Lecture 6

Asymptotic Analysis

6.1 Overview

Asymptotic methods take advantage of a small parameter in a problem that perturbs the solution from one that is known. An approximate solution is obtained in which the influence of the small term is determined to highest order. Often, real insight into the behaviour of complicated problems can be gained. Sometimes the approximate problem can be solved analytically, or at least the problem is much better conditioned for numerical approximation. Let us begin with a simple example that illustrate the ideas and their limitations.

Example 1. *Consider the approximation of e^x for x near zero. It is a little artificial here, but let x be ϵ and consider e^ϵ. We have $e^0 = 1$, which we will consider to be the base solution ($\epsilon = 0$) and see how small values of ϵ change this solution. We can use a linear approximation*

$$e^\epsilon = 1 + \epsilon. \tag{6.1}$$

We know what the error expression is for this approximation from Lecture #2 but for the asymptotic approximations we will develop in the rest of the lecture, this will not be known. What we can say about (6.1) without knowing anything about the size of the second derivatives of e^x is that $e^\epsilon = 1 + \epsilon$ is asymptotically better that any other linear approximation as $\epsilon \to 0$:

$$\lim_{\epsilon \to 0} \max_{x \in [-\epsilon, \epsilon]} \frac{|e^x - (1 + cx)|}{\epsilon} = (c - 1). \tag{6.2}$$

In practice, parameters in an application problem determine the "ϵ" after scaling and there is no easy way to determine how accurate the approximation is for this particular value. There is even no easy way to determine whether adding a term to the asymptotic expansion increases the accuracy. Consider e^{-2} for example. Here $1 - 2 = -1$ (from the linear approximation) is a worse approximation to e^{-2} than 1 (the base solution). Of course $\epsilon = -2$ does not seem “small” but it is often not so easy to recognize when an asymptotic approximation is accurate or not.

So it is often difficult to show the asymptotic nature of an approximation, like the result in (6.2) and it is also hard to show the accuracy of the approximation for any specific valie of “ϵ”. For this reason, this approximation technique is often called *formal asymptotics*. However, although these are valid criticisms, when formal asymptotics are used with care and backed with some numerical comparisons in model situations, they are a valuable tool to the applied mathematician to recognize the structures in problems and find approximate solutions accurate enough for many applications.

6.2 Regular Asymptotics

Consider the roots $y = x(\epsilon)$ of

$$f(y, \epsilon) = 0 \tag{6.3}$$

and suppose that $x_0 = x(0)$ that solves $f(x_0, 0) = 0$ is known. Then if we wanted to find the asymptotic expansion for $x(\epsilon)$ we could do implicit differentiation of $f(x(\epsilon), \epsilon)$ to get

$$\frac{\partial f}{\partial y}(x_0, 0)\frac{dx}{d\epsilon}(0) + \frac{\partial f}{\partial \epsilon}(x_0, 0) = 0$$

and then solve for

$$\frac{dx}{d\epsilon}(0) = -\left[f_y\right]^{-1} f_\epsilon \tag{6.4}$$

and then the asymptotic expansion is the linear approximation, exactly as you expect

$$x(\epsilon) \approx x_0 + \epsilon\frac{dx}{d\epsilon}(0) \tag{6.5}$$

This cartoon also applies when x is a continuous function and f involves the solution of a differential equation. When f_ϵ and f_y are well-behaved in (6.4) and so (6.5) is well-defined, we say that the asymptotic expansion is *regular*. We will see two types of expansions, ones with multiple scales in Section 6.3 and one with singular terms in Section 6.4, that are not regular.

