx}\int_0^x f(s,x)\,ds \text{ exists and}$$
$$= \int_0^x \frac{\partial f}{\partial x}(s,x)\,ds + f(x,x).$$

Re #5: On a grid with spacing h, $U_j \approx u(jh)$

$(A(x)u')'$ can be approximated by

$$\frac{A_{j+\frac{1}{2}}(U_{j+1}-U_j) - A_{j-\frac{1}{2}}(U_j - U_{j-1})}{h^2}$$

where $A_{j+\frac{1}{2}} = A((j+\frac{1}{2})h)$.

Re #4: This energy isn't quite right for the problem (sorry) but derive the equation of the function that minimizes it anyway. It will have a nonlocal (integral) term in it.

Math 401, Spring 2007
Assignment I solutions.

#1. $u' + au = f(x) \qquad u(0) = 0.$

multiply by e^{ax}

$$(e^{ax}u)' = e^{ax}f.$$

integrate $\quad e^{ax}u(x) = \int_0^x e^{as} f(s)\,ds.$

$$u(x) = \int_0^x e^{a(s-x)} f(s)\,ds. \qquad (1)$$

if we took the general form for a Green's function

$$u(x) = \int_0^\infty G(s,x)\, f(s)\,ds.$$

$\uparrow$ soln point $\qquad \uparrow$ integrate over all data points

then $\quad G(s,x) = \begin{cases} e^{a(s-x)} & \text{if } s \le x \\ 0 & s > x \end{cases}$

to fit the form of (1). Note that the condition $G=0$ for $s > x$ is an expression of <u>causality</u> (that is, x is a timelike variable, and future forcing cannot affect the past).

#2. $$u(x) = (x-1)\int_0^x s f(s)\,ds + x\int_x^1 (s-1)f(s)\,ds. \quad (2)$$

Note: you don't need the fancy theorem I quoted after all, just the product rule and the fundamental theorem of calculus.

$$u' = \int_0^x s f(s)\,ds + \cancel{x(x-1)f(x)}$$

$$+ \int_x^1 (s-1) f(s)\,ds - \cancel{x(x-1)f(x)}$$

$$u'' = x f(x) - (x-1)f(x) = f(x)$$

From form (2), clearly $u(0)=0$ & $u(1)=0$.

#3.

$$A = N^2 \begin{bmatrix} -2 & 1 & 0 & \cdots & & \\ 1 & -2 & 1 & 0 & \cdots & \\ 0 & 1 & -2 & 1 & 0 & \cdots \\ \vdots & & \ddots & \ddots & \ddots & \\ & & & 1 & -2 & 1 \\ 0 & \cdots & 0 & 0 & 1 & -2 \end{bmatrix}. \quad (3)$$

To find $\underline{v}$, the j'th column of $A^{-1} = G$, solve

$$A\underline{v} = \underline{e}_j = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow \text{j'th column}$$

The components v_i of $\underline{v}$ satisfy

$$v_{i-1} - 2v_i + v_{i+1} = \begin{cases} 1/N^2 & \text{if } i=j \\ 0 & \text{otherwise.} \end{cases} \quad (4)$$

Note: (4) is consistent with (3) at $i=1$ and $i=N-1$ if we formally consider

$$V_0 = V_N = 0.$$

Now $V_{i-1} - 2V_i + V_{i+1} = 0$ (5). is a constant coefficient diffence equation. These are solved like constant coefficient DE's (where you try solutions e^{ax} and get a polynomial for a) by trying $V_i = G^i$. For the equation above we find $G = 1$ (repeated root) and so

$$V_i = A + Bi \qquad (6)$$

Note: form (6) can also be seen by rewriting (5) as $(V_{i+1} - V_i) - (V_i - V_{i-1}) = 0$.

So for $i < j$

$$V_i = B_1 i \qquad \text{(recall } V_0 = 0\text{)}.$$

and for $i > j$

$$V_i = B_2 (N - i) \qquad (V_N = 0)$$

For the two definitions to match up at V_j we need

$$B_1 j = B_2 (N - j)$$

and to satisfy (4) at $i = j$ we require

$$B_1 (j-1) - 2 B_1 j + B_2 (N - j - 1) = 1/N^2.$$

Solving the 2 linear equations above we get

$$B_1 = -(N-j)/N^3$$

$$B_2 = -j/N^3.$$

Thus
$$\mathbb{G}_{ij} = \begin{cases} -\dfrac{i(N-j)}{N^3} & i \le j \\ -\dfrac{j(N-i)}{N^3} & i > j. \end{cases}$$

Notes:

1. $\mathbb{G}$ has very similar structure to the exact Green's function. In fact, finding $\underline{U} = \mathbb{G}\underline{F}$ can be shown to be an integral approximation of the exact, continuous Green's function representation.
2. Even knowing the exact $\mathbb{G}$, it is faster to compute $\mathbb{A}\underline{U} = \underline{F}$ using sparse (tridiagonal) system solvers than using the form $\underline{U} = \mathbb{G}\underline{F}$.
3. Another way to proceed with this question is to see the form of $\mathbb{G}$ above from numerical computation with small N, then show $\mathbb{A}\mathbb{G} = \mathbb{I}$ for all N.

#4. $E[v] = \frac{k}{2}\left[\int_0^L \sqrt{1+v'(x)^2} - L\right]^2 + g\int_0^L \mu(x)\, v(x)\,dx.$ (7)

Forgot the $\frac{1}{2}$ in Q statement.

$v(0) = v(L) = 0$

Consider $v(x) = v_0(x) + \varepsilon \eta(x)$ $\eta(0) = \eta(L) = 0$

minimizer

Putting this form in (7) gives.

$$E = \frac{k}{2}\left[\int_0^L \sqrt{1+[v_0'+\varepsilon\eta']^2}\,dx - L\right]^2 + g\int_0^L \mu(v_0+\varepsilon\eta)\,dx.$$

$$\frac{dE}{d\varepsilon} = k\left[\int_0^L \sqrt{1+[v_0'+\varepsilon\eta']^2}\,dx - L\right]\int_0^L \frac{[v_0'+\varepsilon\eta']\eta'}{\sqrt{1+[v_0'+\varepsilon\eta']^2}}\,dx + g\int_0^L \mu\,\eta\,dx.$$

$$\left.\frac{dE}{d\varepsilon}\right|_{\varepsilon=0} = T\int_0^L \frac{v_0'\eta'}{\sqrt{1+[v_0']^2}}\,dx + g\int_0^L \mu\eta\,dx \qquad (8)$$

where $T = k\left[\int_0^L \sqrt{1+[v_0']^2}\,dx - L\right]$ is the tension in the wire.

After integration by parts, (8) becomes

$$\left.\frac{dE}{d\varepsilon}\right|_{\varepsilon=0} = \int_0^L \left\{ -T\left(\frac{V_0'}{\sqrt{1+[V_0']^2}}\right)' + g\mu \right\} \eta \, dx.$$

if $\left.\frac{dE}{d\varepsilon}\right|_{\varepsilon=0} = 0$ for every η, then

$$-T\left(\frac{V_0'}{\sqrt{1+[V_0']^2}}\right)' + g\mu = 0 \qquad V_0(0) = V_0(L) = 1.$$

where $T = k\left[\int_0^L \sqrt{1+[V_0']^2}\, dx - L\right]$ is a nonlocal term.

Note: As noted on the web site, the starting energy (7) is not physically correct. The difficulty is that the potential energy in (7) assumes that the wire is uniformly stretched, consistent with the constant tension in the equation derived from it. The correct energy would take into account variable amounts of stretching along the length of the wire, and the resulting equation would have variable tension. On Assignment II there is an optional question where you can help me figure out the right equation.

<u>Note</u>: Constant tension is consistent with the linearized equation ($|V'| << 1$) which is why this incorrect energy lead to the correct linearized equation in the class notes.

#5 I set up a simple MATLAB program to do this, posted to my web site. The issue here is that we want to see how accurate the perturbation approximation is

$$u_\varepsilon(x) \approx u_0(x) + \varepsilon u_1(x).$$

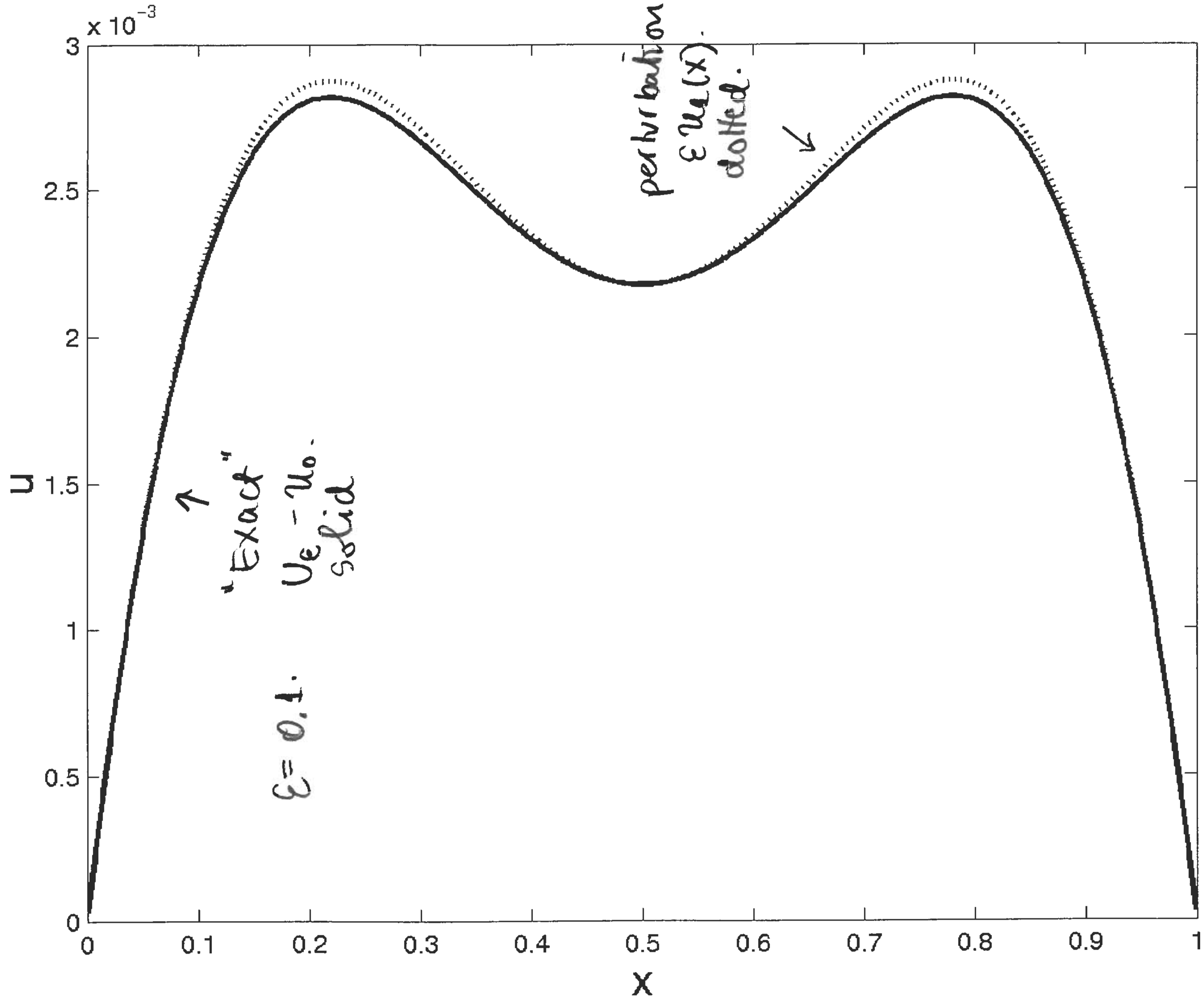
We don't know $u_\varepsilon(x)$ but we can compute the approximation $U_{\varepsilon,N}$ and standard convergence theory shows that

$$\lim_{N\to\infty} U_{\varepsilon,N} = u_\varepsilon(x).$$

On page 8, I plot

$$U_{\varepsilon,N} - u_0(x) \quad \text{against} \quad \varepsilon u_1(x). \qquad (9)$$

for $\varepsilon = 0.1$ I take N large enough so that the error $U_{\varepsilon,N}$ to $u_\varepsilon(x)$ is negligible on this plot ($N = 1{,}000$). Plots of (9) for $\varepsilon = 0.01$ show no visible difference between the curves.


x 10^-3
u
x
"Exact"
$U_\varepsilon - u_0$.
solid
$\varepsilon = 0.1$.
perturbation
$\varepsilon u_1(x)$.
dotted.

Note: As mentioned in the notes, it is very difficult to determine the accuracy of a truncated perturbation expansion. Numerical tests on simple but representative problems can give you some insight.

Optional problem:

Suppose for a given $A(x)$ the maximum occurs at $x = x_0$, so $u'(x_0) = 0$.

$$(Au')' = -1 \quad \text{integrate using} \uparrow$$

$$Au' = -x + x_0.$$

So, $u' > 0$ for $x < x_0$ and $u' < 0$ for $x > x_0$, that is $u(x)$ can have only one critical point x_0.

$$u' = \frac{x_0 - x}{A(x)}.$$

$$u(x_0) = \int_0^{x_0} \frac{x_0 - s}{A(s)}\, ds \underset{\substack{\uparrow \\ \text{also}}}{=} -\int_{x_0}^{1} \frac{x_0 - s}{A(s)}\, ds.$$

so u is minimized by minimizing the following functional over all $A(\cdot)$ and x_0:

$$F(A, x_0) = \frac{1}{2}\left\{ \int_0^{x_0} \frac{x_0 - s}{A(s)}\, ds + \int_{x_0}^{1} \frac{s - x_0}{A(s)}\, ds \right\} \qquad (10)$$

Subject to the constraints:

$$\int_0^{x_0} \frac{x_0 - s}{A(s)}\, ds = \int_{x_0}^{1} \frac{s - x_0}{A(s)}\, ds \qquad (11)$$

and

$$\int_0^1 A(s)\, ds = 1. \qquad (12).$$

For each fixed x_0, the minimum, $f(x_0)$ can be found using variational methods. Then, the x_0 which minimizes f can be found, solving the problem.

Note: Intuitively, we expect $x_0 = \frac{1}{2}$ and the resulting $A(\cdot)$ to be symmetric about $x = \frac{1}{2}$. In this case (11) is satisfied, the two terms in (10) are identical, and the problem reduces to the minimization of

$$\int_0^{1/2} \frac{\frac{1}{2} - s}{A(s)}\, ds \qquad \text{with} \qquad \int_0^{1/2} A(s)\, ds = \frac{1}{2}.$$

This solution has minimum value $1/9$, the details will be left to a later assignment question.

Note: Without the theory, you can come pretty close by (intelligent) trial and error. Using the code from problem #4 and varying ε to find the smallest maximum temperature, you can get a maximum of ≈ 0.1140 at $\varepsilon \approx -0.659$

This is larger than the optimal $\frac{1}{9}$, but much better than the uniform rod $1/8$.

<u>Note</u>: The shape of the optimal A is much different than the A_ε for x near $\frac{1}{2}$.

Math 401, Spring 2007
Assignment II
due Monday, February 5

#1. Consider the following problem for $u(x)$, $x \in [0, 2\pi]$:

$$-u'' + u = f(x)$$

$$\left.\begin{aligned} u(0) &= u(2\pi) \\ u'(0) &= u'(2\pi) \end{aligned}\right\} \text{ i.e. } u \text{ is } 2\pi\text{-periodic}$$

a) Show that this problem is self-adjoint
Note: The BC's are not in the form given in class.

b) Show the problem has ! solutions, i.e. that if $f \equiv 0$, $u \equiv 0$.

c) Find the normalized eigenfunctions of this problem.

d) Find the Green's function for this problem.

#2. Consider the following problem for $u(x)$, $x \in [0,1]$.

$$-u'' = f, \quad u(0) = 0, \quad u'(0) + u(1) = 0.$$

a) This problem is not self adjoint (the differential part is, but not the BC'S). Find the BC's of the adjoint problem

b) Show that the problem has ! solutions.

c) Find the Green's function for this problem.

#3 Consider the following problem for $u(x)$, $x \in (-\infty, \infty)$.

$$-u'' + u = f \qquad \lim_{|x| \to \infty} u(x) = 0.$$

Write u & f in terms of their Fourier transforms and plug into the DE above. Use the result to give an alternate derivation of the Green's function for this problem shown in class.

#4 Consider the following problem:

$$-u'' = f(x) \qquad u'(0) = 0, \; u'(1) = 0.$$

a) Show that this problem only has solutions when $f_{ave} := \int_0^1 f(x)\,dx = 0$, and then $u(x)$ is only determined up to a constant.

b) Find the Green's function $G(s,x)$ for this problem

$$u(x) = \underbrace{\int_0^1 G(s,x)\, f(s)\,ds}_{u_0(x)} + C \quad \leftarrow \text{the arbitrary constant}$$

where $f_{ave} = 0$ and $u_{0,ave} = 0$.

#5 If $f(x)$ is differentiable for all x with $f'(x) < K$ for all x, show that

$$f_\varepsilon = \Psi_\varepsilon * f$$

Ψ_ε: mollifiers in §2.1 notes; $*$: convolution

is C^∞ and that $f_\varepsilon'(x) < K$ for all $\varepsilon > 0$ and x.

Do either #6 or #7.

#6 (Modelling) Help me out, and properly derive the nonlinear equation describing the elastic wire with attached mass.

#7 (analytic) Consider the self-adjoint problem for $u(x)$, $x \in [0,1]$:

$$(a(x)u')' + c(x)u = f(x)$$

$$u(0) = 0, \quad u(1) = 0$$

with $a(x) > 0$, $c(x) < 0$ for all x.

a) Show this problem has ! solutions.

b) Write down the conditions that the Green's function for this problem must satisfy.

c) Show that the Green's fn with the properties in (b) exists.

Math 401, Spring 2007
Assignment II solutions

#1 a) Proceeding as in the class notes,

$$\int_0^{2\pi} (-u''+u)\, v\, dx = \int_0^{2\pi} u(-v''+v)\,dx + \left(-u'v + uv'\right)\Big|_0^{2\pi} \qquad (1)$$

The boundary terms are

$$+u'(0)v(0) - u'(2\pi)v(2\pi) - u(0)v'(0) + u(2\pi)v'(2\pi) \qquad (2)$$

Using $u'(2\pi) = u'(0)$, $u(2\pi) = u(0)$ this becomes:

$$u'(0)\left(v(0) - v(2\pi)\right) - u(0)\left(v'(0) - v'(2\pi)\right).$$

For this to be zero for all possible values of $u'(0)$, $u(0)$, we require

$$v(0) = v(2\pi), \quad v'(0) = v'(2\pi).$$

This, and the form (1) show that the problem is self-adjoint.

b). Consider $-u'' + u = 0$.

$$u = Ae^{x} + Be^{-x}.$$

To satisfy the BC's, $u(0) = u(2\pi)$, $u'(0) = u'(2\pi)$

$$(1-e^{2\pi})A + (1-e^{-2\pi})B = 0.$$

$$(1-e^{2\pi})A + (1+e^{-2\pi})B = 0.$$

Note that this system is ! soluable with $A=B=0$.
Therefore, solutions are unique.

c) This is a standard excercise, see the course references or your old copy of Boyce & DiPrima. Outline of argument:

Consider $-u'' + u = \lambda u$,

$$-u'' = (\lambda - 1)u \quad , \quad u \text{ periodic.}$$

$\lambda - 1 < 0$ leads to no eigenfunctions, like the argument in b) above.

$\lambda - 1 > 0$, $\lambda - 1 = \mu^2$ has eigenfunctions when $\mu = n\pi$.

$\lambda_0 = 1$, $\quad u_0 = \frac{1}{\sqrt{2\pi}}$ $\quad$ normalizing factors to make $\int_0^{2\pi} (u_n)^2 dx = 1$.

$\lambda_n = (n\pi)^2 + 1$, $n > 0$ $\quad u_n = \frac{1}{\sqrt{\pi}} \cos n\pi$

$\quad v_n = \frac{1}{\sqrt{\pi}} \sin n\pi$ $\quad \Big\}$ both eigenfn's with λ_n.

