Survey of Applied Mathematics Techniques

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July 8, 2018

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Lecture 7

The Finite Element Method

7.1 Reminder of the Weak Formulation

Recall from Lecture #1 that the weak form of $-u'' + u = f$ is

$$(u', \phi') + (u, \phi) = (f, \phi) \tag{7.1}$$

for all test functions $\phi \in L_2$. The inner products above are in L_2. We can recognize the terms on the right as the H^1 inner product of u and ϕ, which we will denote as $\langle u, \phi \rangle$ and so have an even more compact form for (7.1):

$$\langle u, \phi \rangle = (f, \phi) \quad \text{for all } \phi \in H^1$$

We know from Lecture #1 that with suitable boundary conditions, there is a unique solution $u \in H^2$ for $f \in L_2$ and that

$$\|u\|_{H^2} \leq C\|f\|_2. \tag{7.2}$$

7.2 Galerkin Approximation

Consider now an N dimensional subspace $\mathcal{S}$ of H^1. We can make a numerical method by finding $U(x) \in \mathcal{S}$ such that

$$\langle U, \phi \rangle = (f, \phi) \quad \text{for all } \phi \in \mathcal{S} \tag{7.3}$$

This is known as a *Galerkin Approximation*, characterized by the subspace the solution is in is the same as the space of test functions. With $\mathcal{S} \subset H^1$ we say that $\mathcal{S}$ is *conforming.*

Note 1. *We can characterize collocation methods in this context by saying that dirac delta functions are used as test functions.*

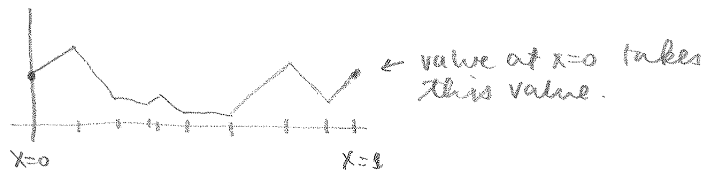


Figure 7.1: Piecewise linear functions.

We can take a basis $\{\Phi_i(x)\}$ of $\mathcal{S}$ and so write

$$U = \sum_{i=1}^{N} a_i \Phi_i \tag{7.4}$$

for a unique choice of coefficients $\mathbf{a}$ for every $U \in \mathcal{S}$. It would then be sufficient to satisfy (7.3) for all $\phi \in \{\Phi_i(x)\}$. This generates N linear equations for the N unknown coefficients in the approximate solution (7.4). We will show below that this system is always solvable, for any choice of $\mathcal{S}$.

7.3 Some Types of Elements

A simple choice for $\mathcal{S}$ is piecewise linear, continuous functions on subintervals between grid points $\{x_i\}$ as shown in Figure 7.1. Since $\mathcal{S} \subset H^1$ this a conforming approximation. We are still considering the periodic boundary case. Implementing boundary conditions for the FEM is discussed below in Section 7.6. A basis for this space consists of the functions $\Phi_i(x)$ that have value 1 at a grid point i and 0 at the others, and are piecewise linear in between. An example of such a basis function is shown in Figure 7.2. Now

$$U \in \mathcal{S} \;\; \text{can be written uniquely as} \quad U = \sum_{i=1}^{N} a_i \Phi_i.$$

In this case, a_i are the values $U(x_i)$ so it makes sense to label them as U_i. However it is not always the case that basis coefficients correspond to values of the approximating function.

Other choices of elements in 1D are described below, and shown in Figure 7.3.

Piecewise Quadratic: and continuous on N subintervals. This is a $2N$ dimensional subspace with an additional degree of freedom in each subinterval. The basis vectors come in two types as shown in Figure 7.3.

Cubic Hermite: and C_1 on N subintervals. This is also a $2N$ dimensional subspace with two types of basis vectors.

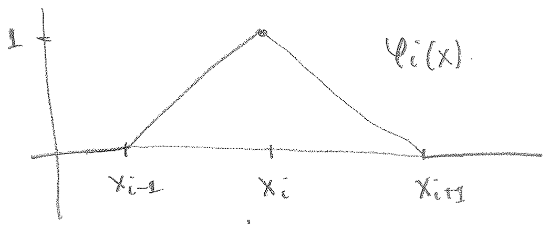


Figure 7.2: A basis function for the set of piecewise linear functions.