Example 2. *Approximate the root of*

$$f(y, \epsilon) := y^4 + \epsilon y - 1 = 0 \tag{6.6}$$

near $x = 1$. We have $x_0 = 1$ and can use (6.4) and (6.4) directly.

$$\frac{\partial f}{\partial y} = 4y^3 + \epsilon$$

so $f_y(x_0, 0) = 4$.

$$\frac{\partial f}{\partial \epsilon} = y$$

so $f_\epsilon(x_0, 0) = 1$. *Using (6.4) we have*

$$\frac{dx}{d\epsilon}(0) = -\frac{1}{4}$$

and so the asymptotic expansion is $x(\epsilon) \approx 1 - \epsilon/4$. *We could do a second implicit differentiation to find a correction at order* ϵ^2. *When* $\epsilon = 0.01$ *the approximation to the root is* $x \approx 0.9975$, *the exact root (approximated using Newton's method) is* $x \approx 0.997497$.

However, this is not how formal asymptotics is actually done. We would have taken the expression

$$y = x_0 + x_1\epsilon$$

where $x_0 = 1$ as before and x_1 is to be determined, and plugged this into (6.6), expanded in powers of ϵ and equated terms of similar powers of ϵ:

$$\begin{aligned} O(1): &\quad 1 - 1 = 0 \ \text{ so we got } x_0 \text{ right} \\ O(\epsilon): &\quad 4x_1 + 1 = 0 \ \text{ so } x_1 = -1/4 \end{aligned}$$

so we end up with the same asymptotic expression as before and the steps are the same but just presented in a different way.

Example 3. *Consider the heat conduction problem for* $u(x, t; \epsilon)$*:*

$$(1 + \epsilon x)u_t = u_{xx} \tag{6.7}$$

with boundary conditions $u(0) = u(1) = 0$. *When* $\epsilon = 0$ *we have that the fastest decaying mode is*

$$e^{-\mu^2 t}\phi(x)$$

with $\mu = \pi$ *and* $\phi = \sin(\pi x)$. *We want to see how the value of* μ *and the eigenfunction* ϕ *depend on* ϵ. *Take*

$$\mu = \pi + \mu_1\epsilon$$

and

$$\phi = \sin(\pi x) + \epsilon\phi_1(x)$$

with μ_1 *and* $\phi_1(x)$ *to be determined. We plug the two expressions above into (6.7) and separate out terms by powers of* ϵ *as we did in the discussion after the first example in this section. At order 1 we confirm the base solution. At order* ϵ *we have*

$$y'' + \pi^2 y = -2\pi\mu_1 \sin(\pi x) - \pi^2 x \sin(\pi x) \quad \textit{with } y(0) = y(1) = 0. \tag{6.8}$$

where we have use y *for* ϕ_1. *Now (6.8) can be solved analytically with the method of undetermined coefficients. Because of the resonant right hand sides we take the form for the solution as*

$$y(x) = A\sin(\pi x) + B\cos(\pi x) + ax\sin(\pi x) + bx\cos(\pi x) + cx^2\sin(\pi x) + dx^2\cos(\pi x)$$

where the terms with A and B are the homogeneous solution and a, b, c, d are determined by the non-homogeneous terms. Solving for the coefficients we have

$$y(x) = A\sin(\pi x) + B\cos(\pi x) - \frac{1}{4}x\sin(\pi x) + \mu_1 x\cos(\pi x) + \frac{\pi}{4}x^2\cos(\pi x).$$

When we match to the boundary conditions we get $B = 0$ and $\mu_1 = -\pi_4$. Thus the first order corrections to the base values are

$$\begin{aligned} \mu &\approx \pi - \frac{\pi}{4}\epsilon \\ \phi(x) &\approx \sin(\pi x) + \epsilon\left\{-\frac{1}{4}x\sin(\pi x) - \frac{\pi}{4}x\cos(\pi x) + \frac{\pi}{4}x^2 + A\sin(\pi x)\right\} \end{aligned}$$

It might seem alarming that the coefficient A is not determined. However, on closer notice this term just scales the size of the base solution (an eigenfunction) and this size is arbitrary. We could keep A and then this term will affect the order ϵ^2 asymptotic term, but it is consistent to take $A = 0$. If we take $\epsilon = 0.01$ we get $\mu \approx 3.1337$ from the asymptotic result, which is correct to all digits shown.