<u>Note</u>: These are the Fourier series basis functions.

d) $G(s,x)$ would solve

$$-G_{ss} + G = \delta(s-x) \quad , \quad G \ 2\pi\text{-periodic in } s.$$

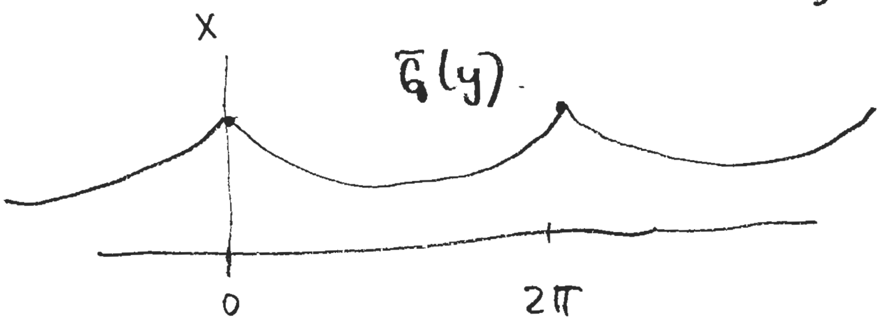
It is possible to make the algebra much easier with the observation that

$$G(s,x) = \hat{G}(s-x) \quad , \quad \hat{G}(y).$$

where $-\hat{G}'' + \hat{G} = \delta(y)$, $\hat{G}$ 2π-periodic.

$$\tilde{G}(y) = \frac{\cosh(y-\pi)}{2\sinh(\pi)} \quad \left.\right\} \quad \text{for } 0 \le y \le 2\pi$$

extended periodically for other y.



2.a) Proceed as in #1 a) to get the boundary terms (2):

$$u'(0)v(0) - u'(1)v(1) - u(0)v'(0) + u(1)v'(1)$$

Use $u(0)=0$, $u'(0) = -u(1)$:

$$u(1)\left(-v(0) + v'(1)\right) - u'(1)v(1).$$

It follows that

$$v'(1) - v(0) = 0, \quad v(1) = 0$$

are boundary conditions for the adjoint problem. Since these are different from the original BC's, we can say the problem is not self-adjoint.

b) $-u'' = 0 \quad \Rightarrow \quad u = a + bx.$

$u(0) = 0 \Rightarrow a = 0, \quad u = bx.$

$u'(0) + u(1) \Rightarrow 2b = 0 \Rightarrow b = 0.$

So $u \equiv 0$, solutions are unique.

c) $G(s,x)$ solves the adjoint problem

$$-G_{ss} = \delta(s-x), \quad G(1,x)=0, \quad G_s(1,x) - G(0,x)=0$$

i.e. $[G]=0$, $[G_s] = -1$ at $s=x$.

$$G = \begin{cases} A + Bs & s<x. \\ C(s-1) & s>x. \end{cases} \quad (\text{satisfies } G(1,x)=0)$$

$[G]=0 \Rightarrow A+Bx = C(x-1)$.

$[G_s]=-1 \Rightarrow B = C+1$.

$G_s(1,x) - G(0,x) = 0 \Rightarrow C - A = 0$.

Solution, $A = -\frac{x}{2}$, $C = -\frac{x}{2}$, $B = -\frac{x}{2}+1$

$$G(s,x) = \begin{cases} -\frac{x}{2} + \left(-\frac{x}{2}+1\right)s & s \le x. \\ -\frac{x}{2}(s-1) & s > x \end{cases}$$

<u>Note</u>: $G(s,x) \neq G(x,s)$ for all x,s since the problem is not self adjoint.

#3. $-u'' + u = f$.

$$(\alpha^2+1)\hat{u} = \hat{f}$$

$$\hat{u} = \frac{\hat{f}}{\alpha^2+1}.$$

Note that

$$\int_{-\infty}^{\infty} \frac{e^{-i\alpha x}}{\alpha^2+1}\, d\alpha = \pi e^{-|x|},$$

by Tables or by contour integration limits, taking into account the pole at $\pm i$. Taking into account the scaling noted in Ch 4, § 4.1, we see that u is obtained by convolving f with

$$\hat{G}(y) = \frac{1}{2} e^{-|y|}$$

as found in the Ch 2 notes.

#4. a) $-u'' = f(x) \qquad u'(0) = u'(1) = 0.$

Clearly u is only determined up to a constant. Integrating the equation using the BC's shows that

$$\int_0^1 f(x)\, dx = 0.$$

b). We want to have the following result:

$$u(x) - u_{ave} = \int_0^1 G(s,x)\, u''(s)\, ds \qquad (3).$$

for every function $u(x)$ with $u'(0) = u'(1) = 0$.

$$u(x) = \int_0^1 \left(-G(s,x)\, u''(s) + u(s)\right) ds.$$

Formally integrating by parts

$$= \int_0^1 \left(-G_{ss} + 1\right) u(s)\, ds.$$

Where $G_s(0,x) = G_s(1,x) = 0$ to eliminate the boundary terms. Thus, formally, we want

$$-G_{ss} + 1 = \delta(s,x)$$

i.e. $G_{ss} = +1, \ s \neq x \ ; \ [G] = 0, \ [G_s] = -1$ at $s = x$.

$$G = \begin{cases} \dfrac{s^2}{2} + A & s < x \quad (\text{satisfies } G_s(0,x) = 0). \\[2ex] \dfrac{(s-1)^2}{2} + B & s > x \quad (\text{satisfies } G_s(1,x) = 0). \end{cases}$$

$G_s(x-,x) = x, \quad G_s(x+,x) = x-1$, so jump condition satisfied.

Continuity $\quad \dfrac{x^2}{2} + A = \dfrac{(x-1)^2}{2} + B.$

$$A = B - x + \tfrac{1}{2} \qquad (4).$$

Note: Any A & B that satisfy (4) can be used to get a Green's function that has the desired properties. Different solutions just shift $G(\cdot, x)$ by a constant, and so the RHS of (3) is not changed since $\int_0^1 u''(s)\,ds = 0$.

Although not required by the question, the "right" $G(\cdot, x)$ to pick satisfies

$$\int_0^1 G(s,x)\,ds = 0 \qquad \text{for all } s \Rightarrow$$

$$\frac{x^3}{6} + Ax - \frac{(x-1)^3}{6} + B(1-x) = 0. \qquad (5).$$

(4) & (5) together determine A & B uniquely.
The resulting G has the property that

$$u(x) = \int_0^1 G(s,x)\, f(s)\, ds.$$

solves $\quad u'' = f - f_{ave} \qquad u'(0) = u'(1) = 0$

with $u_{ave} = 0$.

#5 $$f_\varepsilon(x) = \int_{-\infty}^{\infty} \psi_\varepsilon(x-s)\, f(s)\, ds \qquad (6)$$

also changing $y = x-s$, $s = x-y$, $dy = ds$.

$$= \int_{-\infty}^{\infty} \psi_\varepsilon(y)\, f(x-y)\, dy \qquad (7).$$

From form (6) it can be seen that f_ε is C^∞ and from form (7),

$$f_\varepsilon'(x) = \int_{-\infty}^{\infty} \psi_\varepsilon(y)\, f'(x-y)\, dy$$

Note: integral is actually over a finite interval since ψ_ε has compact support.

$$< K \int_{-\infty}^{\infty} \psi_\varepsilon(y)\, dy = K.$$

#6. Preliminaries: We need to make some assumptions about the material properties of the wire. A wire of unstretched length L_0 under uniform tension T is assumed to stretch an amount

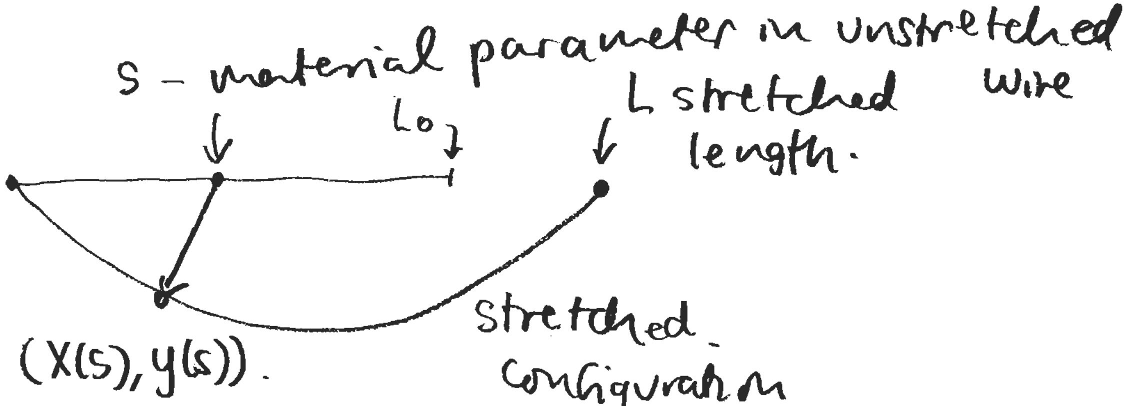
(8) $\Delta L = \tilde{k} T L_0$ Note that this $\tilde{k}$ is not the same k as before (it's the old $\frac{1}{k}$ per unit length). This is a reasonable assumption, but could be extended to a nonlinear relationship

$$\frac{\Delta L}{L_0} = f(T)$$

But we'll only consider (8) in what follows, and capture the nonlinear geometric effects. The energy required to stretch the wire the distance ΔL under the tension T is

$$E = \frac{1}{2} T \Delta L = \frac{1}{2} \tilde{k} T^2 L_0 = \frac{1}{2} (\Delta L)^2 / (\tilde{k} L_0). \quad (9).$$

<u>Approach 1</u> Energy minimization in Lagrangean coordinates. Consider s to be the coordinate $s \in [0, L_0]$ in the unstretched wire of length L_0. Mass density $\mu(s)$ is attached to the unstretched wire. In this model, the mass follows the material point s to its stretched configuration $x(s), y(s)$, shown in the figure below:



$$x(0) = 0, \quad y(0) = 0, \quad x(L_0) = L, \quad y(L_0) = 0 \qquad (10).$$

Consider a small segment of unstretch wire, $s \in [s_0, s_0 + \Delta s]$. It will have stretched length approximately

$$\sqrt{\dot{x}^2(s_0) + \dot{y}^2(s_0)}\,\Delta s \qquad \text{or}$$

$$\Delta L \approx \left(\sqrt{\dot{x}^2 + \dot{y}^2} - 1\right)\Delta s.$$

Using the last term in (9) with unstretched length $L = \Delta s$ we have the potential energy in this segment

$$E_k \approx \frac{1}{2\tilde{\varepsilon}}\left[\sqrt{\dot{x}^2 + \dot{y}^2} - 1\right]^2 \Delta s.$$

The gravitational potential energy is

$$E_g \approx \mu g y \,\Delta s.$$

Summing E_k & E_g, then summing over all segments, taking the limit as $\Delta s \to 0$ gives the following for the total energy.

$$E[x, y] = \int_0^1 \left\{\frac{1}{2\tilde{\varepsilon}}\left[\sqrt{\dot{x}^2 + \dot{y}^2} - 1\right]^2 + \mu g y\right\} ds.$$

Consider $x = x_0 + \varepsilon \eta(x)$ $\qquad \eta(0) = \eta(L_0) = 0.$ $\qquad$ (10.)

$y = y_0 + \delta \xi(x)$ $\qquad \xi(0) = \xi(L_0) = 0.$

↑

x_0, y_0 minimizers.

$$\left.\frac{dE}{d\varepsilon}\right|_{\varepsilon=0} = 0 \text{ for all } \eta \Rightarrow$$

$$\left(\frac{\left[\sqrt{\dot{x}^2+\dot{y}^2} - 1\right]}{\sqrt{\dot{x}^2+\dot{y}^2}} \dot{x} \right)^{\cdot} = 0. \qquad (11)$$

$$\left(\frac{\left[\sqrt{\dot{x}^2+\dot{y}^2} - 1\right]}{\sqrt{\dot{x}^2+\dot{y}^2}} \dot{y} \right)^{\cdot} = \hat{k}\mu g. \qquad (12)$$

Equations (11) & (12) with conditions (10) are the boundary value system that can be solved for $X(s)$, $y(s)$.

<u>Note</u>: The local tension in the wire can be shown to be

$$T(s) = \frac{1}{\hat{k}}\left(\sqrt{\dot{x}^2+\dot{y}^2} - 1\right). \qquad (13)$$

(11) & (12) can be derived as horizontal & vertical force balances.

<u>Additional check</u>: Consider the case where $|\dot{y}(s)|$ is very small. (11) now reads (approximately)

$$\left(\frac{|\dot{x}| - 1}{|\dot{x}|} \dot{x} \right)^{\bullet} \approx 0.$$

integrate, assume $\dot{x} > 0$ (reasonable),

$$\dot{x} \approx \text{Constant}$$

and $x(s) \approx \frac{L}{L_0} s$ to satisfy $x(0) = 0$, $x(L_0) = L$.

Thus, to highest order, the wire is just uniformly stretched. Consider now (12) in this case,

$$\left(\frac{1}{\hat{k}} \frac{\left(\frac{L}{L_0} - 1 \right)}{(L/L_0)} \dot{y} \right)^{\bullet} = \mu(s)\, g. \qquad (14)$$

We consider now the problem in the form $y(x)$. Here, $\mu(s) = \frac{\hat{\mu}(x) L_0}{L}$ where $\hat{\mu}$ is the mass per unit stretched length.

$$\frac{d}{ds} = \frac{L}{L_0} \frac{d}{dx},$$

so (14) becomes.

$$\frac{1}{\hat{k}} \frac{(L - L_0)}{L} \frac{d^2 y}{dx^2} \cdot \frac{L^2}{L_0^2} = \frac{\hat{\mu}(x) L_0}{L} g$$

or $$\underbrace{\frac{1}{\tilde{k}} \frac{L-L_0}{L_0}} \frac{d^2y}{dx^2} = \tilde{\mu}(x) g.$$

this is the uniform tension T in the linearized problem, so

$$\frac{d^2y}{dx^2} = \frac{g}{T} \hat{\mu}(x).$$

This is the linearized problem we had before (check ✓).

<u>Approach 2</u> Consider $y(x)$ and tension $T(x)$ in the wire. Consider a segment of length Δx, matching horizontal and vertical forces, then taking limits:

$$\left(\frac{T}{\sqrt{1+(v')^2}} \right)' = 0. \qquad (15).$$

$$\left(\frac{T v'}{\sqrt{1+(v')^2}} \right)' = \mu g \qquad (16).$$

<u>Note</u>: in this approach $\mu(x)$ is given. It's a model of a different situation, where the mass attached to the wire is forced to move only vertically.

(16) has boundary conditions $V(0) = V(L) = 0$.
An additional condition has to be given for (15) to determine the constant C,

$$T = C\sqrt{1+(V')^2}. \qquad (17)$$

This constant is determined so that the unstretched length L_0 matches the stretched L and tension, as follows. Consider an interval of length Δx. The stretched length is approximately

$$\sqrt{1+(V')^2}\,\Delta x.$$

It is under tension T, and it's unstretched length is given by

$$\frac{\sqrt{1+(V')^2}}{1+\hat{k}T}\,\Delta x.$$

summing, taking $\Delta x \to 0$ we obtain

$$L_0 = \int_0^L \frac{\sqrt{1+(V')^2}}{1+\hat{k}T}\,dx$$

$$L_0 = \int_0^L \frac{\sqrt{1+(V')^2}}{1+\hat{k}C\sqrt{1+(V')^2}}\,dx. \qquad (18).$$

Combining (16) & (18) gives

$$(CV')' = \mu g \qquad (17).$$

where C must satisfy the nonlocal condition (18)

It can be shown that approaches 1 & 2 match up, when the difference in physical models noted at the bottom of p.12 is taken into account.

#7. $(a(x)\,u')' + c(x)\,u = f(x)$.

$u(0) = 0, \quad u(1) = 0.$

a) $(a(x)u')' + c(x)\,u = 0$. multiply by u and integrate

$$\int \{ (au')'\,u + c u^2 \}\,dx = 0. \quad \text{integrate first term by parts.}$$

$$\int \{ -a\,(u')^2 + c\,u^2 \}\,dx = 0.$$

Note that the integral will be negative unless $u \equiv 0$ ✓.

b). Want $u(x) = \int_0^1 G(s,x)\,\{ (au')' + cu \}\,ds \quad (18)$

in s variables

for every u with $u(0) = u(1) = 0$. Integrate (18) by parts, assuming that $G(\cdot, x)$ is piecewise differentiable (so continuous).

$$u(x) = \int_0^1 \left\{ -a G_s u' + c u G \right\} ds$$

If we take $G(0,x) = G(1,x) = 0$. (19).

We will allow $G(\cdot, x)$ to have a discontinuity in derivative at $s = x$.

$$u(x) = \int_0^x \left\{ -a G_s u' + c u G \right\} ds \; +$$

$$\int_x^1 \left\{ -a G_s u' + c u G \right\} ds.$$

integrate by parts, using $u(0) = u(1) = 0$,

$$u(x) = \int_0^x ((a G_s)_s + c G) u \, ds - a(x) G_s(x_-, x) u(x)$$

$$+ \int_x^1 ((a G_s)_s + c G) u \, ds + a(x) G_s(x_+, x) u(x).$$

This can be made true if for each x,

$$\left.\begin{aligned} & (a G_s)_s + c G = 0 \qquad s < x \ \& \ s > x. \\ & G(x_-, x) = G(x_+, x) \\ & G_s(x_+, x) - G_s(x_-, x) = \frac{1}{a(x)} \end{aligned}\right\} \quad (20)$$

Together with (19), this specifies $G(\cdot, x)$ for each x.

c) For given x, let $v(s) = G(s,x)$.

Let v_1 solve $\quad (a v_1')' + c v_1 = 0,$

$$v_1(0) = 0, \quad v_1'(0) = 1.$$

v_2 solve $\quad (a v_2')' + c v_2 = 0$

$$v_2(1) = 0, \quad v_2'(1) = 1.$$

Then
$$v(s) = \begin{cases} A v_1 & s < x \\ B v_2 & s > x. \end{cases}$$

where A & B must be chosen to satisfy the matching conditions of (20), i.e.

$$\left.\begin{aligned} A v_1(x) - B v_2(x) &= 0 \\ A v_1'(x) - B v_2'(x) &= -\frac{1}{a(x)} \end{aligned}\right\} \quad (21)$$

Consider the case when this cannot be uniquely solved (this leads to a contradiction), that is when the determinant

$$W(x) = -v_1 v_2' + v_1' v_2 = 0.$$

Note the following identity, however:

$$\begin{aligned} W' &= -\cancel{v_1' v_2'} - v_1 v_2'' + v_1'' v_2 + \cancel{v_1' v_2'} \\ &= -v_1 \left(\frac{1}{a}\right)(-a' v_2' - c v_2) + v_2 \left(\frac{1}{a}\right)(-a' v_1' - c v_1). \end{aligned}$$

since $aV_j'' + a'V_j' + cV_j = 0$ for $j=1,2$.

$$W' = \frac{a'}{a}(V_1V_2' - V_1'V_2) = -\frac{a'}{a}W.$$

This is a linear equation, solved as

$$W(x) = W(0)\, e^{-\int_0^x \frac{a'(s)}{a(s)}\,ds}.$$

So if $W(x)=0$, then $W(1)=0$ also $\Rightarrow$

$$-V_1(1)\underbrace{V_2'(1)}_{\uparrow\, 1} + V_1'(1)\underbrace{V_2(1)}_{\uparrow\, 0} = 0$$

by construction,

so $V_1(1)=0$. Now, however, V_1 satisfies

$$(aV_1')' + cV_1 = 0, \quad V_1(0) = V_1(1) = 0$$

but we showed in a) there were no nonzero solutions to this problem. Thus (21) can be solved for $A(x)$, $B(x)$ for every x.

Math 401, Spring 2007
Assignment #3, due February 26

#1. Consider the following problem for $u(x,t)$:

$$u(x,0) = u_0(x) \quad \text{given}$$
$$u_t(x,0) \equiv 0.$$

$$u_{tt} - u_{xx} = 0 \quad x > 0, \quad u_{tt} - 4u_{xx} = 0 \quad x < 0$$

with u and u_x continuous at $x=0$ for all t. Find an explicit form of the solution $u(x,t)$ in terms of u_0.

Note: this problem describes waves in two strings of different densities tied together (at $x=0$).

#2. Consider the form of the solution to the Dirichlet problem in the upper half plane derived in class:

$$u(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y}{(s-x)^2 + y^2}\, g(s)\,ds.$$

Show by explicit computation that

$$\Delta u = 0 \quad \text{for } y > 0$$

and $\lim_{y\to 0} u(x,y) = g(x)$. Be as rigorous as you are able.