Note 1. *It is possible to determine the exact μ and ϕ in the example above using Airy functions.*

6.3 Multiple Scales: Homogenization

Consider

$$c(x)u'' = f(x) \quad \text{with } u(0) = u(1) = 0 \tag{6.9}$$

where $c(x)$ varies over very short spatial scales, that is $c(x) = w(x/\epsilon)$ where w is 1-periodic in its argument. We expect c to also vary over these spatial scales, but also have a local average value that changes over $x \in [0, 1]$. It is only this local average, or "macroscopic" behaviour that is interesting in many applications. This is a problem with multiple scales. We assume an asymptotic expansion in functions $v(x, y)$ where $y = x\epsilon$ and v is 1-periodic in y:

$$u(x) = u_0(x, \frac{x}{\epsilon}) + \epsilon u_1(x, \frac{x}{\epsilon}) + \epsilon^2 u_2(x, \frac{x}{\epsilon}) + \ldots$$

It will turn out that u_0 does not depend on y and this will be the macroscopic behaviour we are interested in. To proceed we write the expansion of d^2u/dx^2 in the terms above

$$\begin{aligned} \frac{d^2u}{dx^2} \quad = \quad & u_{0,xx} + \frac{2}{\epsilon}u_{0,xy} + \frac{1}{\epsilon^2}u_{0,yy} + \\ & \epsilon u_{1,xx} + 2u_{0,xy} + \frac{1}{\epsilon}u_{1,yy} + \\ & \epsilon^2 u_{2,xx} + 2\epsilon u_{0,xy} + u_{0,yy} \end{aligned}$$

and collect terms in $w(y)u'' = f(x)$:

$$
\begin{aligned}
O(1/\epsilon^2): &\quad u_{0,yy} = 0 && (6.10)\\
O(1/\epsilon): &\quad 2u_{0,xy} + u_{1,yy} = 0 && (6.11)\\
O(1): &\quad w(y)(u_{0,xx} + 2u_{1,xy} + u_{2,yy}) = f(x) && (6.12)\\
O(\epsilon): &\quad (u_{1,xx} + 2u_{2,xy} + u_{3,yy}) = 0 && (6.13)
\end{aligned}
$$

From (6.10) we see that u_0 is independent of y, $u_0(x)$ as we foreshadowed. This is the base, macroscopic solution we are interested in, but it is determined by an equation at lower order in ϵ. With $u_0(x) = 0$ we see that $u_{0,xy} = 0$ and then from (6.11) we see that $u_1(x)$, also a function of x only. Now consider (6.12) with $u_{1xy} = 0$:

$$u_0'' + u_{2,yy} = \frac{1}{w} f(x) \tag{6.14}$$

where for $u_0(x)$ we can use primes for x derivatives again. To proceed, we integrate (6.14) over $y \in [0, 1]$ (remember that u_2 is periodic in y so the middle term above will be zero) to obtain

$$u_0'' = \left[\int_0^1 \frac{dy}{w(y)} dy\right] f(x).$$

Note that this has the same form as the original equation

$$c_{\text{ha}} u_0'' = f(x)$$

but with c_{ha} the harmonic average over the fine scale variations

$$c_{\text{ha}} = \left[\int_0^1 \frac{dy}{w(y)} dy\right]^{-1}.$$

Note 2. *When asymptotics gets "tricky" it is usually a good idea to identify the form that the next term will take. It is a good check that all dominant terms are accounted for and also to know the order at which the next correction happens.*