#3. Find the Green's function representations of the solution to the semi-infinite rod heat conduction problem

$$u_t = u_{xx} + f(x,t) \quad t \geq 0, \quad x \geq 0.$$

$$u(x,0) = g(x) \quad x \geq 0$$

$$u(0,t) = 0. \quad \text{for all } t \geq 0.$$

#4. The 2D free space Poisson problem had solution

$$u(\underline{x}) = \frac{1}{2\pi} \iint_{\underline{S}} \log|\underline{s} - \underline{x}| \, f(\underline{s}) \, d\underline{s}.$$

Let x be the complex variable $x = x_1 + i x_2$, $s = s_1 + i s_2$. Then

$$u(\underline{x}) = \frac{1}{2\pi} \iint_{\underline{S}} \mathrm{Re}\left\{ \mathrm{Log}(s - x) \right\} f(\underline{s}) \, d\underline{s}.$$

is equivalent to the form above. Show how this form leads to a simple form for the multipole expansion for $u(\underline{x})$, $\underline{x}$ in the far field. Show that the first two terms match the ones in the notes. Find the next term in the expansion.

Do <u>two</u> of # 5, 6, 7, 8

#5. A wire 10 m long stretches 1 m when kept under uniform tension of 1 N. The wire has mass 0.1 kg. Calculate (analytically if you can, otherwise numerically) the shape it attains when tied between poles of equal height 10 m apart. See the solutions to assignment # 2 for the equations to solve.

#6. Find the solution to the problem

$$u_t = u_{xx} \qquad x \geq 0,\ t \geq 0. \qquad u \to 0 \text{ as } x \to \infty \text{ for each } t.$$

$$u(x,0) \equiv 0 \qquad x \geq 0$$

$$u(0,t) = 1 \qquad t \geq 0$$

<u>Note</u>: this problem is discontinuous at $x=0, t=0$. It is a model for what happens when initial & boundary values have a mismatch in parabolic equations.

#7. Consider the nonlinear wire of assignment #2. Derive a time dependent version of the model under the assumption that the unstretched wire has uniform density and no external mass is added. Show that under appropriate conditions and scaling, the wave equation is obtained for the vertical displacement.

#8 Consider $\Delta u = f(x,y)$, the 2D poisson problem, with $f \in C_0^\infty$. Show that this problem has unique solutions if the following additional conditions are added:

$$V = u - \frac{Q}{2\pi} \log |\underline{x}|, \quad Q = \iint_{\underline{s}} f(\underline{s})\, ds$$

satisfies $|\underline{x}||V|$, $|\underline{x}|^2 |\nabla V|$ bounded for $|\underline{x}| > 1$.

Note: Our Green's function representation satisfies these conditions, but to prove this is the only one that does may be difficult. I don't know how to show this (yet) - it is worth \$2 or a beer for the best proof.

Math 401, Spring 2007
Assignment #3 solutions

#1. $$u(x,t) = \begin{cases} f_1(x-2t) + g_1(x+2t) & x \le 0. \\ f_2(x-t) + g_2(x+t) & x \ge 0. \end{cases}$$

(D'Alembert's solution).
The standard argument shows that

$$f_1(\xi) = g_1(\xi) = \tfrac{1}{2} u_0(\xi) \qquad \xi < 0.$$

$$f_2(\xi) = g_2(\xi) = \tfrac{1}{2} u_0(\xi) \qquad \xi > 0.$$

$g_1(\xi)$, $\xi > 0$ and $f_2(\xi)$, $\xi < 0$ still need to be determined.

At $x=0$, continuity of u gives.

$$f_1(-2t) + g_1(2t) = f_2(-t) + g_2(t) \qquad (1).$$

Continuity of u_x gives.

$$f_1'(-2t) + g_1'(2t) = f_2'(-t) + g_2'(t)$$

integrating gives

$$-\tfrac{1}{2} f_1(-2t) + \tfrac{1}{2} g(2t) = -f_2(-t) + g_2(t) \qquad (2).$$

(integration constant taken to be zero ... we assume that $u_0(x)$ has compact support).

(1) & (2) are linear equations for unknown values $g_1(2t)$ & $f_2(-t)$ in terms of known values $f_1(-2t)$ & $g_2(t)$, $t \geq 0$.

The solutions are

$$g_1(2t) = -\frac{1}{3} f_1(-2t) + \frac{4}{3} g_2(t).$$

$$f_2(-t) = f_1(-2t) + \frac{1}{2} g_2(t).$$

or

$$g_1(\xi) = -\frac{1}{3} u_0(-\xi) + \frac{4}{3} g_2(\xi/2). \qquad \xi > 0.$$

$$f_2(\xi) = \frac{1}{2} u_0(-\xi) + u_0(2\xi) \qquad \xi < 0.$$

These have the interpretation of reflected plus transmitted waves.

#2 Take $g \in C_0^\infty$ to justify taking derivatives under the integral. Milder conditions on g would suffice. Consider

$$f(x,y) = \frac{y}{(s-x)^2 + y^2}, \text{ fixed } s. \qquad (3).$$

$$f_x = 2y\left[(s-x)^2 + y^2\right]^{-2}(s-x)$$

$$f_{xx} = -2y\left[(s-x)^2 + y^2\right]^{-2} + 8y\left[(s-x)^2 + y^2\right]^{-3}(s-x)^2$$

$$= \left[(s-x)^2 + y^2\right]^{-3}\left\{-2y\left[(s-x)^2 + y^2\right] + 8y(s-x)^2\right\}$$

$$f_y = \left[(s-x)^2 + y^2\right]^{-1} - 2y^2\left[(s-x)^2 + y^2\right]^{-2}$$

$$f_{yy} = -2y\left[(s-x)^2 + y^2\right]^{-2} - 4y\left[(s-x)^2 + y^2\right]^{-2}$$
$$+ 8y^3\left[(s-x)^2 + y^2\right]^{-3}.$$

$$= \left[(s-x)^2+y^2\right]^{-3}\left\{-6y\left[(s-x)^2+y^2\right]+8y^3\right\}.$$

Consider $\left[(s-x)^2+y^2\right]^3(f_{xx}+f_{yy})$ above

$$= 6y(s-x)^2-2y^3-6y(s-x)^2+2y^3=0$$

Thus, the $u(x,y)$ from the formula is harmonic.

Analytic fn.

<u>Note</u>: f in (3) is $-\operatorname{Im}\left\{\frac{1}{z-s}\right\}$ and so trivially is known to be harmonic.

$x+iy$

Consider $\frac{1}{\pi}\int_{-L}^{L}\frac{1}{s^2+1}\,ds = \frac{1}{\pi}\left\{\arctan L-\arctan(-L)\right\}$

$$= \frac{2}{\pi}\arctan L.$$

As $L\to\infty$, $\frac{2}{\pi}\arctan L \to 1.$ (4).

With $g\in C_0^\infty$, let M_0 and M_1 be bounds on $|g|$ and $|g'|$ respectively. Let $\varepsilon>0$. Considering (4), choose L large enough such that

$$\frac{4M_0}{\pi}\int_L^\infty \frac{1}{s^2+1}\,ds < \varepsilon/2 \qquad (5)$$

Consider now

$$u(x,y) - g(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y}{(s-x)^2+y^2}\, g(s)\,ds \quad - g(x)$$

change variables in the integral to

$$\xi = (s-x)/y \quad d\xi = ds/y.$$

$$s = x + y\xi.$$

$$u(x,y) - g(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1}{\xi^2+1}\{g(x+y\xi) - g(x)\}\,d\xi.$$

$$|u(x,y) - g(x)| \le \left|\frac{1}{\pi}\int_{-L}^{L} \frac{1}{\xi^2+1}\{g(x+y\xi) - g(x)\}\,d\xi\right| + \varepsilon/3$$

using (5). Now consider $y < \frac{M_1}{L}\cdot\frac{\varepsilon}{2}$ (6) so that

$|g(x+y\xi) - g(x)| \le \varepsilon/2$ in the integral above.

Now $|u(x,y) - g(x)| < \varepsilon$ for all y sufficiently small ((6) with L from (5)).

#3. Consider $u\{f,g\} = u\{f,0\} + u\{0,g\}$, as usual. The condition $u(0,t) = 0$ is satisfied if we extend f & g to be odd functions (in x) and solve the infinite rod problem (the method of images).

$u\{0, g\}$:

$$u(x,t) = \int_{-\infty}^{\infty} \frac{1}{2\sqrt{\pi t}} e^{-(s-x)^2/4t} \hat{g}(s)\,ds$$

where $\hat{g}(s) = \begin{cases} g(s) & s \geq 0 \\ -g(-s) & s < 0. \end{cases}$ ← image.

This can be rewritten as

$$u(x,t) = -\int_{-\infty}^{0} \frac{1}{2\sqrt{\pi t}} e^{-(s-x)^2/4t} g(-s)\,ds + \int_{0}^{\infty} \frac{1}{2\sqrt{\pi t}} e^{-(s-x)^2/4t} g(s)\,ds.$$

changing the parametrization of the first integral from $s \to -s$ allows the integrals to be combined,

$$u(x,t) = \int_0^{\infty} \underbrace{\frac{1}{2\sqrt{\pi t}} \left\{ e^{-(s-x)^2/4t} - e^{-(s+x)^2/4t} \right\}}_{\text{Green's fn for the initial value problem.}} g(s)\,ds$$

Either using the same technique or using Duhamel's principle on the result above, we obtain

$$u(x,t) = \int_0^t \int_0^{\infty} \underbrace{\frac{1}{2\sqrt{\pi(t-\tau)}} \left\{ e^{-(s-x)^2/4(t-\tau)} - e^{-(s+x)^2/4(t-\tau)} \right\}}_{\text{G's fn}} F(s,\tau)\,ds\,d\tau$$

for $u\{F, 0\}$.

#4 $\mathrm{Re}(\mathrm{Log}(s-x)) = \mathrm{Re}(\mathrm{Log}(x-s))$

easier to use, sorry.

$$\mathrm{Log}(x-s) = \mathrm{Log}\, x + \mathrm{Log}(1-s/x). \qquad (7)$$

x far field, $|s/x| = \varepsilon$ is small.

$$\mathrm{Log}(1-s/x) = -s/x + \frac{1}{2}\frac{s^2}{x^2} - \cdots \qquad (8)$$

$$\mathrm{Re}\left\{-s/x\right\} = \mathrm{Re}\left\{\frac{-s\bar{x}}{|x|^2}\right\} = \frac{-s_1x_1 - s_2x_2}{x_1^2+x_2^2}$$

$$\mathrm{Re}\left\{\frac{s^2}{x^2}\right\} = \mathrm{Re}\left\{\frac{s^2\bar{x}^2}{|x|^4}\right\} = \frac{(s_1^2-s_2^2)(x_1^2-x_2^2)+4x_1x_2s_1s_2}{(x_1^2+x_2^2)^2}$$

Using (7) & (8)

$$u(\underline{x}) \approx \frac{1}{2\pi} Q_0 \log|x| \qquad Q_0 = \iint_\Omega f(\underline{s})\, d\underline{s} \checkmark$$

$$-\frac{1}{2\pi}\underline{Q_1}\cdot\frac{\underline{x}}{|\underline{x}|^2} \qquad \underline{Q_1} = \iint \underline{s}\cdot f(\underline{s})\, d\underline{s} \checkmark$$

$$\left.\begin{aligned} &+\frac{(x_1^2-x_2^2)}{4\pi(x_1^2+x_2^2)^2}\iint_\Omega (s_1^2-s_2^2)\, f(\underline{s})\, d\underline{s}. \\ &+\frac{(x_1x_2)}{\pi(x_1^2+x_2^2)^2}\iint_\Omega s_1s_2\, f(\underline{s})\, d\underline{s} \end{aligned}\right\} \text{next order term.}$$

#5 The wire has mass $\mu = 0.01$ kg/m of unstretched length. The $\hat{k}$ parameter can be found by

$$\hat{k} = \frac{\Delta L}{T L_0} = \frac{1m}{1N \cdot 10m} = 0.1 \quad N^{-1}.$$

The equations to solve are

$$\left(\frac{\sqrt{\dot{x}^2+\dot{y}^2} - 1}{\sqrt{\dot{x}^2+\dot{y}^2}} \, \dot{x} \right)^{\cdot} = 0 \qquad (9)$$

$$\left(\frac{\sqrt{\dot{x}^2+\dot{y}^2} - 1}{\sqrt{\dot{x}^2+\dot{y}^2}} \, \dot{y} \right)^{\cdot} = \hat{k} \mu g. \qquad (10)$$

To discretize, consider $X_i \approx x(ih)$, $Y_i \approx y(ih)$, $i = 1, \dots N-1$ where $h = L_0 / N$ (L_0 unstretched length). Define forward & backward difference approximations to derivatives:

$$D_+ X_i := \frac{X_{i+1} - X_i}{h}, \quad D_- X_i := \frac{X_i - X_{i-1}}{h}.$$

Equation (9) is approximated as

$$D_- \left\{ \frac{\sqrt{(D_+ X_i)^2 + (D_+ Y_i)^2} - 1}{\sqrt{(D_+ X_i)^2 + (D_+ Y_i)^2}} \, D_+ X_i \right\} = 0 \qquad i = 1, \dots N-1$$

where $X_0 = 0$, $X_N = L$, the stretched length. Discrete approximation of (10) is done similarly. A Nonlinear system in $2(N-1)$ variables X_i, Y_i, $i = 1, \dots N-1$ results.

The system can be solved using Newton's method, with an explicit Jacobian.

When $L = L_0$, starting with $x = s$, $y = 0$ as an initial guess gives a singular Jacobian. For our problem with $L = L_0 = 10$, computing the problem for $L > 10.1$ is possible starting at $x = \frac{L}{L_0} s$, $y = 0$. I used this to get a starting guess for the $L = L_0 = 10$ problem. The solution is shown on the next page. The MATLAB programme "nlwire.m" is posted on the web site.

#6 It is possible to use the result of problem #3 to write a form of the solution $u - 1$. It gives more insight, however, to solve the problem using the Laplace Transform in time.

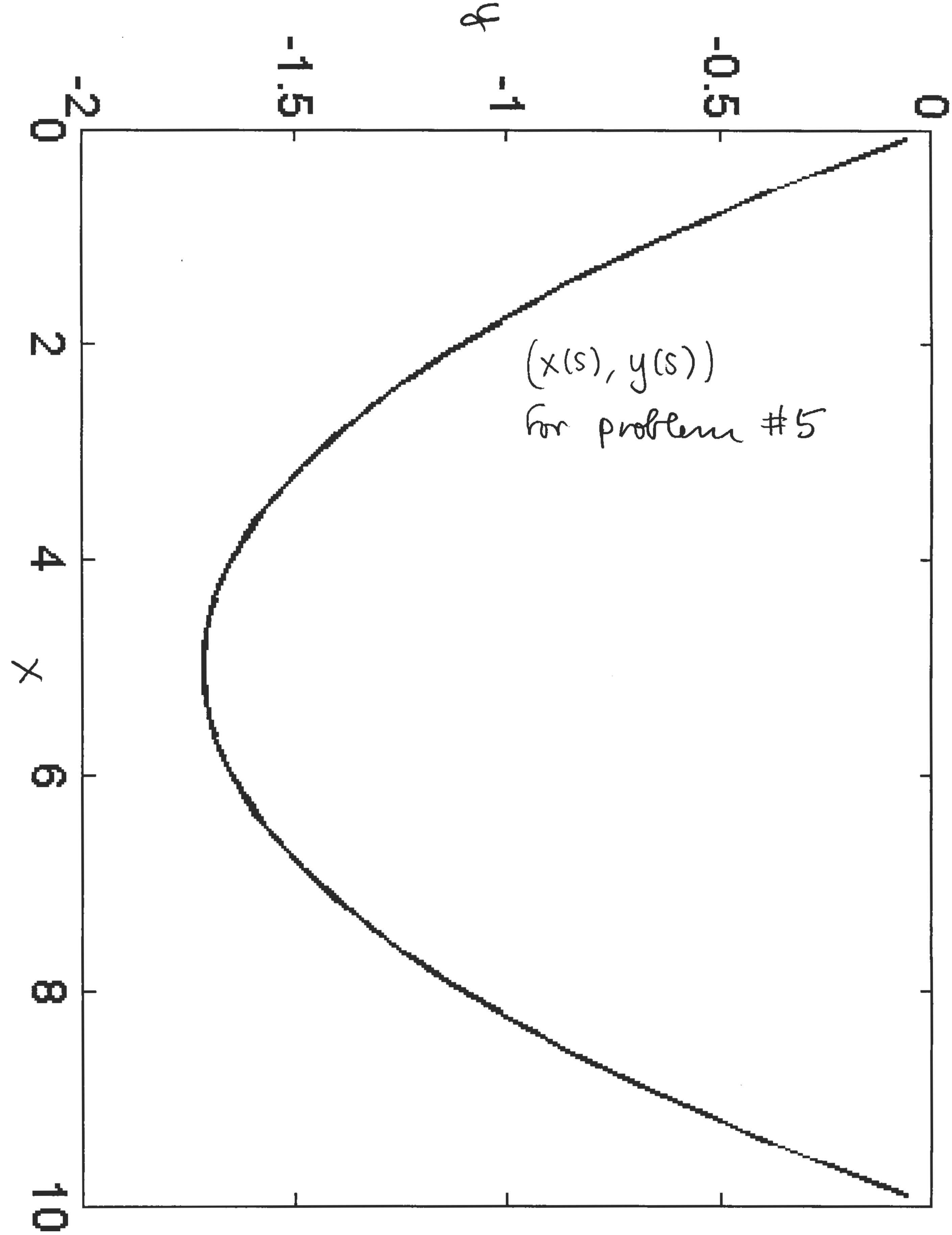
$$s\hat{u} = \hat{u}_{xx} \qquad \hat{u}(0,s) = \mathcal{L}\{1\} = \frac{1}{s}. \quad (11).$$

Consider s with $\mathrm{Re}(s) > 0$, take $\sqrt{s}$ to have $|\mathrm{Arg}(\sqrt{s})| < \pi/4$. (11) has solution

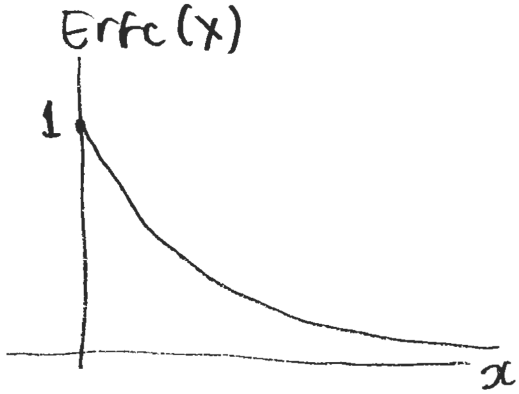
$$\hat{u} = \frac{1}{s} e^{-\sqrt{s}\,x}$$

Tables give the corresponding

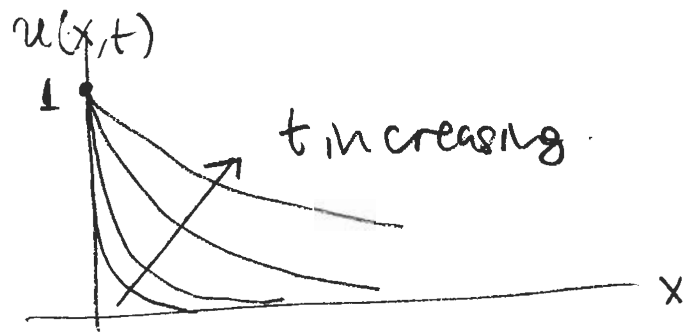
$$u(x,t) = \mathrm{erfc}\left\{\frac{x}{2\sqrt{t}}\right\}. \qquad (12)$$


(x(s), y(s))
for problem #5
y
x
0
-0.5
-1
-1.5
-2
0
2
4
6
8
10

Since



We see that (12) describes how the initial discontinuity at $x=0$ is smoothed out.



#7. As noted in Assignment #2 solutions, (9) & (10) from #7 describe horizontal & vertical force balances. Allowing a force imbalance to give a mass an acceleration leads to

$$\left(\frac{\sqrt{(x_s)^2+(y_s)^2} - 1}{\sqrt{(x_s)^2+(y_s)^2}} \, x_s \right)_s = \hat{k}\mu x_{tt} \tag{13}$$

$$\left(\frac{\sqrt{(x_s)^2+(y_s)^2} - 1}{\sqrt{(x_s)^2+(y_s)^2}} \, y_s \right)_s - \hat{k}\mu g = \hat{k}\mu y_{tt}. \tag{14}$$

So far, this is a coupled nonlinear hyperbolic system for $x(s,t)$, $y(s,t)$.

Scale x, y & s by the unstretched length L_o. Introduce the time scale $T = t/\sqrt{L_o^2 \hat{k} \mu}$ to obtain

$$\left(\frac{\sqrt{(X_s)^2 + (Y_s)^2} - 1}{\sqrt{(X_s)^2 + (Y_s)^2}} X_s \right)_s = X_{TT} \qquad (15)$$

$$\left(\frac{\sqrt{(X_s)^2 + (Y_s)^2} - 1}{\sqrt{(X_s)^2 + (Y_s)^2}} Y_s \right)_s - \hat{k} \mu g = Y_{TT} \qquad (16).$$

Consider now the situation where

$$X = \frac{L}{L_o} s + \varepsilon X_1(s,T) + O(\varepsilon^2).$$

$$y = \varepsilon y_1(s,T) + O(\varepsilon^2)$$

"small" displacements from uniformly stretched wire. To highest order, (16) reads

$$\left(1 - \frac{L_o}{L}\right) y_{ss} = y_{TT}$$

where it has also been assumed that $\hat{k} \mu g = O(\varepsilon^2)$ and can be neglected ($\hat{k}$ small, "stiff" spring). To highest order, $s = \frac{L_o}{L} x$, so we can make the final expression

$$\left(1 - \frac{L_o}{L}\right) \frac{L^2}{L_o^2} y_{xx} = y_{TT}$$

or $$y_{xx} = \frac{L_o^2}{L^2(1 - L_o/L)} y_{TT}$$

A further change of time scaling to

$$\mu = \tau \, L\sqrt{1 - L_0/L}\,/L_0.$$

leads to

$$y_{xx} = y_{\mu\mu}$$

#8 Consider the possibility of two such solutions leading to different V_1, V_2. Consider $W = V_1 - V_2$, satisfying $\Delta W = 0$

$|\underline{x}|\,|W|$ bounded.

The argument to show $W \equiv 0$ (unique solutions of the given form) is straightforward.

on S_R, $|W| < C/R$.

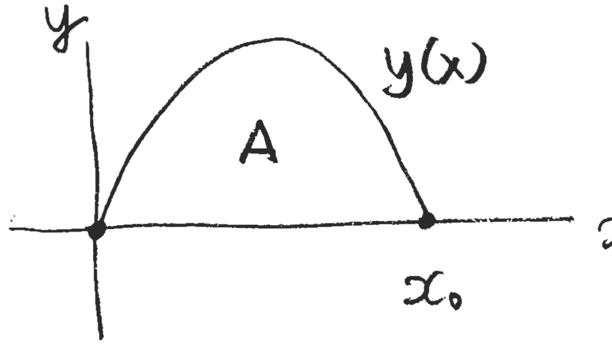
Using the maximum principle for harmonic functions, $|W| < C/R$ for all $\underline{x}$ in B_R. Taking $R \to \infty$ gives $W \equiv 0$.

Math 401, Assignment #4

Due March 12

For questions #1&2, 7/10 marks will be given for finding the equation for $y(x)$.

#1. Find the curve $y(x)$ with given length L containing the greatest area between it and the x-axis:



ie. Maximize $\int_0^{x_0} y(x)\,dx$ subject to the constraint $L = \int_0^{x_0} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$ with

$y(0)=0$, $y(x_0)=0$.

Note: x_0 must be determined as part of the problem.

#2 Find the shape $y(x)$ of the hanging chain of length L between two poles distance D apart, ie. minimize

$$\int_0^D y\sqrt{1+(y')^2}\,dx$$

subject to the constraint $\int_0^D \sqrt{1+(y')^2}\,dx = L$

#3. The largest (least negative) eigenvalue λ_1 for $\Delta u = \lambda u$ in a domain Ω in 2D with $u|_{\partial\Omega} = 0$ is known.

Let $\tilde{\lambda}_1$ be the largest eigenvalue of $\Delta u = \lambda u$ in $\tilde{\Omega}$ with $u|_{\partial\tilde{\Omega}} = 0$ and $\tilde{\Omega} \subset \Omega$, and $\hat{\lambda}_1$ be the largest eigenvalue of $\Delta u = \hat{\lambda} u$ in $\hat{\Omega}$ with $u|_{\partial\hat{\Omega}} = 0$ and $\Omega \subset \hat{\Omega}$.

Show that $\tilde{\lambda}_1 \leq \lambda_1 \leq \hat{\lambda}_1$

Do <u>one</u> of #4, 5, 6.

#4. Give a more rigorous proof of the equations with constraint in §5.2.
<u>Note</u>: This is a "\$2" question.

#5. Look in the references and describe the analytic solution of the Brachristochrone problem.

#6. Consider the constrained Brachristochrone problem of §5.5 with $H=1$, $L=1$, and

$$\phi(x) = \frac{1}{2} - 8\left(x - \frac{1}{2}\right)^2$$

Compute approximate solutions to this problem using one of the ideas discussed in class.

Math 401, Spring 2007
Assignment #4 solutions.

#1. $F[u] = \int_0^{x_0} u(x)\,dx$, $u(0)=0$, $u(x_0)=0$,

constraint $\int_0^{x_0} \sqrt{1+\left(\frac{du}{dx}\right)^2}\,dx = L$, given.

There are a couple of approaches:

A). Consider $y(x)$ as the equations are presented. The E-L equations are

$$1 + \lambda\left(\frac{u'}{(1+(u')^2)^{1/2}}\right)' = 0. \qquad u(0) = u(x_0) = 0.$$

I can follow from here to show that u must be an arc of a circle. It seems difficult to show that u describes the half circle that is the optimal solution.

B) Consider instead the curve parametrized by arc length $(x(s), y(s))$ with $s \in [0, L]$ and

$$(\dot{x})^2 + (\dot{y})^2 = 1. \qquad y(0) = 0, \; y(L) = 0.$$

where we denote $\frac{d}{ds}$ by a $\dot{}$.

$$\dot{x} = \sqrt{1-(\dot{y})^2}$$

and if $F[y] = \int_0^{x(L)} y\,dx$, we can change the variable of integration to s and obtain

$$F[y] = \int_0^L y\sqrt{1-(\dot{y})^2}\, ds.$$

Find the E-L equations here (no constraint). I'll use the Beltrami identity in the notes,

$$y\sqrt{1-(\dot{y})^2} + \frac{(\dot{y})^2 y}{\sqrt{1-(\dot{y})^2}} = C \text{ , constant.}$$

which simplifies to

$$\frac{y}{\sqrt{1-(\dot{y})^2}} = C. \qquad y(0)=0, \; y(L)=0.$$

It can be explicitly shown that this problem has solutions

$$y(s) = C \sin(s/c + \delta).$$

Taking $\delta = 0$ satisfies $y(0)=0$,

$C = L/\pi$ " $y(L)=0$.

Thus $y(s) = \frac{L}{\pi}\sin(\pi s/L)$.

$X(s)$ can be recovered from

$$X(s) = \int_0^s \sqrt{1-(\dot{y})^2}\, d\tau = \frac{L}{\pi}\left(1-\cos(\pi s/L)\right)$$

Thus the optimal solution is a half-circle, as expected.

C) Historically, various geometrical arguments were used to solve this problem.

#2 $$F[u] = \int_0^D u\sqrt{1+(u')^2}\,dx \qquad u(0)=0, \quad u(D)=0.$$

constraint $$\int_0^D \sqrt{1+(u')^2}\,dx = L.$$

E-L equations with constraints

$$\sqrt{1+(u')^2} - \left(\frac{uu'}{\sqrt{1+(u')^2}}\right)' - \lambda\left(\frac{u'}{\sqrt{1+(u')^2}}\right)' = 0.$$

$$\sqrt{1+(u')^2} = \left(\frac{(u+\lambda)\,u'}{\sqrt{1+(u')^2}}\right)' \qquad u(0) = u(D) = 0.$$

Note: You can derive a Beltrami identity for functional minimization with constraints that gives a simpler form,

$$\frac{u+\lambda}{\sqrt{1+(u')^2}} = c, \quad \text{constant.}$$

Both forms have solutions

$$u = C\left[\cosh\left(\frac{x-D/2}{C}\right) - \cosh\left(\frac{D}{2C}\right)\right].$$

It's a bit unsatisfying, but I couldn't derive these solutions, just check them by direct substitution, using the identities

$$\frac{d}{dx}\cosh x = \sinh x, \quad \frac{d}{dx}\sinh x = \cosh x.$$

$$\sqrt{1+\sinh^2 x} = \cosh x.$$

The value of λ is $C\cosh(D/2C)$ in this parametric form. C is determined by matching the constraint

$$\int_0^D \sqrt{1+(u')^2}\,dx = \int_0^D \cosh\left(\frac{x-D/2}{C}\right)dx$$

$$= 2C\sinh\left(\frac{D}{2C}\right) = L.$$

This is an implicit equation for $C(L)$.

#3. Analagously to the 1D case, it can be shown that

$$-\lambda_1 = \min_{u\in H_0^1(\Omega)} \int_\Omega |\nabla u|^2 dx, \text{ subject to } \int_\Omega |u|^2 d\underline{x} = 1. \quad (1)$$

Similarly,

$$-\hat{\lambda}_1 = \min_{u\in H_0^1(\hat{\Omega})} \int_{\hat{\Omega}} |\nabla u|^2 d\underline{x}, \text{ subject to } \int_{\hat{\Omega}} u^2 d\underline{x} = 1 \quad (2).$$

Let u be the minimizer of (1) [normalized λ_1 eigenvector]. Extend u by zero to $\hat{u}$ on the larger domain $\hat{\Omega}$. $\hat{u}$ is an $H_0^1(\hat{\Omega})$ function [some technical work is needed to really show this].

$$\int_{\hat{\Omega}} \hat{u}^2 dx = \int_{\Omega} u^2 dx = 1$$

So $\hat{u}$ is a candidate in (2) so

$$-\hat{\lambda}_1 \le \int_{\hat{\Omega}} |\nabla \hat{u}|^2 dx = \int_{\Omega} |\nabla u|^2 dx = -\lambda_1 .$$

Thus $-\hat{\lambda}_1 \le -\lambda_1$. Similarly $-\lambda_1 \le -\hat{\lambda}_1$ and the result

$$\hat{\lambda}_1 \le \lambda_1 \le \hat{\lambda}_1$$

is shown.

#4. There were a couple of good attempts at this by students, but I haven't seen a completely convincing argument yet.

#5 The conventional solution has y moving downwards from a rest height of y.

$$F[y] = \int_0^L \sqrt{\frac{1+(y')^2}{y}}\, dx .$$

Use the Beltrami identity,

$$\sqrt{\frac{1+(y')^2}{y}} - y' \frac{y'}{\sqrt{y(1+(y')^2)}} = C, \text{ constant.}$$

Simplifying $\quad \dfrac{1}{\sqrt{y(1+(y')^2)}} = C$

$$y\,(1+(y')^2) = C\,,$$ different constant (positive).

parametric solutions of this equation with $y(0)=0$ are

$$x = \frac{C}{2}(\theta - \sin\theta)$$

$$y = \frac{C}{2}(1-\cos\theta).$$

to check: $\frac{dy}{d\theta} = \frac{C}{2}\sin\theta\,, \quad \frac{dx}{d\theta} = \frac{C}{2}(1-\cos\theta).$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\sin\theta}{(1-\cos\theta)}. \qquad (3)$$

$$\left(1+\left(\frac{dy}{dx}\right)^2\right) = \frac{(1-\cos\theta)^2 + \sin^2\theta}{(1-\cos\theta)^2}$$

$$= \frac{1 - 2\cos\theta + \overbrace{\cos^2\theta + \sin^2\theta}^{1}}{(1-\cos\theta)^2} = \frac{2}{(1-\cos\theta)}.$$

Then $y\,(1+(y')^2) = C$, as required.

#6. I'll follow the notation from #5 above and take y positive downwards from the release height (although the plot will show the original coordinates).

I'll use the "Andrew conjecture": A brachristochrone solution applies from $[0, x_1]$ and from $[x_2, 1]$. The solution lies along the constraint in $[x_1, x_2]$ as shown on p. 8. The points x_1 & x_2 are determined such that the Brachristochrone solutions meet the constraint tangentially.

<u>Determining x_1</u>: Use the parametrized curves from the last question. Consider θ_1 to be the parameter that gives the appropriate point.

$$\frac{dy}{dx} = \frac{\sin\theta_1}{(1-\cos\theta_1)} = -\phi'(x_1).$$

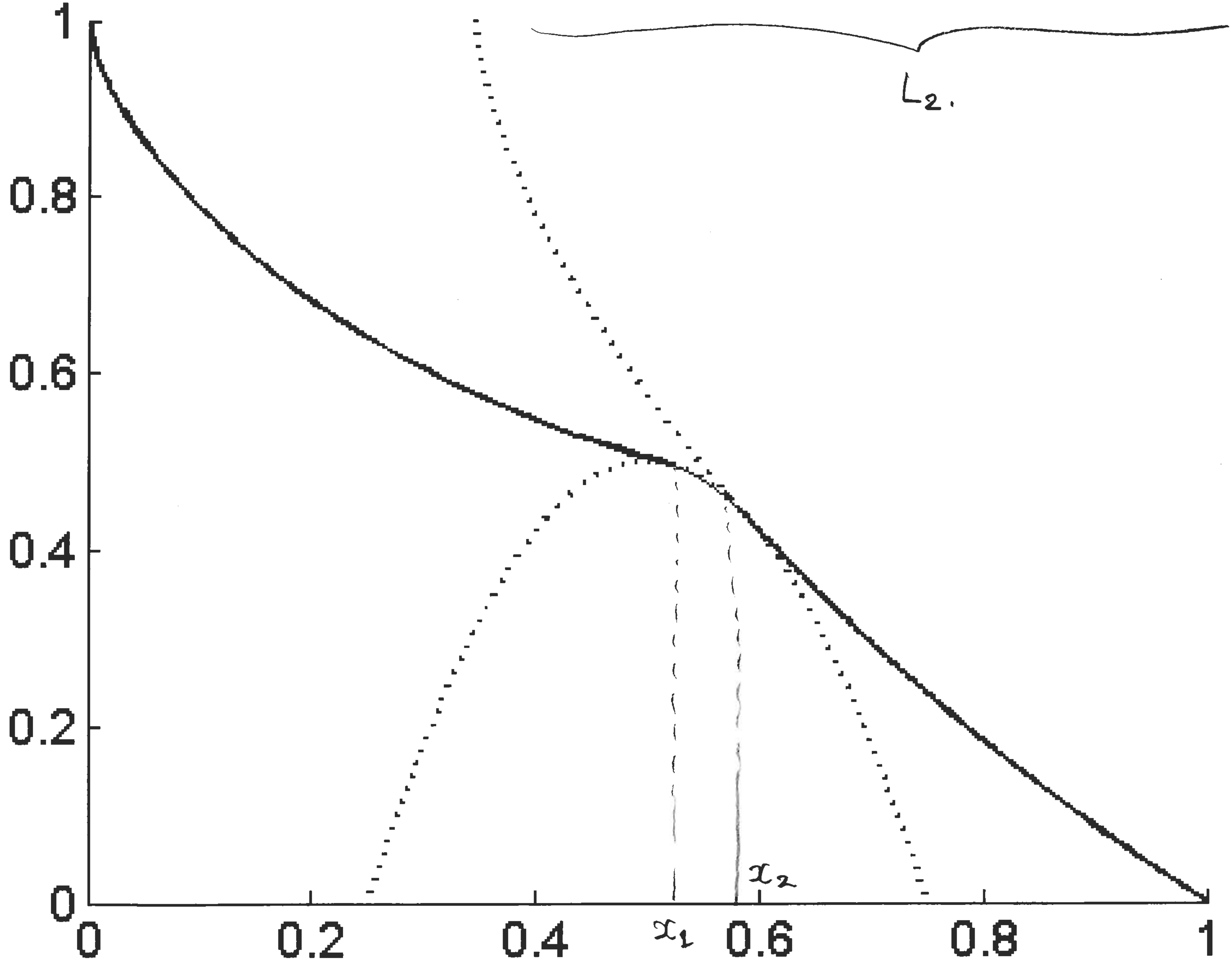
(3) p. 6. we've changed $y \to H-y$.

This determines x_1 from θ_1. Now with x_1 and θ_1 given, we can solve for C_1 in the parametrization

$$C_1 = \frac{2x_1}{\theta_1 - \sin\theta_1}.$$

Now we can determine $y_1 = \frac{C_1}{2}(1-\cos\theta_1)$. The correct θ_1 gives $y_1 = H-\phi(x_1)$ in the calculation above. This becomes a scalar root find for θ_1, whose derivative can be found analytically. I solved using Newton's method, giving $x_1 \approx 0.5218$.

8/



Determining X_2: The second curve could be offset as shown in the figure on p.8.

$$X = \frac{C_2}{2}(\theta - \sin\theta) + (L - L_2)$$

$$y = \frac{C_2}{2}(1 - \cos\theta)$$

Iterate on θ_2, solving for x_2:

$$\frac{\sin\theta_2}{(1-\cos\theta_2)} = -\phi'(X_2).$$

then $y_2 = H - \phi(X_2)$, then C_2:

$$C_2 = 2y/(1-\cos\theta_2).$$

Then L_2: $L_2 = L - X_2 + \frac{C_2}{2}(\theta_2 - \sin\theta_2)$.

There is then a nested iteration to determine the θ_3 such that $X = L$:

$$\frac{C_2}{2}(\theta_3 - \sin\theta_3) = L_2.$$

Then the residual for θ_2 is

$$\frac{C_2}{2}(1-\cos\theta_3) - \underset{H}{\underset{\uparrow}{1}} = 0.$$

This iteration (and the nested one for θ_3) is done with Newton's iterations, leading to

$x_2 \approx 0.5882$

The final soln is shown on p.8. MATLAB code is posted.

Math 401, Assignment #5

Due March 26.

#1. Show that

$$f(x) = |x - \tfrac{1}{2}| - \tfrac{1}{2}$$

is in $H_0^1[0,1]$.

#2. Consider the initial curve $y = \sin x$ moving upward with normal speed 1. Estimate (analytically or using numerical computations) the time at which a derivative discontinuity (a sharp corner) first appears in the curve.

#3. Consider the functional of functions on $[0,1]$:

$$F[u] = \int_0^1 \{(u')^2 + u^2 + ug\}\,dx \qquad g(x) \text{ given.}$$

$u(0) = 0$, no conditions at $x = 1$.

a) Determine the equations and boundary conditions that the minimizer must solve.

b) Consider dividing the interval $[0,1]$ into N equal subintervals of length $h = \frac{1}{N}$ and minimizing F over piecewise linear functions on this grid. Show that this leads to a linear system

$$A\underline{U} = \underline{F}$$

where $\underline{U}$ are the values at the end points of the subintervals. Show explicitly the entries in $\mathbb{A}$ & $\underline{F}$.

#4. Consider the problem

$$-u'' = f(x) \quad , \; u(0) = 0, \; u(1) = 0.$$

Use functions that are piecewise quadratic on N equal subintervals of Length $h = \frac{1}{N}$ to approximate u. Show the discrete problem that this idea leads to.

Do one of #5 or #6.

#5. Determine how accurate the method you describe in #4 is. You may assume that $f \in C^1$ so $u \in C_0^3$ ← continuous third derivatives, zero @ $x = 0, 1$.

continuously differentiable

#6. Solve the problem #6 from the last assignment using the level set numerical method described in §6.4. Compare to the solution from Assignment #4.

Math 401, Spring 2007
Assignment V solutions.

#1 $f(x) = |x - \frac{1}{2}| - \frac{1}{2} = \begin{cases} -x & x \le \frac{1}{2} \\ x-1 & x \geq \frac{1}{2}. \end{cases}$

Note: Can't just show $\int_0^1 |f'|^2 dx$ is bounded, using the almost everywhere defined $f'(x)$. Consider $g(x) = H(x-\frac{1}{3}) - H(x-\frac{2}{3})$ where $H(x)$ is the Heavyside function. $g'(x) = 0$ a.e. but g is not in H_0^1.

Consider $f_\varepsilon = f * \psi_\varepsilon$
$\uparrow$
Mollifier from §2.1.