With that note in mind, let us return to (**??**) which has only been satisfied in an average sense. Write it as

$$u_{2,yy} = \left(\frac{1}{w(y)} - \frac{1}{c_{\text{ha}}}\right) f(x)$$

and consider this equation at fixed x. Solutions will only exist when the right hand side has average value of zero over y, which is true by construction. In this case, solutions are only determined up to a constant, that can depend on x. Thus we can write

$$u_2(x, y) = f(x)\phi(y) + u_2(x)$$

where ϕ solves

$$\frac{d^2\phi}{dy^2} = \left(\frac{1}{w(y)} - \frac{1}{c_{\text{ha}}}\right)$$

with average value zero. Considering now (6.13) we have

$$u_1'' + u_{3,yy} = -2\frac{d\phi}{dy}f'$$

where after integrating over y we see that $u_1 \equiv 0$. We can now see the pattern that the y dependence of u_n is determined from $O(\epsilon^{n-2})$ equations and the x dependence at $O(\epsilon^n)$.

6.4 Singular Asymptotics

6.4.1 A simple example

Consider the ODE BVP for $w(x)$:

$$\epsilon w'' + w' = 1 \text{ with } w(0) = w(1) = 0.$$

This is not a regular asymptotics problem as can be seen by trying to find the base solution as we did before with $\epsilon = 0$, which is $w = c + x$ which cannot match both boundary conditions. A typical signature of singular asymptotics is that the small parameter appears as the coefficient of the highest derivative term. In fact

$$w = c + x \tag{6.15}$$

will be the solution in an "outer region" with c chosen appropriately, but there will also be a boundary layer of width ϵ. If we consider the equation in terms of the variable $y = x/\epsilon$ then we see we can get a higher order balance from the first two terms that will lead to something interesting. In this case, the "right" balance is easy to see but in more complicated problems, there may be some trial and error in seeing what are the right terms to combine. Now in this scaling the equation becomes

$$\frac{1}{\epsilon}\left(\frac{d^2w}{dy^2} + \frac{dw}{dy}\right) = 1$$

and we have the highest order "inner" solution

$$w = ae^{-y} + b = ae^{-x/\epsilon} + b. \tag{6.16}$$

Note that this decays rapidly in x. If it were to have any influence at $x = 1$ then C would have to be exponentially large, which we do not expect of the solution. Thus we expect that (6.16) describes the solution near $x = 0$. In that case, the outer solutions (6.15) must match the boundary condition at $x = 1$, so

$c = -1$. The inner solution (6.16) must match the boundary condition at $x = 0$ so $a = -b$ and it must also match the outer solution is the sense that

$$\lim_{y\to\infty} w_{\text{inner}}(y) = w_{\text{outer}}(x = 0+)$$

which gives $b = -1$. We can put this together and write

$$w(x) \approx (x - 1) + e^{-x/\epsilon}.$$

For this problem it is possible to find the exact solution and you can verify that the expression above matches it to exponentially small terms.

6.4.2 2D Allen Cahn Dynamics

Consider now the Allen Cahn equation for $u(x, y)$:

$$u_t = \Delta u - \frac{1}{\epsilon^2} f(u) \quad \text{with } f(u) = u^3 - u. \tag{6.17}$$

We consider the model in a doubly periodic domain so there are no boundary conditions to consider. This is a model from materials science, describing the interaction of two crystals of the same material with with different orientations and how they interact when annealed. We could have rescaled time to get $u_\tau = \epsilon^2 \Delta u - (u^3 - u)$ to realize this is a singular perturbation problem, but there is a reason why the scaling in (6.17) is preferable as you will see below. We look for the outer solution and at $O(1/\epsilon^2)$ we have that $u^3 - u = 0$ at every point. This gives three solutions $u = 0, \pm 1$. If we do consider the fast τ time scale, we can see that $u = 0$ is an unstable equilibrium and $u = \pm 1$ are two stable equilibria of an autonomous ODE. Thus, the outer solution is a mix of regions of $u \equiv +1$ and $u \equiv -1$ as shown in Figure 6.1. We could consider these as base solutions in the outer region and looks for asymptotic corrections. Consider

$$u = 1 + \epsilon u_1(x, y).$$

Now, u_1 solves

$$-f'(1)u_1 = -2u_1 = 0$$

so $u_1 \equiv 0$ in the outer solution. In fact it can be shown that there are no asymptotic corrections at any order to the outer solutions $u = \pm 1$ for Allen Cahn.