Since ψ_ε is an even function, $f_\varepsilon(x) = f(x)$ for $|x - \frac{1}{2}| > \varepsilon$. In particular, $f_\varepsilon(0) = 0$, $f_\varepsilon(1) = 1$ for all $\varepsilon \le \frac{1}{2}$. Consider $\{f_{1/n}\}_{n=2}^{\infty}$. This sequence has limit $f(x)$, we need to show that it is a Cauchy sequence in H_0^1. Boundary conditions are satisfied, and if $n, m > N$, then

$$\|f_n - f_m\|_{L^2}^2 \le \int_{\frac{1}{2}-\frac{1}{N}}^{\frac{1}{2}+\frac{1}{N}} (f_n(x) - f_m(x))^2 \, dx$$

but since $|f(x)| \le \frac{1}{2}$, $|f_\varepsilon(x)| \le \frac{1}{2}$ for all ϵ, so we have the expression above is $\le \frac{1}{N}$

Consider now

$$f_\varepsilon(x) = \int_{-\infty}^{\infty} \psi_\varepsilon(s) f(s-x)\, ds$$

$$= \int_{-\infty}^{x+\frac{1}{2}} \psi_\varepsilon(s)(-s+x)\, ds + \int_{x+\frac{1}{2}}^{\infty} \psi_\varepsilon(s)(s-x-1)\, ds.$$

$$f_\varepsilon'(x) = \int_{-\infty}^{x+\frac{1}{2}} \psi_\varepsilon(s)\, ds + \cancel{\psi_\varepsilon(x+\tfrac{1}{2})(-\tfrac{1}{2})}$$

$$- \int_{x+\frac{1}{2}}^{\infty} \psi_\varepsilon(s)\, ds - \cancel{\psi_\varepsilon(x+\tfrac{1}{2})(-\tfrac{1}{2})}.$$

$$|f_\varepsilon'(x)| \le \int_{-\infty}^{\infty} \psi_\varepsilon(s)\, ds = 1.$$

Note: This is an extension of the Assignment II, #5 result to piecewise differentiable functions.

With this bound on f_ε', we show similarly to above that

$$\|(f_n - f_m)'\|_{L^2}^2 \to 0 \quad \text{as} \quad n,m \to \infty.$$

Thus, $\{f_n\}$ is a Cauchy sequence in H_0^1.

#2. Begin with the equation for $y(x,t)$ derived in class,

$$y_t - \sqrt{1+(y_x)^2} = 0.$$

Take the derivative with respect to x and let $u = y_x$:

$$u_t + (\underbrace{-\sqrt{1+u^2}}_{f(u)})_x = 0 \qquad u(x,0) = \cos x.$$

Borrow a result from Math 400, that solutions to this problem are constant along lines with slope $f'(u)$, i.e.

$$u(s + f'(u_0(s))t, t) = u_0(s).$$

Take the derivative with respect to x:

$$u_x \cdot (1 + f''(u_0(s)) \cdot u_0'(s)t) = u_0'(s).$$

$$u_x = \frac{u_0'(s)}{1 + f''(u_0(s)) \cdot u_0'(s)t}$$

A singularity occurs at s at a time

$$t = \frac{-1}{f''(u_0(s)) \cdot u_0'(s)}$$

The first singularity in the soln appears at

$$t_* = \frac{1}{M}$$

where $M = \max_s \left[-f''(u_0(s)) \cdot u_0'(s)\right]$.

<u>Note</u>: if $M \leq 0$, no singularity forms in the soln.

For our problem, $-f'(u) = \dfrac{u}{\sqrt{1+u^2}}$

$$-f''(u) = \frac{1}{(1+u^2)^{3/2}}$$

$$-f''(u_0)\cdot u_0' = \frac{-\sin s}{(1+\cos^2 s)^{3/2}}$$

By inspection, the maximum of this expression is $M = 1$, attained at $s = -\frac{\pi}{2} + 2n\pi$, $n \in \mathbb{Z}$.

Note: Graphically, this is the location we expect singularities to form.

Thus, a singularity forms at $t_* = 1$.

Note: It would have been fine to estimate t_* using numerical computations.

#3. a) Standard E-L application gives

$$-2u'' + 2u + g = 0.$$

$$u(0)=0, \quad u'(1) = 0.$$

b) Consider an approximation

$$U(x) = \sum_{i=1}^{N} U_i B_i(x) \qquad (1)$$

where $B_i(x)$, $i=1,\dots N-1$ are as in the notes and

$$B_N(x) = \begin{cases} 1-(1-x)/h & x \in [1-h, 1]. \\ 0 & \end{cases}$$

So $B_N(x)$ is "half" a hat function. This must be included in (1) since $u(1)$ is not specified.

Put (1) into the functional and proceed as in the class notes:

$$F[\underline{U}] = \underline{U}\cdot \mathbb{A}_1 \underline{U} + \underline{U}\cdot \mathbb{A}_2 \underline{U} + \underline{U}\cdot \underline{G} \qquad (2).$$

where $G_i = \int_0^1 B_i(x)\, g(x)\, dx.$

$$A_{1,ij} = \int_0^1 B_i' B_j' \, dx.$$

$$A_{2,ij} = \int_0^1 B_i(x) B_j(x)\, dx.$$

From the class notes,

$$A_{1,ij} = \begin{cases} \frac{2}{h} & \text{if } i = j < N. \\ \frac{1}{h} & \text{if } i = j = N \quad \text{"half" a hat.} \\ -\frac{1}{h} & \text{if } |i-j| = 1. \end{cases}$$

For $i < N$,

$$A_{2,ii} = A_{2,11} = \int_0^h \left(\frac{x}{h}\right)^2 dx + \int_h^{2h} \left(2 - \frac{x}{h}\right)^2 dx.$$

$$= 2h/3.$$

$A_{2,NN} = h/3$ "half" a hat.

If $|i-j| = 1$,

$$A_{2,ij} = A_{2,12} = \int_h^{2h} \left(2 - \frac{x}{h}\right)\left(\frac{x}{h} - 1\right) dx = h/6.$$

Minimizing (2) we find

$$(2\mathbb{A}_1 + 2\mathbb{A}_2)\,\underline{U} = -\underline{G}.$$

<u>Discussion</u> [not needed for full marks]: For $1<i<N$, the i'th row of the system above is

$$\frac{2}{h}\left(2U_i - U_{i+1} - U_{i-1}\right) + \frac{h}{3}\left(4U_i + U_{i+1} + U_{i-1}\right)$$
$$\approx -h\,g(ih) \qquad (3).$$

using a trapezoidal rule approximation of the integrals in $\underline{G}$ as done in class. Dividing by h, (3) can be recognized as an approximation of

$$-2u'' + 2u + g = 0.$$

Considering the N'th equation:

$$\frac{2}{h}\left(U_N - U_{N-1}\right) + \frac{h}{3}\left(2U_N + U_{N-1}\right) \approx -\frac{h}{2}\,g(1)$$

This is formally equivalent to (3) if we considered $U_{N+1} = U_{N-1}$, but this can be rewritten

$$\frac{U_{N+1} - U_{N-1}}{2h} = 0,$$ an approximation of $u'(1) = 0$.

#4 Use the weak formulation,

$$(u', v') = (f, v) \qquad \forall\, v \in H_0^1. \qquad (4).$$

Let $C_i(x)$ be the quadratic function on the i'th interval with $C_i = 0$ on the subinterval ends and $C_i'' = 1$. Then

$$\{B_i(x)\}_{i=1}^{N-1} \cup \{C_i(x)\}_{i=1}^{N}$$

↑ hats from the last question.

is a basis for the space of H_0^1 piecewise quadratic functions we are considering. Order them

$$(C_1, B_1, C_2, B_2, \ldots B_{N-1}, C_N).$$

and let $D_i(x)$, $i = 1, \ldots 2N-1$ be this ordered basis.

Consider

$$U(x) = \sum_{i=1}^{2N-1} U_i\, D_i(x).$$

<u>Note</u>: For odd i, U_i does not describe a grid value of $u(x)$ – that is OK.

The Galerkin approximation of (4) is

$$\mathbb{A}\, \underline{U} = \underline{F}$$

where $A_{ij} = \int_0^1 D_i'(x)\, D_j'(x)\, dx$

and $F_i = \int_0^1 D_i(x)\, f(x)\, dx.$

Let us identify the nonzero entries of A:

i odd : $A_{ii} = A_{11} = \int_0^h (C_1')^2 \, dx$ $\qquad$ $C_1(x) = \frac{1}{2}x(x-h)$.

$C_1' = x - \frac{1}{2}h$.

$$= \int_0^h \left(x - \frac{h}{2}\right)^2 dx = \frac{h^3}{12}.$$

$$A_{i,i+1} = A_{12} = \int_0^h C_1' B_1' \, dx$$

$$= \int_0^h \left(x - \frac{h}{2}\right)\left(\frac{1}{h}\right) = 0.$$

similarly $A_{i,i-1} = 0$, thus all non-diagonal entries of A are zero in odd rows.

i even : $A_{ii} = \frac{2}{h}$

$A_{ij} = -\frac{1}{h}$ $\quad$ if $\quad |i-j| = 2$.

Using the known properties of the $B_i(x)$'s.

#5 Begin with the result in the notes:

$$\|e'\|_{L^2} \leq \|(u - \varphi)'\|_{L^2} \quad \text{for all } \varphi \in V_h.$$

On each subinterval $[ih, (i+1)h]$ consider the quadratic function $\varphi(x)$ such that φ matches u at the end and midpoint of the subinterval. Let $w = u - \varphi$.

$w(ih) = w((i+\frac{1}{2})h) = w((i+1)h) = 0$

so (Rollé) $\quad w'(\xi_1) = w'(\xi_2) = 0, \quad \xi_1 \in (ih, (i+\frac{1}{2})h)$

and (Rollé) $\quad w''(\xi_3) = 0 \quad\quad \xi_2 \in ((i+\frac{1}{2})h, (i+1)h)$

$\xi_3 \in (\xi_1, \xi_2)$.

Now $w''(x) = \int_{\xi_3}^{x} w'''(s)\,ds = \int_{\xi_3}^{x} u'''(s)\,ds$ since $\psi''' \equiv 0$ (ψ quadratic).

$$|w''(x)| \leq \int_{ih}^{(i+1)h} |u'''(s)|\,ds \leq Ch.$$

where $C = \max_{x \in [0,1]} |u'''(x)|$.

Now $w'(x) = \int_{\xi_1}^{x} w''(s)\,ds$.

so $\quad |w'(x)| \leq \int_{ih}^{(i+1)h} |w''(x)|\,dx \leq Ch^2$.

$$\int_{ih}^{(i+1)h} |w'(x)|^2 \leq C h^5.$$

Summing over all i, recalling $N = 1/h$,

$$\| (u-\psi)' \|^2 \leq C h^4$$

or $\| (u-\psi)' \| \leq Ch^2$, hence we see second order convergence in H_0^1.

<u>Discussion</u>: With $f \in H^1$ we obtain solutions in H^3 with $\|u\|_{H^3} \leq C\,\|f\|_{H^1}$. It is possible to obtain second order convergence in H_0^1 and third order convergence in L_2 by modifying the argument above (following the notes) in this case.

#6. An $(N+1) \times N$ grid is laid out in (x,y) with spacing $h = \frac{1}{N}$:

$$x_i = (i-1)h. \qquad i = 1, \dots N+1.$$

$$y_j = (j-1)h \qquad j = 1, \dots N.$$

A matrix V_{ij} of speeds is constructed, with

$$V_{ij} = \begin{cases} \sqrt{1-y_j} & \text{if } y_j \geq \phi(x_i) \\ \underset{\text{"small"}}{\varepsilon} & \text{if } y_j < \phi(x_i). \end{cases}$$

An initial boolean matrix R_{ij} ("reached") and time matrix T_{ij} are constructed:

$$R_{ij} = \begin{cases} \text{true} & i = 1,\ j = N. \\ \text{false} & \text{otherwise} \end{cases}$$

$T_{1N} = \sqrt{h}$, rest of entries are arbitrary.

At each step of the algorithm below, one entry of $\mathbb{R}$ is changed from false to true, and that entry of $\mathbb{T}$ is set. When $i=N+1$, $j=1$ is set (the target), the algorithm moves to step 2 below.

Step 1. Let $N_{i,j}$ be the set of (vertical & horizontal) neighbors of i,j for which R is true. For each i,j for which $R_{i,j}$ = false and $N_{i,j}$ is not empty, consider the following equation for $T_{i,j}$:

$$\sum_{(i',j')\in N_{ij}} \left(T_{ij} - T_{i',j'} \right)^2 = h^2/(N_{ij})^2.$$

$\uparrow$ previously set values.

$$\underbrace{|N_{ij}|}_{\#\text{ of neighbours.}} T_{ij}^2 - 2\left(\sum T_{i',j'}\right) T_{ij} + \sum (T_{i',j'})^2 = \frac{h^2}{(N_{ij})^2}.$$

$$T_{ij}^2 - \overbrace{\boxed{\frac{2\left(\sum T_{i',j'}\right)}{|N_{i,j}|}}}^{b} T_{ij} + \overbrace{\frac{\sum (T_{i',j'})^2 - h^2/(N_{ij})^2}{|N_{i,j}|}}^{c.} = 0.$$

$$T_{ij} = b \pm \sqrt{b^2 - c}$$

Take the larger root here $T_{ij} = b + \sqrt{b^2 - c}$.

Of all the T's computed in this way, let i, j be where the smallest occurs. Set R_{ij} = true, and T_{ij} equal to the computed value. Repeat step 1 until you reach the final point.

<u>step 2</u> Construct a sequence $\{(i_n, j_n)\}$ of grid points beginning at $(N+1, 1)$ and ending at $(1, N)$ such that the next point in the sequence has the smallest T value adjacent to the previous point. The path $\{(x_{i_n}, y_{j_n})\}$ is a (crude) approximation of the optimal path.

Computational results are shown on p. 13. MATLAB code is posted on the course web page.

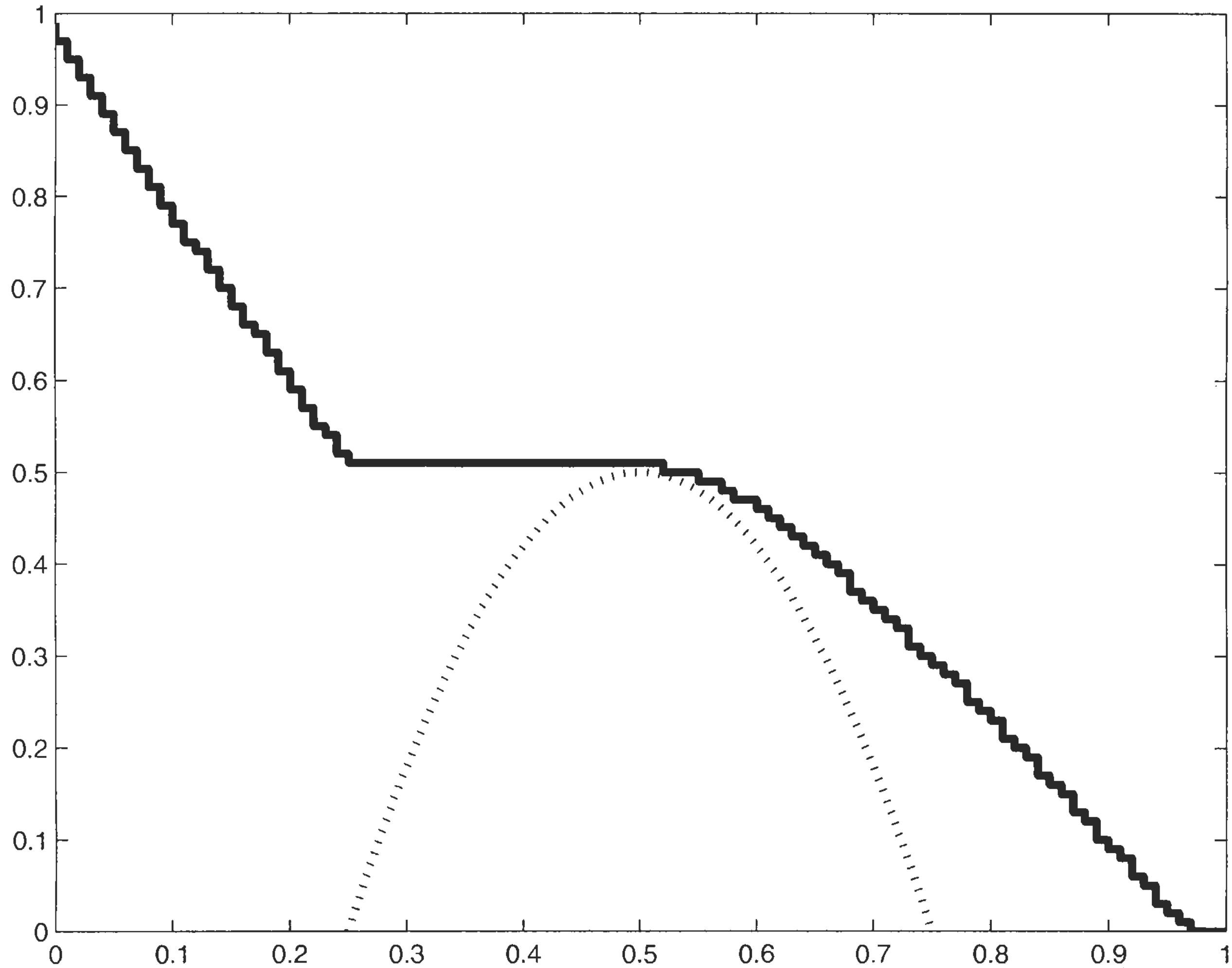
Final times, unscaled by $\sqrt{2g}$, computed with various N:

N	T(1,1)
10	2.85
20	2.72
40	2.67
100	2.65

← looks convergent, but I did not compute the time from my Assignment IV soln to compare...

<u>Note</u>: Efficient implementations would use data structures to keep track of the interface between reached & unreached points.

Note: My path reconstruction idea is too crude, I think, but it shows that we're getting basically the right thing.



Math 401, Assignment #6.
Due April 11, 2007 [Hard dead line].

#1 $u(x)$ is the solution of the ODE BVP:

$$u'' - u + \varepsilon u^2 = 1 \qquad u(0)=0, \; u(1)=0.$$

Find the $O(1)$ and $O(\varepsilon)$ terms in a regular expansion for $u(x)$.

#2. Consider the BV eigenvalue problem

$$\lambda(1+\varepsilon x)\, u = ((1+\varepsilon x)\, u')' \qquad u(0)=0, \; u(1)=0.$$

When $\varepsilon = 0$, the first eigenvalue is $\lambda = -\pi^2$ with eigenfunction $u(x) = \sin \pi x$. Find the next nonzero term in a perturbation expansion for this λ.

#3. Let Ω be a given 2D domain. If $u(x,y)$ with $u|_{\partial\Omega}$ given, ≥ 0 minimizes

$$F[u] = \int_D f(x, u, \overset{p}{u_x}, \overset{q}{u_y})\, dA$$

where D is the part of $\Omega \cap \{(x,y) : u(x,y) > 0\}$ connected to $\partial\Omega$, show that u satisfies

$$f_u - (f_p)_x - (f_q)_y = 0 \qquad \text{in } D.$$

and $f_p u_x + f_q u_y - f = 0$ on $\partial D \cap \{(x,y) : u(x,y) = 0\}$

as stated in class.

#4 Consider $u(x,t)$ that satisfies

$$u_{tt} = c^2 u_{xx}$$

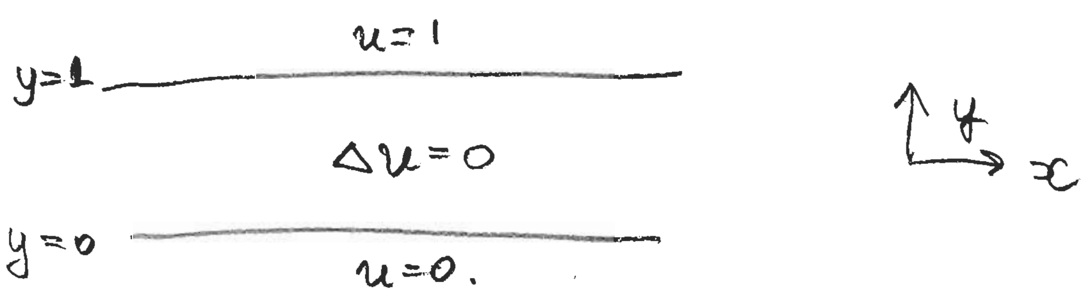
with
$$c(x) = \begin{cases} 1 & x \le 0,\ x \geq 1 \\ w & x \in [0,1]. \end{cases}$$

where $w > 0$ is a given constant. Determine the solution of this problem of the form

$$u(x,t) = \text{Re}\left\{ A(x) e^{-it} \right\}$$

corresponding to a wave incident from the left of the form $\cos(x-t)$. Note any values of w that are exceptional cases. For this problem, w is not necessarily $1 + O(\varepsilon)$.