Now consider a boundary between outer regions of different signs. Suppose it is parametrized by a smooth curve $\mathbf{x}(s, t)$ where s is the arc length parametrization. We can assume that $u(\mathbf{x}) \equiv 0$. Consider coordinates (s, z) near the boundary where

$$(x, y) = \mathbf{x}(s, t) + \epsilon z \hat{n}$$

where $\hat{n}(s)$ is the unit normal to the curve. Notice that we allow the boundary to move in time and the normal speed will be captured in the asymptotic analysis

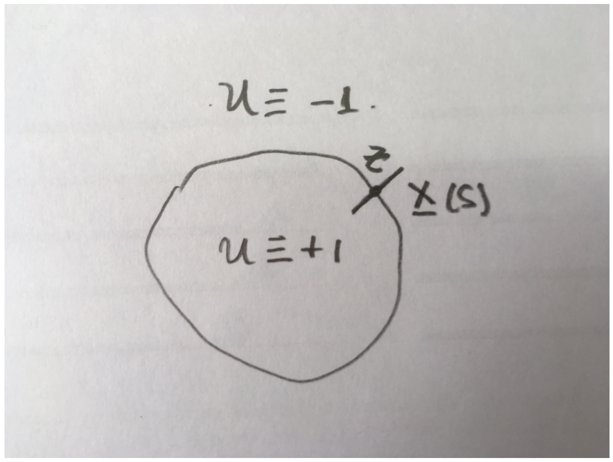

Figure 6.1: 2D Allen Cahn solution showing inner and outer regions

below. The scaling in z is chosen as it will give a balance between the Laplacian and the f/ϵ^2 terms in the equation. In these coordinates, the highest two order terms in Δu are

$$\Delta u = \frac{1}{\epsilon^2}\frac{\partial^2 u}{\partial z^2} + \frac{\kappa}{\epsilon}\frac{\partial u}{\partial z} \tag{6.18}$$

where κ is the curvature of the interface. We now look for an inner solution to (6.17) of the form $u(s, z)$. At highest order $O(1/\epsilon^2)$ we have

$$u_{zz} - (u^3 - u) = 0$$

and to match the outer solution we must have

$$\lim_{z\to\infty} u(z) = 1 \quad \text{and} \quad \lim_{z\to-\infty} u(z) = -1.$$

There are dynamical systems arguments that show such a heteroclinic solution exists, and in this case there is an exact solution

$$u(z) = g(z) := \tanh(z/\sqrt{2}).$$

There is translational invariance in the solution but recall we specified the interface location ($z = 0$) to satisfy $u = 0$. Now we have an inner solution

$$u(\mathbf{x}(s,t) + \epsilon z\hat{n}) = g(z) + \epsilon u_1(s, z).$$

Now consider the next order terms to get the interface motion. As a preliminary, consider the terms involving u_1:

$$u_{1,zz} + f'(g(z))u_1 := \mathcal{L}u_1 \tag{6.19}$$

and since g solves $g'' + f(g) = 0$, $u_1 = g'$ is in the nullspace of the the symmetric operator $\mathcal{L}$ defined above. Thus, $\mathcal{L}u_1$ will be perpendicular to $g'(z)$ at every s.