#5 Consider the following problem for the steady state temperature $u(x,y)$ in an infinite plate held at $u=1$ at the top $(y=1)$ and $u=0$ at the bottom:



The solution to this problem is $u(x,y) = y$. Consider now a perturbation in the upper boundary of the form $y(x) = 1 + \varepsilon f(x)$. Assume the form of the solution

$$u(x;y) = y + \varepsilon u_1(x,y).$$

Derive equations and boundary conditions for $u_1(x,y)$. Solve for u_1 explicitly for the case $f(x) = \sin x$.

Do either #6 <u>or</u> #7.

#6 Derive the equations and free boundary conditions that $u(x)$ must satisfy to be a minimizer of

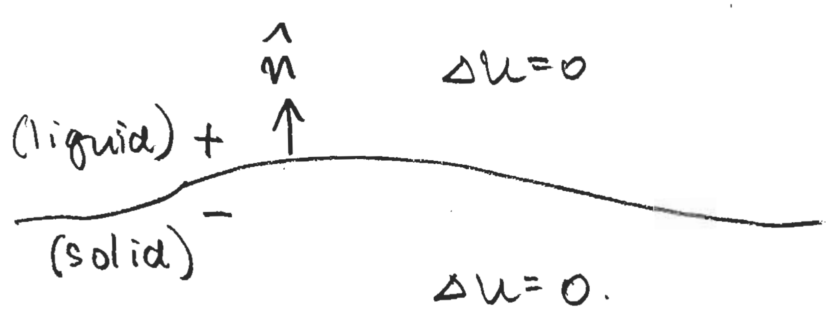
$$F[u] = \int_I f(x, u, u_x)\, dx$$

subject to $u(0)$ having a given value and also the constraint

$$\int_I g(x, u, u_x)\, dx = L.$$

Here, $I = [0, x_0]$ where x_0 is the first nonzero root of u. Apply your result to Assignment IV, #1 (some interpretation will be required).

#7. Consider the following Stefan solidification problem for $u(x,y,t)$ with a free boundary $\Gamma(t)$:



$\Gamma(t)$, described by $y = \eta(x,t)$.

Additional far field conditions on u_+ and u_- are given.

Note: Time dependence in the PDEs has been assumed to be much faster than the movement of Γ here.

on Γ, $u_+ = u_- = 0$ and Γ moves with normal speed V given by

$$V = -\left(\frac{\partial u}{\partial n_+} - a \frac{\partial u}{\partial n_-} \right)$$

where $a > 0$ is a given constant.

physical constant removed by scaling time

Note: Physically, this corresponds to the net heat flux into Γ contributes to phase change, hence movement of the boundary.

A steady state soln of the problem is given below: $\eta(x) \equiv 0$,

$$u_0(x,y) = \begin{cases} Qy & y \geq 0 \\ \frac{Q}{a} y & y \leq 0. \end{cases}$$

Q given constant.

Determine if this solution is stable to perturbations in Γ in the following way: Let

$$\eta(x,t) \approx \varepsilon \eta_1(x,t)$$

with $\eta_1(x,0)$ given but arbitrary.

$$u(x,y,t) \approx u_0(y) + \varepsilon u_1(x,y,t).$$

Determine if η_1, u_1 grow or decay in time.

Hint: The problem for η_1, u_1 can be simplified by taking the Fourier Transform in x.

Math 401, Assignment VI solutions

#1. $u'' - u + \varepsilon u^2 = 1$. $\qquad u(0) = u(1) = 0$.

Use $c(\cdot)$ & $s(\cdot)$ as short hand for $\cosh(\cdot)$ & $\sinh(\cdot)$ below:

$$u_0(x) = -1 + \frac{c(x-1/2)}{c(\frac{1}{2})}$$ is a convenient form.

$$u_0^2(x) = 1 - \frac{2c(x-\frac{1}{2})}{c(\frac{1}{2})} + \frac{c^2(x-\frac{1}{2})}{c^2(\frac{1}{2})}.$$

$$= 1 - \frac{2}{c(\frac{1}{2})}c(x-\tfrac{1}{2}) + \frac{1}{2c^2(\frac{1}{2})}\left\{1 + c(2x-1)\right\}.$$

using the identity
$c^2(x) = \frac{1}{2}(1 + c(2x))$.

$$= \left(1 + \frac{1}{1+c(1)}\right) - \frac{2}{c(\frac{1}{2})}c(x-\tfrac{1}{2}) + \frac{1}{1+c(1)}c(2x-1).$$

At $O(\varepsilon)$,

$$u_1'' - u_1 = -u_0^2.$$

particular $$u_{1,p} = \left(1 + \frac{1}{1+c(1)}\right) - \frac{1}{c(\frac{1}{2})}x\,s(x-\tfrac{1}{2})$$

$$+ \frac{1}{3(1+c(1))}c(2x-1).$$

$$u_1 = u_{1,p} + A\,c(x) + B\,s(x). \qquad (1).$$

$$u_1(0) = 0 \Rightarrow 1 + \frac{1}{1+c(1)} + \frac{c(1)}{3(1+c(1))} + A = 0$$

$$\Rightarrow \quad A = -\left(\frac{2 + 4C(1)/3}{1 + C(1)} \right).$$

$$u_1(1) = 0 \quad \Rightarrow \quad \left(1 + \frac{1}{1+C(1)}\right) - \frac{1}{C(\frac{1}{2})} S\left(\tfrac{1}{2}\right)$$

$$+ \frac{C(1)}{3(1+C(1))} + A\,C(1) + B\,S(1) = 0.$$

$$\Rightarrow B = \frac{1}{S(1)} \left\{ \frac{S(\frac{1}{2})}{C(\frac{1}{2})} - \frac{2 + 4C(1)/3}{1 + C(1)} + C(1)\left(\frac{2 + 4C(1)/3}{1 + C(1)} \right) \right\}.$$

$u_1(x)$ is given by (1) with these coefficients.

#2 $\quad \lambda(1+\varepsilon x)u = ((1+\varepsilon x)u')' \qquad u(0)=0, \ u(1)=0.$

$$u_0(x) = \sin \pi x, \quad \lambda_0 = -\pi^2.$$

At $O(\varepsilon)$:

will cancel

$$-\pi^2 u_1 - \pi^2 x \sin \pi x + \lambda_1 \sin \pi x$$

$$= (x u_0' + u_1')'$$

$$= (x \pi \cos \pi x + u_1')' = \pi \cos \pi x - x\pi^2 \sin \pi x + u_1''.$$

Rewriting:

$$u_1'' + \pi^2 u_1 = \lambda_1 \sin \pi x - \pi \cos \pi x. \qquad (2)$$

$u_{1,p} = a x \sin \pi x + b x \cos \pi x \qquad$ (resonant).

$u_{1,p}' = a(\sin \pi x + \pi x \cos \pi x) + b(\cos \pi x - x\pi \sin \pi x)$

$u_{1,p}'' = a(2\pi \cos \pi x - \pi^2 x \sin \pi x) + b(-2\pi \sin \pi x - \pi^2 x \cos \pi x)$

$u_{1,p}'' + \pi^2 u_{1,p} = 2\pi a \cos \pi x - 2\pi b \sin \pi x.$

Matching (2) we see that

$$a = -\tfrac{1}{2}, \qquad b = -\frac{\lambda_1}{2\pi}. \qquad \text{homogeneous, Forget } \sin\pi x = u_0.$$

$$u_1 = -\tfrac{1}{2}\, x \sin\pi x - \frac{\lambda_1}{2\pi}\, x \cos\pi x + A\cos\pi x$$

$u_1(0) = 0 \Rightarrow A = 0$

$u_1(1) = 0 \Rightarrow \lambda_1 = 0.$

Since $\lambda_1 = 0$, we must continue to $O(\varepsilon^2)$ to find the nonzero term λ_2.

$$u_1 = -\tfrac{1}{2}\, x \sin\pi x, \qquad u_1' = -\tfrac{1}{2}\sin\pi x - \tfrac{\pi}{2}\, x\cos\pi x.$$

at $O(\varepsilon^2)$:

$$\lambda_2 u_0 + \lambda_0 x u_1 + \lambda_0 u_2 = \left(u_2' + x u_1'\right)'$$

$$u_2'' + \pi^2 u_2 = \overbrace{\lambda_2 \sin\pi x}^{\lambda_2 u_0} + \overbrace{\frac{\pi^2}{2} x^2 \sin\pi x}^{\lambda_0 x u_1.}$$

$$\underbrace{+\tfrac{1}{2}\left(x\sin\pi x + \pi x^2\cos\pi x\right)'}_{-(x u_1')'.}$$

$$= \left(\lambda_2 + \tfrac{1}{2}\right)\sin\pi x + \frac{3\pi}{2}\, x\sin\pi x. \qquad (3).$$

$$u_{2,p} = a x\sin\pi x + b x\cos\pi x + c x^2\cos\pi x$$

(resonant, I know that the $d x^2 \sin\pi x$ term will be zero).

$$u_{2,p}' = \qquad\qquad + c\left(2x\cos\pi x - x^2\pi\sin\pi x\right)$$

$$u_{2,p}'' = \underbrace{\qquad\qquad}_{\text{same } a,\, b \text{ terms as on p. 2}} + c\left(-4\pi x\sin\pi x - \pi^2 x^2\sin\pi x + 2\cos\pi x\right)$$

$$u_{2,p}'' + u_{2,p} = 2\pi a \cos\pi x - 2\pi b \sin\pi x$$
$$+ c(-2\pi x \sin\pi x + 2\cos\pi x).$$

matching in (3):

$x \sin\pi x$: $\quad -2\pi c = 3\pi/2 \quad \Rightarrow \quad c = -3/4.$

$\cos\pi x$: $\quad 2\pi a + 2c = 0 \quad \Rightarrow \quad a = \frac{3}{4\pi}.$

$\sin\pi x$: $\quad -2\pi b = (\lambda_2 + \frac{1}{2}) \quad \Rightarrow \quad b = -\left(\frac{\lambda_2 + \frac{1}{2}}{2\pi}\right)$

So $u_2 = \frac{3}{4\pi} x \sin\pi x - \frac{\lambda_2 + \frac{1}{2}}{2\pi} x \cos\pi x - \frac{3}{4} x^2 \cos\pi x.$

$$+ A \cos\pi x.$$

$u_2(0) = 0 \Rightarrow A = 0.$

$u_2(1) = 0 \Rightarrow \quad -\frac{\lambda_2 + \frac{1}{2}}{2\pi} - \frac{3}{4} = 0.$

$$\Rightarrow \lambda_2 = -\frac{3\pi}{2} - \frac{1}{2}.$$

So $\lambda \approx -\pi^2 - \left(\frac{3\pi}{2} + \frac{1}{2}\right)\varepsilon.$

<u>Note</u>: The algebra for Q's #1 & 2 was worse than I had expected it to be.

#3. Consider $F[u + \varepsilon\eta]$. Let Γ be the free boundary of the minimizer:

$$\Gamma = \partial D \cap \{u = 0\}.$$

Let $\tilde{D}$ be the perturbed domain $\tilde{D} = \Omega \cap \{u + \varepsilon\eta \geq 0\}$ and $\tilde{\Gamma}$ the perturbed boundary, $\partial\tilde{D} \cap \{u + \varepsilon\eta = 0\}$.

$$F[u + \varepsilon\eta] = \int_D f(x, u + \varepsilon\eta, u_x + \varepsilon\eta_x \,;\, u_y + \varepsilon\eta_y)\, dA \qquad (\mathrm{I})$$

$$+ \int_{\tilde{D}/D} f(\quad \text{"} \quad)\, dA. \qquad (\mathrm{II})$$

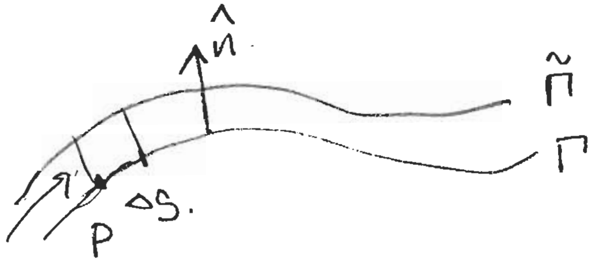
where it has been assumed here for simplicity that $D \subset \tilde{D}$ ($\eta > 0$ on Γ), although the result derived below does not depend on this fact.

Considering (I) we obtain (to order ε):

$$F[u] + \varepsilon \int_D \left(f_u - (f_p)_x - (f_q)_y\right)\eta \, dA$$

$$+ \varepsilon \int_\Gamma (f_p, f_q)\cdot\hat{n}\, \eta \, ds.$$

Considering (II) we have:



Using level set ideas, this $O(\varepsilon)$ distance ℓ can be approximated

$$(u + \varepsilon\eta)(P + \ell\hat{n}) = 0, \text{ to } O(\varepsilon):$$

$$\hat{n} = -\nabla u / |\nabla u|$$

$$\cancel{u(P)} + \nabla u \cdot (\ell\hat{n}) + \varepsilon\eta(P) \approx 0.$$

That is $\quad -\ell|\nabla u| + \varepsilon\eta(P) \approx 0$

or $\quad \ell \approx \varepsilon \dfrac{\eta(P)}{|\nabla u|}.$

Thus, (II) can be approximated to $O(\varepsilon)$ by

$$\varepsilon \int_\Gamma \frac{f\eta}{|\nabla u|}\, ds.$$

Note that in this expression, the effect of negative η is also handled correctly.

Combining (I) & (II) we have for all η:

$$F[u+\varepsilon\eta] \approx F_{min} + \varepsilon \int_D \left(f_u - (f_p)_x - (f_q)_y\right)\eta\, dA$$
$$+ \varepsilon \int_\Gamma \left[(f_p, f_q)\cdot\hat{n} - \frac{f}{|\nabla u|}\right]\eta\, ds$$

Thus, for u to be a minimizer,

$$f_u - (f_p)_x - (f_q)_y = 0 \qquad \text{in } D$$

and $\quad (f_p, f_q)\cdot\hat{n} + \dfrac{f}{|\nabla u|} = 0 \quad \text{on } \Gamma.$

Recalling $\hat{n} = -\nabla u/|\nabla u|$ we have

$$-f_p u_x - f_q u_y + f = 0 \qquad \text{on } \Gamma.$$

#4 We have

$$A(x) = \begin{cases} e^{ix} + b e^{-ix} & x \le 0. \\ c e^{ix/w} + d e^{-ix/w}. & 0 \le x \le 1. \\ f e^{i(x-1)} & x \ge 1. \end{cases}$$

At $x=0$, A & A_x are continuous:

$$1+b = c+d$$

$$i - ib = \frac{ic}{w} - \frac{i}{w} d.$$

Also at $x=1$:

$$c e^{i/w} + d e^{-i/w} = f$$

$$\frac{i}{w} c e^{i/w} - \frac{i}{w} d e^{-i/w} = if.$$

We can represent the four equations above as a linear system:

$$\begin{pmatrix} 1 & -1 & -1 & 0 \\ 1 & 1/w & -\frac{1}{w} & 0 \\ 0 & e^{i/w} & e^{-i/w} & -1 \\ 0 & e^{i/w} & -e^{-i/w} & -w \end{pmatrix} \begin{pmatrix} b \\ c \\ d \\ f \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

To determine which w do not give a unique solution, find the determinant of the matrix:

$$\frac{1}{w} e^{-i/w}(-1-w) + \frac{1}{w} e^{i/w}(-w+1) +$$
$$e^{-i/w}(-1-w) - e^{i/w}(-w+1).$$

$$= \left(w + \frac{1}{w}\right)\left(e^{i/w} - e^{-i/w}\right).$$

no real roots

zero when $e^{2i/w} = 1$,

$w = \frac{1}{n\pi}$, n integer.

For these w, the time harmonic solution does not exist. There is a resonance in the interval $[0,1]$.

#5 $u(x, 1+\varepsilon f(x)) = 1 \Rightarrow$

$1: \quad u_0(x,1) = 1 \quad \checkmark \qquad (u_0 = y).$

$\varepsilon: \quad \underbrace{u_{0,y}}_{1} f(x) + u_1(x,1) = 0$

i.e. $u_1(x,1) = -f(x)$, $u_1(x,0) = 0$.

$\Delta u_1 = 0$ in $0 \le y \le 1$.

When $f(x) = \sin x$, consider u_1 of the form

$$u_1(x,y) = V(y)\sin x$$

$\Delta u_1 = 0 \Rightarrow V'' - V = 0 \qquad V(1) = -1,\ V(0) = 0.$

ie. $V(y) = -\sinh y / \sinh 1$.

and $u_1(x,y) = -\sin x \sinh y / \sinh 1$.

#6. Consider $u + \varepsilon \eta$
minimizer

As in the notes, $\tilde{x}_0 \approx x_0 - \varepsilon \dfrac{\eta(x_0)}{u'(x_0)}$.

$\tilde{x}_0$ zero of $u + \varepsilon\eta$ near x_0

Consider $\displaystyle\int_0^{\tilde{x}_0} g(x, u+\varepsilon\eta, u' + \varepsilon\eta')\,dx$

$$\approx \int_0^{x_0} g(x,u,u')\,dx + \varepsilon \int_0^{x_0} (g_u - g_p')\eta\,dx -$$

$$\frac{\varepsilon \eta(x_0)}{u'(x_0)} g(x_0, u(x_0), u'(x_0)) + \varepsilon g_p \eta \Big|_{x_0} \qquad (4).$$

to $O(\varepsilon)$. As in the notes,

$$F[u+\varepsilon\eta] - F[u] \approx$$

$$\varepsilon \int_0^{x_0} (f_u - f_p')\eta \, dx + \varepsilon\left(f_p - \frac{f}{u'}\right)\eta \Big|_{x_0} = 0. \quad (5).$$

Consider first η that are zero at $x = x_0$. Then the boundary terms in (4) and (5) are not present, and the argument in the notes shows that

$$f_u - f_p' + \lambda(g_u - g_p') = 0.$$

Note: We use the fact that the L_2 closure of $C_0^\infty \cap S$ is S, where S is the span of $\{g_u - g_p'\}$.

Now consider (5) + λ (4):

$$\eta\left[f_p - \frac{f}{u'} + \lambda\left(g_p - \frac{g}{u'}\right)\right]\Big|_{x_0} = 0.$$

so the term above $[\cdot] = 0$.

For the Assignment IV #1,

$f \equiv u$, $g = \sqrt{1+(u')^2}$.

The interior equations are

$$1 + \lambda\left(\frac{u'}{(1+(u')^2)^{1/2}}\right)' = 0 \qquad (6)$$

which show that $y = u(x)$ describe the arc of a circle.

The free boundary condition above reduces to

$$\lambda \left(\frac{-1}{u' \sqrt{1+(u')^2}} \right) = 0.$$

Note from (6) that $\lambda \neq 0$. Thus, the equation above cannot be satisfied. However, we could consider this problem in a limiting sense and realize that $u' \to \infty$ at x_0. This gives the desired half circle solution.