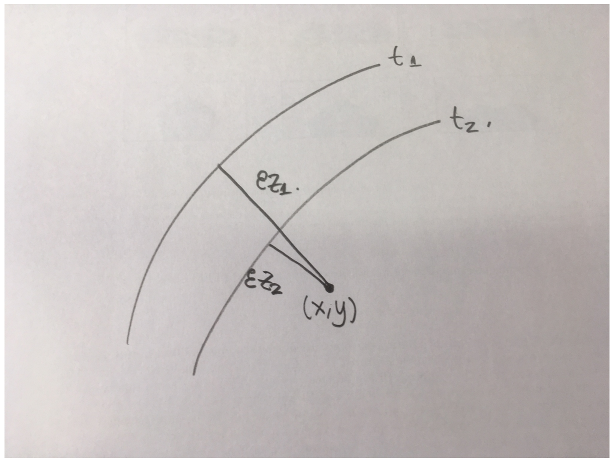

Figure 6.2: Figuring out what z_t is for fixed (x, y).

Now consider $u_t = g'(z)z_t$. We will find that $z_t = O(1/\epsilon)$ so this term will appear in this next order asymptotics. This term is a little tricky to get right. Here z_t is the partial derivative of z with t holding (x, y) fixed. To get this term correctly is difficult with algebra, but if you consider Figure 6.2 you can believe that $z_t = -V/\epsilon$ where V is the normal velocity of the curve, so

$$u_t = -g'(z)V/\epsilon. \tag{6.20}$$

Putting together (6.18), (6.19) and (6.20) in the $O(1/\epsilon)$ terms in (**??**) and obtain

$$-g'(z)V = \kappa g' + \mathcal{L}u_1.$$

Since $\mathcal{L}u_1 \perp g'$ we must have $V = -\kappa$, curvature motion. Note that this is the well-posed curvature motion with this sign the way the coordinates have been defined. All of the components of u_1 in $\{g'\}^\perp$ are determined at a lower order equation.

6.5 Lecture #6 Problems

Problem 1. *Consider the following boundary value problem for $u(x)$:*

$$u'' - u + \epsilon u^2 = 1 \quad \textit{with } u(0) = u(1) = 0$$

Find the $O(1)$ and the $O(\epsilon)$ terms in a regular expansion for $u(x)$.

Problem 2. *Consider the boundary eigenvalue problem for $u(x)$ below*

$$((1 + \epsilon x)u')' = \lambda(1 + \epsilon x)u \quad \textit{with } u(0) = u(1) = 0$$

When $\epsilon = 0$ the first eigenvalue is $-\pi^2$ with eigenfunction $\sin \pi x$. Find the next term in an asymptotic expansion for this eigenvalue.

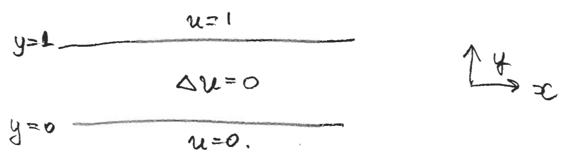

Figure 6.3: The base geometry for Problem 3.

Problem 3. *Consider the following problem for the steady state temperature $u(x, y)$ with $\Delta u = 0$ for $0 \leq y \leq 1$ and all x and boundary conditions $u(x, 0) = 0$ and $u(x, 1) = 1$, as shown in Figure 6.3. Consider now a perturbation of the upper boundary of the form $y(x) = 1 + \epsilon f(x)$. Assume an asymptotic expansion for the solution of the form*

$$u(x, y) = y + \epsilon u_1(x, y)$$

Derive equations and boundary conditions for $u_1(x, y)$. Solve for u_1 explicitly for the case $f(x) = \sin x$.

Problem 4. *Consider the 2D Cahn Hillard equations*

$$u_t = -\Delta \left(\epsilon \Delta u - \frac{1}{\epsilon} f(u) \right) \quad \textit{with } f(u) = u^3 - u.$$

Develop a singular asymptotic theory for this problem. Hint: *regions of $u = \pm 1$ form for this equation as they did for Allen Cahn and they move with $O(1)$ velocity in the scaling above.*

Problem 5. *Derive the complete expression for the Lapacian in the curved coordinates (s, z) presented in Section 6.4.2.*