#7. $u \approx u_0(y) + \varepsilon\, u_1(x,y,t)$.

where u_1 has a 2-part definition $y>0$, $y<0$, $u_{1,\pm}$. The interface is $y \approx \varepsilon\, \eta_1(x)$. On the interface, $u=0$,

$$u_\pm (x, \eta(x,t), t) = 0.$$

At $O(\varepsilon)$ this gives.

$$\left.\begin{aligned} u_{1,+}(x,0,t) &= -Q\eta_1. \\ u_{1,-}(x,0,t) &= -\frac{Q}{a}\eta_1. \end{aligned}\right\} \quad (7)$$

Now $$\hat{n} = \frac{(-\eta',1)}{\sqrt{1+(\eta')^2}} \approx (0,1) - \varepsilon(\eta_1',0).$$

Some care is required to justify the simple $O(\varepsilon)$ term for V:

$$V = -\left(\frac{\partial u_+}{\partial n} - a\frac{\partial u_-}{\partial n}\right)$$
$$= -(\nabla u_+ - a\nabla u_-)\cdot \hat{n}.$$

$$= -\left(\nabla u_+(x,\eta,t) - \nabla u_-(x,\eta,t)\right)\cdot\hat{n}$$

$$= -\cancel{\frac{\partial^2 u_{0,+}}{\partial y^2}}\eta_1 - \frac{\partial u_{1,+}}{\partial y}\ \cancel{\frac{\partial u_{0,+}}{\partial x}}\eta_1' \qquad \text{to } O(\varepsilon).$$

$$+a\cancel{\frac{\partial^2 u_{0,-}}{\partial y^2}}\eta_1 + a\frac{\partial u_{1,-}}{\partial y} + a\cancel{\frac{\partial u_{0,-}}{\partial x}}\eta_1'$$

$$= -\frac{\partial u_{1,+}}{\partial y} + a\frac{\partial u_{1,-}}{\partial y}$$

This is the normal velocity,

$$\frac{\partial\eta}{\partial t} = V\sqrt{1+(\eta')^2} = -\frac{\partial u_{1,+}}{\partial y} + a\frac{\partial u_{1,-}}{\partial y} \quad \text{to } O(\varepsilon). \qquad (8).$$

Now $\Delta u_{1,\pm} = 0$. Taking the transform in x gives

$$\hat{u}_{1,\pm,yy} - \alpha^2\hat{u}_{1,\pm} = 0.$$

$$\hat{u}_{1,+} = -Q\hat{\eta}_1 e^{-|\alpha|y}$$

$$\hat{u}_{1,-} = -\frac{Q}{a}\hat{\eta}_1 e^{|\alpha|y}.$$

↙ use BC's (7) and decay at $|y|\to\infty$.

Use these forms in (8) to get

$$\frac{d\hat{\eta}_1}{dt} = -Q|\alpha|\left(1+\frac{1}{a}\right)\hat{\eta}_1.$$

Thus, we see decay in all Fourier modes (stability to base solution) if $Q \geq 0$ (this is the physically reasonable case when the water above is colder than the ice below).

Math 401, Spring 2007
Midterm Review.

Format is the same as the posted practice midterm:

- do 4 of 6 questions
- each question out of 10 - total 40.
- no notes, calculators, books.

Material covered: to the end of §5.5 of the posted notes.

Ch 2 Green's functions for ODE BVP's:

+ Constructing Green's functions

also
- mollifiers, generalized functions
- adjoint operators
- unique solutions
- eigen analysis.

Ch 3 Classification of PDEs

+ Determining the type of a general, 2D, 2nd order PDE.

also
- implications for appropriate extra conditions (initial, boundary) that lead to a well-posed problem.

Ch 4 Green's functions for PDE's:

+ Basic Green's fn's for heat, wave, poisson problems.

tricks
- Duhamel's principle
- relation of interior Green's fn to boundary G fn's for Poisson problems
- Fourier transform
- D'Alembert soln.
- method of images.

also
- multipole expansions
- single & double layer potentials.

Ch5 (to §5.5) Functional Minimization

+ Minimization of $\int_0^1 F(u,x)\,dx$ or $\int_0^1 F(u,u',x)\,dx$, possibly with integral constraints.

also
- method of continuation
- capturing methods.
- inequality constraints.

Math 401, Spring 2007
Practice Midterm.
Solutions posted Monday, February 26

Instructions for midterms:

+ no calculators, notes, or books.
+ Each question worth 10 marks.
+ Do 4 of the 6 questions (total 40 marks).
+ If you attempt more than 4 questions, the best 4 marks will be used to make your grade.

#1a) Find the maximum (and the maximizing function $u(x)$) of the functional

[8]
$$F[u] = \int_0^1 x^2 u(x)\,dx. \qquad (1)$$

subject to the constraint

$$\int_0^1 [u(x)]^2 dx = 1 \qquad (2).$$

b) Describe what happens when $F[u]$ is [2] maximized, subject to the constraint

$$\int_0^1 |u(x)|\,dx = 1.$$

instead of (2). It is not necessary to show the answer rigorously in this part.

#2 Consider the problem for $u(x)$, $x \in [0,1]$:

$$-u'' = f, \quad u(0) = 0, \quad u'(1) = 0.$$

a) Show the problem above has unique solutions [2]

b) Find the Green's function representation of the solutions to the problem. [8]

#3 Consider the nonlinear problem for $u(x)$, $x \in [0,1]$.

$$-u'' + u^3 = f(x), \quad u(0) = 0, \quad u(1) = 0 \qquad (3)$$

Show that this problem has unique solutions.
Hint: Carefully consider the problem that the difference between two solutions solves.

#4. Describe how you would solve the problem (3) numerically. Assume you are writing your code from scratch, that the only library you have is for solving linear systems of equations. Be specific about your discretization and any iterative schemes you use.

Note: $u_{xx}(x) \approx \dfrac{u(x+h) - 2u(x) + u(x-h)}{h^2}$

#5 Write an explicit solution for $u(x,t)$, $x \geq 0$, $t \geq 0$ that solves:

$$u_{tt} = u_{xx}$$

$u(x,0) = u_0(x)$ given

$u(0,t) = 0 \qquad t \geq 0$

The solution $u(x,t)$ should be in terms of u_0.

#6 a) Consider $u(x,t)$, $x \in \mathbb{R}$, $t \geq 0$, with $u(x,0) = u_0(x)$

[4] given such that

$$u_t = \pm u_{xxxx}$$

For which sign $\pm$ above is the problem well-posed? Justify.

b) Consider the functional

[6] $$F[u] = \int_0^1 \left\{ (u')^4 + u^4 + ug \right\} dx$$

where $g(x)$ is a given function. Find the equation and boundary conditions for the function $u(x)$ that minimizes this functional. Do not try to solve the resulting equation.

Math 401, Spring 2007
Practice Midterm Solutions

#1a) Maximize $F[u] = \int_0^1 \overbrace{x^2 u(x)}^{f(u,x)} dx$

subject to $\int_0^1 \underbrace{[u(x)]^2}_{g(u,x)} dx = 1$

$f(u,x) = x^2 u, \qquad g(u,x) = u^2.$

$$f_u + \lambda g_u = 0.$$

$$x^2 + 2\lambda u = 0. \quad \Rightarrow \quad u = Cx^2.$$

Determine C such that

$$\int_0^1 [u(x)]^2 dx = 1 \quad \Rightarrow \quad C^2 \int_0^1 x^4 dx = 1.$$

$$C^2 = 4 \Rightarrow C = 2.$$

$u(x) = 2x^2.$

Don't forget to find the maximum F,

$F_{max} = 2\int_0^1 x^4 dx = 1/2.$

b) Just using intuition, $F_{max} = 1$ in this case, considering u that approach $\delta(x-1)$.

2 $-u'' = f, \quad u(0) = 0, \quad u'(1) = 0.$

a) $-u'' = 0 \Rightarrow u = a + bx.$

$$\left.\begin{array}{l} u(0) = 0 \Rightarrow a = 0 \\ u'(1) = 0 \Rightarrow b = 0 \end{array}\right\} \Rightarrow u \equiv 0.$$

so the problem above has ! solutions.

b) The problem is self adjoint, consider $G(s,x)$ for each x:

$$G_{ss}(s,x) = 0 \Rightarrow G(s,x) = \begin{cases} a_1 + b_1 s. & s \le x \\ a_2 + b_2 s & s \ge x. \end{cases}$$

$G(0,x) = 0 \Rightarrow a_1 = 0.$

$G_s(1,x) = 0 \Rightarrow b_2 = 0.$

$[G]\big|_{s=x} = 0 \Rightarrow b_1 x = a_2.$ $\qquad a_2 = x.$

$[G_s]_{s=x} = -1 \Rightarrow b_2 - b_1 = -1 \Rightarrow b_1 = 1.$

So $G(s,x) = \begin{cases} s & s \le x. \\ x & s \ge x. \end{cases}$

#3 Suppose $-u'' + u^3 = f(x)$ $\quad u(0)=0, u(1)=0.$

$-v'' + v^3 = f(x).$ $\quad v(0)=0, v(1)=0.$

Consider $w = u-v$, subtract equations above,

$$-w'' + u^3 - v^3 = 0.$$

Now $u^3 - v^3 = (u-v)(u^2 - uv + v^2)$

$$= w \, (\underbrace{\tfrac{1}{2}(u-v)^2 + \tfrac{1}{2}u^2 + v^2}_{g(x),\ \text{non-negative.}})$$

$-w'' + g(x) w = 0.$ $\quad w(0)=0, w(1)=0.$

Multiply by w and integrate:

$$\int_0^1 ((w')^2 + g(x) w^2) \, dx = 0.$$

This shows $w \equiv 0$, unique solution.

#4. I would use finite differences on a uniform grid to approximate the derivatives:

$U_i \approx u(ih)$ $\quad i = 1, \ldots N-1.$

$h = \frac{1}{N}.$

Equations approximated by

$$-\left(\frac{U_{i+1} - 2U_i + U_{i-1}}{h^2}\right) + U_i^3 = f(ih)$$

$i=1, \dots N-1$, setting $U_0 = 0$ in the $i=1$ and $U_N = 0$ in the $i = N-1$ expression. This results in the nonlinear system:

$$\mathcal{N}(\underline{U}) = 0 \qquad \underline{U} = (U_1, U_2, \dots U_{N-1})^T.$$

I would use Newton's method to solve the problem, with iterates $\underline{U}^{(j)}$

$$\underline{U}^{(j+1)} = \underline{U}^{(j)} - \left[\nabla\mathcal{N}(\underline{U}^{(j)})\right]^{-1} \mathcal{N}(\underline{U}^{(j)})$$

$$\left[\nabla\mathcal{N}(\underline{U})\right]_{i,j} = \begin{cases} -1/h^2 & j = i-1 \neq 0. \\ 2/h^2 + 3U_i^2 & j = i \\ -1/h^2 & j = i+1 \neq N. \end{cases}$$

I'd try starting with $\underline{U}^{(0)} = \underline{0}$.

#5 $\quad u_{tt} = u_{xx} \qquad x \geq 0,\ t \geq 0.$

$u(x,0) = u_0(x)$

$u(0,t) = 0 \qquad t \geq 0.$

D'Alembert

$$u(x,t) = f(x-t) + g(x+t).$$

Standard argument

$$f(\xi) = g(\xi) = \tfrac{1}{2} u_0(\xi) \qquad \xi \geq 0.$$

However, negative arguments of $f(\cdot)$ must be given:

$$u(0,t) = 0 \quad \Rightarrow \quad f(-t) + g(t) = 0,$$

$$f(\xi) = -g(-\xi) \quad \text{if } \xi < 0$$

$$= -\tfrac{1}{2} u_0(-\xi)$$

Physically, this represents a reflected (inverted) left moving wave now moving to the right.

#6. a) $u_t = \pm u_{xxxx}$.

Use Fourier transform

$$\frac{d}{dt}\hat{u} = \pm (i\alpha)^4 \hat{u} = \pm \alpha^4 \hat{u}, \quad \hat{u}(0) = \hat{u}_0$$

$$\hat{u} = e^{\pm \alpha^4 t} \hat{u}_0.$$

Well posed only if we take the minus sign above,

$$u_t = -u_{xxxx}$$

(describes linearized surface diffusion).

b) $F[u] = \int_0^1 \{ (u')^4 + u^4 + ug \} dx.$

$$\uparrow$$

$$f(u,p,x) = p^4 + u^4 + u\,g(x).$$

E-L $\quad f_u - (f_p)' = 0 \Rightarrow (4u^3 + g) - 4\{(u')^3\}' = 0$

Natural boundary conditions $f_p = 0$,

$$\Rightarrow u'(0) = u'(1) = 0.$$

Math 401, Spring 2007
Midterm Solutions.

#1. $f(u,x) = u \qquad g(u,x) = \sqrt{x}\, u^2$

$f_u + \lambda g_u = 0 \Rightarrow 1 + 2\lambda\sqrt{x}\, u = 0.$

$\Rightarrow u = \dfrac{c}{\sqrt{x}}.$

$\int_0^1 \sqrt{x}\, u^2\, dx = 1 \Rightarrow c^2 \int_0^1 \dfrac{dx}{\sqrt{x}} = 1 \Rightarrow c = \dfrac{1}{\sqrt{2}}.$

Maximizing $u(x) = \dfrac{1}{\sqrt{2x}}$

$F_{max} = \int_0^1 \dfrac{dx}{\sqrt{2x}} = \sqrt{2}.$

#2 a) $u'' = 0 \Rightarrow u = a + bx$

$u(0) = 0 \Rightarrow a = 0.$

$u'(1) + \alpha u(1) = 0 \Rightarrow b(1+\alpha) = 0$

So $b = 0$ (! solutions) if $\alpha \neq -1$.

b) The problem is self adjoint

$u(x) = \int_0^1 G(s,x) f(s)\, ds$

where : (i) $G_{ss} = 0 \quad s \neq x$. ; (iv) $G(0,x) = 0$

(ii) $[G] = 0 \quad s = x$; (v) $G_s(1,x) + G(1,x) = 0$.

(iii) $[G_s] = -1 \quad s = x$

$$G(s,x) = \begin{cases} As & s < x \\ (A-1)s + B. & s > x. \end{cases}$$ satisfies (i), (iii) & (iv).

(ii) $Ax = (A-1)x + B \Rightarrow B = x$

(v) $(A-1) + (A-1) + B = 0. \Rightarrow A = 1 - \frac{x}{2}.$

$$G(s,x) = \begin{cases} s(1 - x/2) = s - sx/2 & s < x. \\ x - sx/2 & s > x. \end{cases}$$

check: symmetric $G(s,x) = G(x,s)$.

#3. a) Highest order terms $u_{tt} - u_{xx}$, hyperbolic.

b) Take Fourier transform in x: $u(x) \to \hat{u}(\alpha)$.

$$\hat{u}_{tt} + \hat{u}_t + \hat{u} = -\alpha^2 \hat{u}.$$

$$\hat{u}_{tt} + \hat{u}_t + (1+\alpha^2)\hat{u} = 0.$$

Has solutions

$$\hat{u} = A(\alpha) e^{r_1 t} + B(\alpha) e^{r_2 t} \qquad (1).$$

where r_1, r_2 solve

$$r^2 + r + (1+\alpha^2) = 0$$

$$\Rightarrow \quad r_{1,2} = \frac{-1 \pm \sqrt{1 - 4(1+\alpha^2)}}{2}$$

Note that $Re\{r_{1,2}\} = -\frac{1}{2}$ so both terms in (1) decay in time. This problem is certainly well posed (not unbounded growth, even has decay in time).

#4.
$$\frac{1}{|\underline{x}-\underline{s}|} = \frac{1}{\sqrt{|\underline{x}|^2 - 2\underline{x}\cdot\underline{s} + |\underline{s}|^2}}$$

$$= \frac{1}{|\underline{x}|}\frac{1}{\sqrt{1 - \frac{2\underline{x}\cdot\underline{s}}{|\underline{x}|^2} + \frac{|\underline{s}|^2}{|\underline{x}|^2}}}$$

$$\approx \frac{1}{|\underline{x}|}\left(\frac{1}{1-(\underline{x}\cdot\underline{s}/|\underline{x}|^2)}\right)$$

$$\approx \frac{1}{|\underline{x}|}\left(1 + \frac{\underline{x}\cdot\underline{s}}{|\underline{x}|^2}\right) = \frac{1}{|\underline{x}|} + \frac{\underline{x}\cdot\underline{s}}{|\underline{x}|^3}.$$

Thus
$$u(\underline{x}) \approx -\frac{1}{4\pi}\frac{1}{|\underline{x}|}\int_\Omega f(\underline{s})\,d\underline{s}$$
$$-\frac{1}{4\pi}\frac{1}{|\underline{x}|^3}\,\underline{x}\cdot\int_\Omega \underline{s}\,f(\underline{s})\,d\underline{s}.$$

#5 Idea: extend f to be even in x_2, u solving the whole plane Poisson problem will be even in x_2, so $\left.\frac{\partial u}{\partial x_2}\right|_{x_2=0} = 0$. This corresponds to adding a positive image of (s_1, s_2) $s_2 > 0$ at $(s_1, -s_2)$:

$$u(\underline{x}) = \frac{1}{4\pi} \int_{-\infty}^{\infty} \int_{0}^{\infty} \Big\{ \log\left[(x_1 - s_1)^2 + (x_2 - s_2)^2 \right] +$$

$$\log\left[(x_1 - s_1)^2 + (x_2 + s_2)^2 \right] \Big\} f(\underline{s}) \, ds_2 ds_1$$

$$G(\underline{s}, \underline{x}).$$

Check: $\dfrac{\partial}{\partial x_2} G = \dfrac{2(x_2 - s_2)}{(x_1 - s_1)^2 + (x_2 - s_2)^2} + \dfrac{2(x_2 + s_2)}{(x_1 - s_1)^2 + (x_2 - s_2)^2}$

$$\left.\frac{\partial G}{\partial x_2}\right|_{x_2 = 0} = 0 \quad \checkmark.$$

#6. $F[u + \varepsilon\eta] = \int_0^1 \left\{ (u'' + \varepsilon \eta'')^2 - (u + \varepsilon\eta) g \right\} dx$

$$\left.\frac{dF}{d\varepsilon}\right|_{\varepsilon=0} = \int_0^1 (2u''\eta'' - g\eta)\, dx = 0 \qquad \forall \eta.$$

integrate by parts

$$= \int_0^1 (-2u'''\eta' - g\eta)\, dx + 2u''\eta' \Big|_0^1$$

again $= \int_0^1 (2u'''' - g)\eta \, dx + 2u''\eta' \Big|_0^1$

$$- 2u'''\eta \Big|_0^1$$

η (and boundary values η & η') are arbitrary,

so $\quad u'''' = g, \quad u''(0) = u''(1) = u'''(0) = u'''(1) = 0.$

Math 401 Midterm, Spring 2007

Instructions:

- No calculators, notes or books
- Each question is worth 10 marks
- Do 4 of the 6 questions (total 40 marks)
- If you attempt more than 4 questions the best 4 marks will be used to make your grade.

1. Find the maximum (and the maximizing function $u(x)$) of the functional

$$F[u] = \int_0^1 u(x)dx$$

subject to the constraint

$$\int_0^1 \sqrt{x}u^2(x)dx = 1.$$

2. Consider the problem below to determine $u(x)$ for given $f(x)$ and α:

$$-u'' = f, \qquad u(0) = 0, \qquad u'(1) + \alpha u(1) = 0.$$

(a) [2 marks] For what value of α does the problem above have unique solutions?

(b) [8] For $\alpha = 1$ find the Green's function representation of the solutions to this problem.

3. Consider the problem for $u(x, t)$, $t \geq 0$:

$$u_{tt} + u_t + u = u_{xx}$$

with initial data $u(x, 0)$ and $u_t(x, 0)$ given.

(a) [2 marks] Classify the PDE above (as elliptic, parabolic or hyperbolic).

(b) [8] Determine if the problem is well-posed. Justify your answer.

4. Given $f(\underline{x})$ in C_0^∞ (with Ω being the support of f) with $\underline{x} \in \mathcal{R}^3$, we know from the class notes that

$$u(\underline{x}) = \frac{1}{4\pi} \int_\Omega \frac{f(\underline{s})}{|\underline{x} - \underline{s}|} d\underline{s}$$

solves $\Delta u = f$ with $u \to 0$ as $|\underline{x}| \to \infty$. Write the first two terms in a multipole expansion for far field values of u in this 3D setting.

5. Given $f(\underline{x})$ with $\underline{x} \in \mathcal{R}^2$, we know from the class notes that

$$u(\underline{x}) = \frac{1}{2\pi} \int \log(|\underline{x} - \underline{s}|) f(\underline{s}) d\underline{s}$$

solves $\Delta u = f$. Use the method of images to find the Green's function representation for the solution $u(x)$ with $\underline{x} = (x_1, x_2)$ for $x_2 > 0$ of the problem:

$$\Delta u = f, \qquad \frac{\partial u}{\partial x_2}(x_1, 0) = 0 \text{ (for all } x_1)$$

6. Consider the functional

$$F[u] = \int_0^1 \left\{ (u'')^2 - ug \right\} dx$$

where $g(x)$ is a given function. Determine the differential equation *and* boundary conditions that u must satisfy to be a minimizer. *Do not* try to solve the equation. **Note:** F involves values of u'' so the standard Euler-Lagrange equations cannot be applied.

Math 401, Spring 2007
Final Review

Format of the final exam:
- do 8 of 10 questions.
- each question out of 10 - total 80.
- if more than 8 questions are attempted, the best 8 marks will be taken.
- no notes, calculators, books.

Material:

<u>Ch 2</u> Green's funtions for ODE BVP's:

+ constructing Green's functions.

also
- mollifiers, generalized functions.
- adjoint operators
- unique solutions
- eigen analysis.

<u>Ch 3</u> Classification of PDE's.

+ Determining the type of a general, 2D, 2nd order PDE.

also
- implications for appropriate extra conditions (initial, boundary) that lead to a well-posed problem.

Ch4 Green's functions for PDE's:

+ Basic Green's fn's for heat, wave and Poisson problems.

tricks
- Duhamel's principle.
- relation of interior Green's fn to boundary G's fn's for poisson problems.
- Fourier transform.
- D'Alembert soln.
- method of images.

also
- multipole expansions.
- single & double layer potentials.

Ch5 Functional minimization

+ Minimization of $\int_0^1 F(u,x)\,dx$ or $\int_0^1 F(u,u',x)\,dx$, possibly with integral constraints.

+ Euler-Lagrange equations, natural BC's, Beltrami identity.

+ Energy arguments - H_0^1, Poincaré Lemma.

+ Eigenfunctions as constrained minimizers.

also
- method of continuation
- capturing methods
- inequality constraints.

Ch 6 level set methods.

+ Arrival time $T(x,y)$, $V\,|\nabla T| = 1$.

  optimal paths $\frac{d(x,y)}{d\sigma} = -\nabla T$.

  appropriate numerical methods to handle derivative discontinuities.

+ Transient level set methods

$$\phi_t + V(x,y)\,|\nabla\phi| = 0.$$

Ch 7 Finite Element Method.

+ Rayleigh Ritz, Galerkin approaches.
+ Explicit construction of $A\,\underline{U} = \underline{F}$ for 1-D problems.
+ convergence analysis.

also • weak formulation.

Ch 8 Perturbation Methods.

+ regular perturbation of ODE BVP and eigenvalues.
+ regular perturbation of elliptic problems in 2D: PDE and boundaries.
+ multiple time scales & homogenization.

also • time harmonic solutions to wave equations (& perturbation).

• free boundary value problems.

In the course, we studied the particular material above, with asides to address:

+ How to model physical systems with DE's
+ General techniques for understanding the structure of solutions to PDEs.
+ Implications for efficient numerical approximation.

We had a "taste" of many topics, all of which have much deeper, more general results. Some topics we could have included are

+ hyperbolic conservation laws and their numerical approximation. This would have related to level set methods and could have been an example of singular perturbations (viscous shock profiles).

+ Theory of Sturm-Liouville problems (ODE BVP's). Could have used Green's fn representation of inverses to show the completeness of eigenfunctions.

+ Financial applications

+ Non-dimensionalization of physical problems to identify small parameters.
+ Convergence proofs for perturbation series.

Send feedback to welton@math.ubc.ca on

+ course content
+ reserve books
+ corrections to posted notes and solutions.

Office hours during exam period

April M16, M23, W25 11-1.

Some practice exam questions from the post-midterm material are posted. Solutions will follow.

Math 401, Spring 2007
Practice Exam Solutions

#1. $-(u_{xx} + 2u_{yy}) = \lambda\, a(x,y)\, u$
with $a(x,y) > 0$ and $u|_{\partial\Omega} = 0$.

a) Multiply the equation by u and integrate by parts:

$$\int_\Omega \{(u_x)^2 + 2(u_y)^2\}\, dA = \lambda \int_\Omega a(x,y)\, u^2\, dA.$$

this shows λ must be positive.

b) Consider

$$F[u] = \frac{1}{2}\int_\Omega \{(u_x)^2 + 2(u_y)^2\}\, dA$$

with the constraint $G[u] = \frac{1}{2}\int_\Omega a(x,y)\, u^2\, dA = 1.$

Minimizing F leads to the E-L equation

$$-u_{xx} - 2u_{yy} \underset{\sim}{-} \lambda(u\, a(x,y)) = 0 \qquad \checkmark.$$

usually $+\lambda$ but use the negative constant to match up with our problem.

Now the minimizer satisfies this equation for some λ,

$$F[u] = \frac{1}{2}\int_\Omega \{(u_x)^2 + 2(u_y)^2\}\, dA \quad \text{by parts}$$

$$= -\frac{1}{2}\int_\Omega u(u_{xx} + 2u_{yy})\, dA = \frac{1}{2}\lambda \int_\Omega u(au)\, dA$$

$= \lambda.$

Thus, the λ from minimizer must be the smallest eigenvalue.

#2 $F[u] = \int_0^{x_0} [(u')^2 + u^2 + 1]\,dx.$

EL $\quad 2u - (2u')' = 0$

$\Rightarrow \quad u'' - u = 0. \qquad\qquad u(0) = 1 \qquad\qquad$ (i)

At the free boundary,

$u(x_0) = 0 \qquad\qquad$ (ii)

$u'(2u') - [(u')^2 + \cancelto{0}{u^2} + 1] = 0$

$\Rightarrow (u')^2 = 1, \quad u' = -1 \quad$ (u decreasing) $\quad$ (iii)

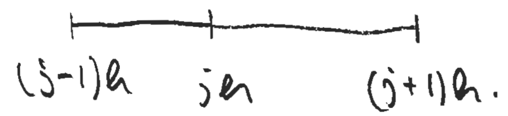
$u(x) = \dfrac{-\sinh(x - x_0)}{\sinh x_0}$ satisfies (i) & (ii).

$u'(x) = -\dfrac{1}{\sinh x_0}\cosh(x - x_0)\,, \quad u'(x_0) = -\dfrac{1}{\sinh x_0}$

$u'(x_0) = -1 \quad \Rightarrow \quad x_0 = \sinh^{-1} 1$

#3.

For $j = 1, \ldots N-1$ let $P_j(x)$ be the piecewise degree 3 polynomial with value 1 at $x = jh$, value 0 at other nodes and derivative zero at other nodes:



in $[(j-1)h, jh]$, $P_j(jh) = 1$, $P_j'(jh) = 0$

$P_j((j-1)h) = 0$, $P_j'((j-1)h) = 0$.

To work out $P(\cdot)$ explicitly in this interval, find the cubic $Q(x)$ with the following properties:

$$Q(0) = 1, \qquad Q'(0) = 0$$

$$Q(1) = 0, \qquad Q'(1) = 0.$$

$Q(x) = (1+bx)(x-1)^2$. (satisfies conditions at 1). & $Q(0) = 1$.

$Q'(x) = b(x-1)^2 + (1+bx)(x-1)$.

$Q'(0) = b - 1 = 0 \Rightarrow b = 1$.

ie. $Q(x) = (1+x)(x-1)^2$.

Now $P(x)$ in the interval at the top of the page is given by

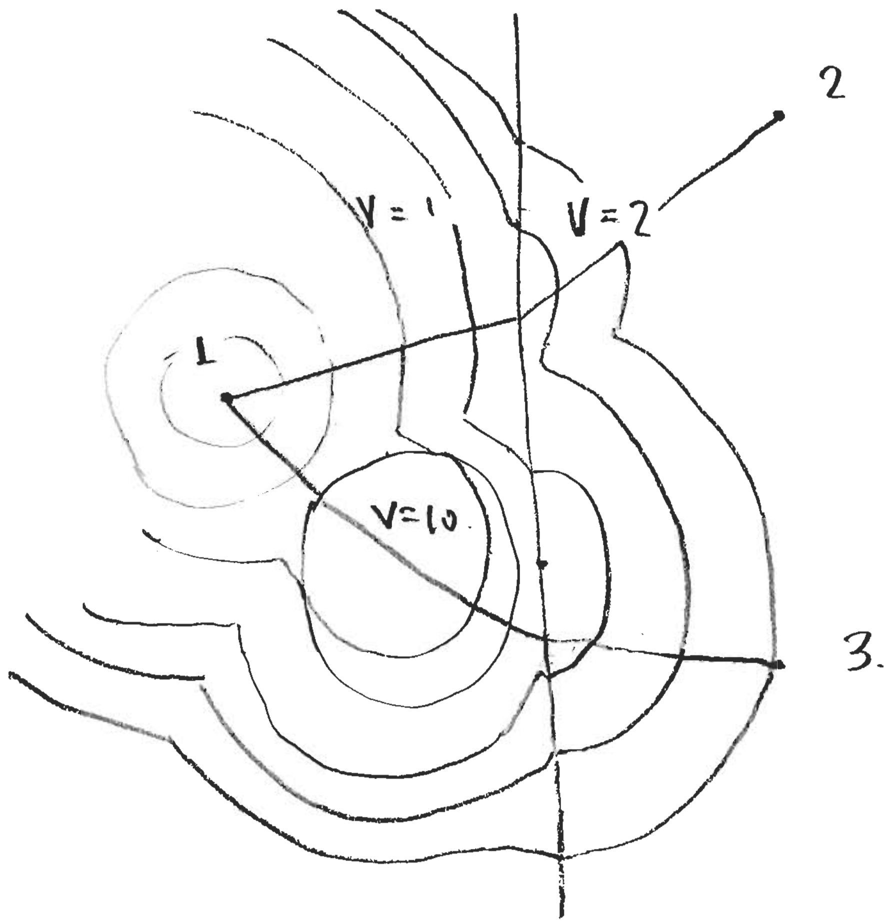
$$P_j(x) = Q\left(\frac{jh-x}{h}\right).$$

Similarly in $[jh, (j+1)h]$, $P_j(x) = Q\left(\frac{x-jh}{h}\right)$

out side of $[(j-1)h, (j+1)h]$, $P_j(x) \equiv 0$.

For $j = 0, \cdots N$ let $R_j(x)$ be the piecewise degree 3 polynomial with derivative value 1 at $x = jh$, value zero at all nodes and derivative zero at other nodes. It can be constructed using a template function $\tilde{Q}(x)$ as above. The subspace has dimension $2N = (N-1) + (N+1)$.

#4.



#5. $-u'' + \varepsilon u^2 = 1 \qquad u(0) = 0,\ u(1) = 0.$

$-u_0'' = 1 \Rightarrow u_0 = \frac{1}{2}x(1-x) = \frac{1}{2}x - \frac{1}{2}x^2$

$-u_1'' = -u_0^2 = -\frac{1}{4}x^2 + \frac{1}{2}x^3 - \frac{1}{4}x^4.$

$u_1'' = \frac{1}{4}x^2 - \frac{1}{2}x^3 + \frac{1}{4}x^4$

$u_1' = \frac{x^3}{12} - \frac{x^4}{8} + \frac{x^5}{20} + a.$

$u_1 = \frac{x^4}{48} - \frac{x^5}{40} + \frac{x^6}{120} + ax + b.$

$u_1(0) = 0 \Rightarrow b = 0 \qquad u_1(1) = 0 \Rightarrow a = \frac{1}{40} - \frac{1}{48} - \frac{1}{120}.$

Math 401, Spring 2007.
Practice Exam Questions

Note: The final exam covers the whole term, but these questions are samples from the material after the midterm only.

#1. Let Ω be a bounded domain in 2D. Consider the following eigenvalue problem on Ω:

$$-(u_{xx} + 2u_{yy}) = \lambda\, a(x,y)\, u$$

where $a(x,y) > 0$ is given and $u|_{\partial\Omega} = 0$.

[4] a) Show all λ are positive

[6] b) Let λ_1 be the smallest eigenvalue and $u_1(x,y)$ the corresponding eigenfunction. Find a functional with a constraint such that λ_1 is its minimum value and u_1 its minimizer.

#2. Consider the free boundary value problem

$$F[u] = \int_0^{x_0} [(u')^2 + u^2 + 1]\, dx.$$

where x_0 is the first positive zero of $u(x)$ and $u(0) = 1$. Find the minimizing function $u(x)$.

Note: Recall that the free boundary condition at x_0 is $u' f_p - f = 0$.

#3. Consider a subdivision of $[0,1]$ into N subintervals of equal length $h = \frac{1}{N}$. Find a basis for the subspace of functions that have the following properties:

(i) C^1 (continuous & continuous first derivatives).

(ii) zero value at $x=0,1$.

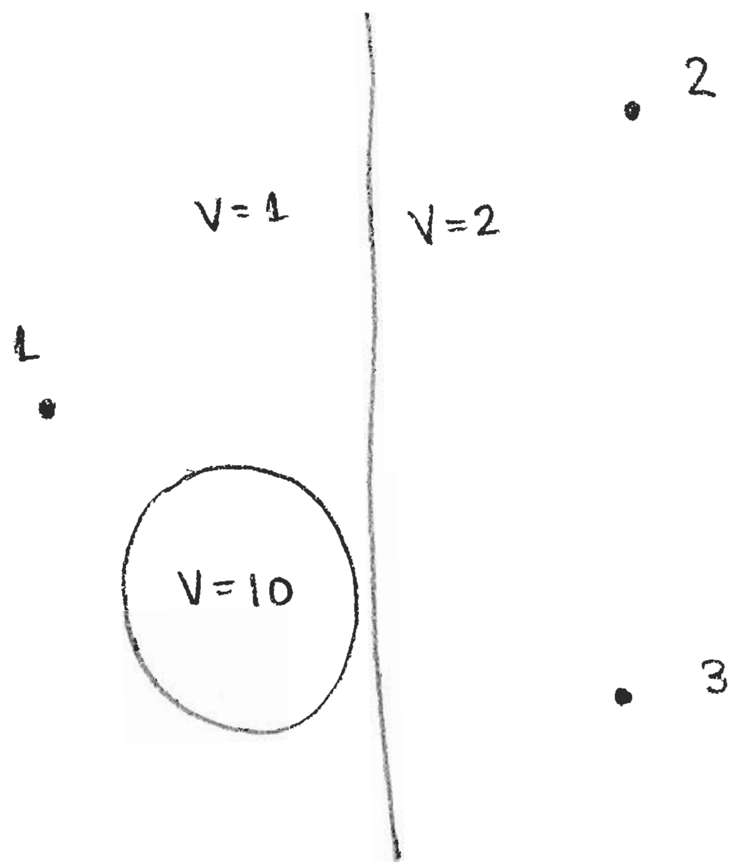
(iii) piecewise cubic on each subinterval.

Be sure to clearly identify the dimension of the subspace.

#4. Consider the following diagram showing a start location (1), two destinations (2 & 3), and travel speeds V by region. Add to the diagram sketches of the following:

(i) shortest travel paths from 1 to 2 and from 1 to 3.

(ii) some contours of $T(x,y)$, the travel time from 1 to (x,y).



#5. Consider the boundary value problem

$$-u'' + \varepsilon u^2 = 1 \qquad u(0) = 0, \; u(1) = 0.$$

Find the $O(1)$ and $O(\varepsilon)$ terms in a perturbation expansion for the solution.

Be sure that this examination has 4 pages including this cover

The University of British Columbia
Sessional Examinations - April 2007

Mathematics 401
Green's Functions and Variational Methods

Closed book examination Time: 2.5 hours

Special Instructions:

Do any 8 of 10 questions. If more than 8 questions are attempted, the best 8 marks will be taken. Each question is out of 10.
No notes, calculators, or books.

Rules governing examinations

[10] **1.** Consider the ODE BVP for $u(x)$, $x \in [0, 1]$ given below:

$$u'' + xu' - u = f(x), \qquad u(0) = 0, u(1) = 0.$$

(a) [5 marks] Find the adjoint of this problem.

(b) [5] Let $G(s, x)$ be the Green's function for the original problem above, i.e.

$$u(x) = \int_0^1 G(s, x) f(s) ds$$

Write the *conditions* that $G(s, x)$ must satisfy. It is not necessary to find G explicitly.

[10] **2.** Determine which of the following problems for $u(x, t)$, $x \in \mathbf{R}$, $t \geq 0$ are well posed. Justify.

(a) [5] $u_{tt} = u_{xxxx}$ with $u(x, 0)$ and $u_t(x, 0)$ given.

(b) [5] $u_t = -u_{xx} - u_{xxxx}$ with $u(x, 0)$ given.

[10] **3.** Consider $u(x, y)$ that solves

$$\Delta u = f(x, y)$$

with f given with compact support Ω contained in the unit disk centred at the origin. Recall that the solution can be written

$$u(x, y) = \frac{1}{2\pi} \int_\Omega \ln |(x, y) - (s, t)| ds dt$$

(a) [5] Consider $\underline{x} = (x, y)$ far from the origin, i.e. $\epsilon = 1/|\underline{x}|$ is small. Reproduce the multipole expansion derived in class for the solution u, showing the terms of O(1) and O(ϵ).

(b) [5] Consider the specific case of Ω the unit disk and $f \equiv 1$ in Ω. Evaluate the terms you found in part (a) above. Simplify.

[10] **4.** Consider $u(x, y)$ in the domain $\Omega = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}$ (a square).

(a) [8] Consider the functional

$$F[u] = \int_\Omega u^2 dA$$

for u with zero values on the boundary of Ω, subject to the constraint

$$\int_\Omega (u_x^2 + u_y^2) dA = 1.$$

Find the maximum of F and the maximizing function.

(b) [2] Discuss what occurs when a minimum of F above with its constraint is sought.

[10] **5.** Consider the ODE eigenvalue problem below for $u(x)$, $x \in [0, 1]$:

$$-u'' = \lambda a(x) u, \qquad u'(0) = 0, \qquad u'(1) = 0$$

with $a(x) > 0$ given.

Continued on page 3

(a) [3] Note that $\lambda = 0$ is an eigenvalue with eigenfunction $u \equiv 1$. Show that all other eigenvalues are positive.

(b) [7] Show that eigenfunctions u_1 and u_2 corresponding to *different* eigenvalues satisfy

$$\int_0^1 a(x)u_1(x)u_2(x)dx = 0$$

[10] **6.** Consider travel in the plane in which the diagram found on the last page of this exam is embedded. Travel speed $V(x, y)$ is very fast in region A (take $V \to \infty$), slow in region B (take $V \equiv 0$ here) and is scaled to a unit speed outside of A and B. Detach the last page of the exam and sketch the following on it:

(a) [3] The shortest-time travel path (or paths) from point 1 to point 2.

(b) [7] Some contours of $T(x, y)$, the shortest travel time from point 1 to (x,y).

[10] **7.** Consider the ODE BVP for $u(x)$, $x \in [0, 1]$:

$$((1 + x^2)u')' = f(x), \qquad u(0) = 0, \qquad u(1) = 0.$$

(a) [3] Write a weak formulation of this problem.

(b) [7] Let $\{B_i(x)\}$, $i = 1, \ldots N - 1$ be the piecewise linear finite element basis functions we described in class, on an equally spaced grid with spacing $h = 1/N$ and points $x_i = ih$. Consider

$$U(x) = \sum_{i=1}^{N-1} U_i B_i(x)$$

Write the coefficients of the linear system $\mathbf{A}\underline{U} = \underline{F}$ for the FE approximation of the problem. You may leave the entries of $\mathbf{A}$ and $\underline{F}$ as integrals.

[10] **8.** Consider the ODE eigenvalue problem

$$-u'' + \epsilon x u' = \lambda u, \qquad u(0) = 0, \qquad u(1) = 0.$$

At $\epsilon = 0$, $u = \sin \pi x$ is an eigenfunction corresponding to $\lambda = \pi^2$. Find the O(ϵ) corrections to this u and this λ.

[10] **9.** Consider the following problem for $u(x, y)$ in the infinite plate $x \in \mathbf{R}$, $0 \le y \le 1$:

$$\epsilon u_{xx} + u_{yy} = 0, \qquad u(x, 0) = 0, \qquad u(x, 1) = T(x)$$

where $T \in C^\infty$ is given.

(a) [8] Find the O(1) and O(ϵ) terms in a perturbation series for u.

(b) [2] For what class of functions for T would your solution to (a) fail?

[10] **10.** Consider $\underline{v}(s, t)$ where for each s and t, $\underline{v} \in \mathbf{R}^3$ (a vector with three components). Initial values $\underline{v}(s, 0)$ are given and $\underline{v}$ satisfies

$$\underline{v}_t = \underline{v} \times \underline{v}_{ss}$$

where $\times$ denotes the cross product.

(a) [5] Show that this *nonlinear* problem is well-posed. *Hint:* take the dot product of the equation with $\underline{v}$.

(b) [5] *Either* find the analytic solution of the problem when $\underline{v}(s, 0) = (\cos s, \sin s, 0)$ *or* describe how you would approximate the solution numerically.

The End

1.

A	B.

2.