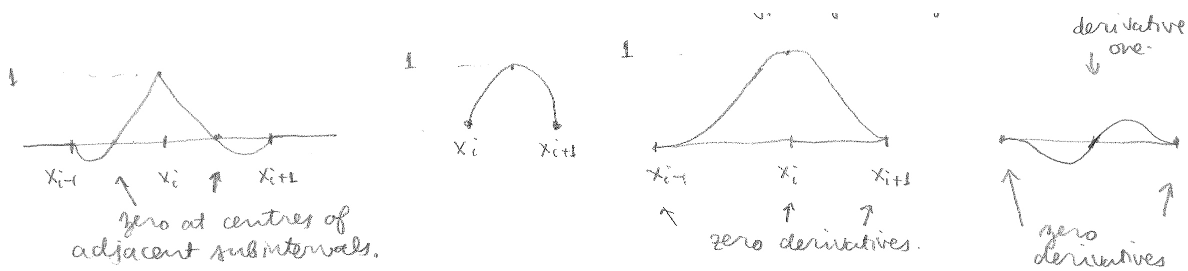


Figure 7.3: Basis functions for the set of piecewise quadratic functions (left) and the Cubic Hermite (right).

7.4 Mass and Stiffness Matrices

Returning to the general case (7.4) we insert this expression into (7.1) and take the Φ_j as test functions leading to a linear system for the coefficients $\mathbf{a}$:

$$\mathcal{A}\mathbf{a} = \mathbf{F}$$

where $\mathcal{A} = \mathcal{K} + \mathcal{M}$, known as the *stiffness* and *mass* matrices, respectively:

$$\begin{aligned} K_{ij} &= \int_0^1 \Phi_i' \Phi_j' \quad \text{Stiffness Matrix} \\ M_{ij} &= \int_0^1 \Phi_i \Phi_j \quad \text{Mass Matrix} \\ F_j &= \int_0^1 \Phi_j f(x) \end{aligned}$$

In Section 7.7 on matrix assembly and quadrature, we will see how the procedure to construct these matrices and right hand sides can be automated. The values of F_j will have to be approximated numerically.

If we consider the piecewise linear case on a uniform grid with spacing h, the following results for $\mathcal{K}$:

$$K_{ij} = \begin{cases} 2\int_0^h \frac{1}{h^2}dx = \frac{2}{h} & \text{if } j = i \\ -\int_0^1 \frac{1}{h^2}dx = -\frac{1}{h} & \text{if } j = i \pm 1 \\ 0 & \text{otherwise} \end{cases}$$

$\mathcal{M}$:

$$M_{ij} = \begin{cases} 2\int_0^h (1-\frac{x}{h})^2 dx = \frac{2h}{3} & \text{if } j = i \\ -\int_0^1 \frac{x}{h}(1-\frac{x}{h})dx = \frac{h}{6} & \text{if } j = i \pm 1 \\ 0 & \text{otherwise} \end{cases}$$

and F:

$$F_j = \frac{1}{h}\int_{x_j-h}^{x_j}(x - x_j - h)f(x)dx + \frac{1}{h}\int_{x_j}^{x_j+1}(x_j - x + h)f(x)dx.$$

In summary,

$$\frac{1}{h}(-U_{j-1} + 2U_j - U_{j-1}) + \frac{h}{6}(U_{j-1} + 4U_j + U_{j+1}) = F_j.$$

If the solution is sufficiently smooth, we can show that $F_j = hf(x_j) + O(h^3)$ and the middle term above is $u(x_j) + O(h^3)$. Dividing by h, we can see this as a recognizable approximation of the problem.

Consider the matrix $\mathcal{A}$ we have constructed using a basis for any subspace $\mathcal{S}$. There is a simple argument to show that it is always invertible. Suppose not, then there is a vector $\mathbf{a} \neq \mathbf{0}$ such that $\mathcal{A}\mathbf{a} = \mathbf{0}$. Since $\{\Phi_j\}$ is a basis, then the function U defined by (7.4) is not zero. Reversing the process that constructed $\mathcal{A}$ we have $\langle U, \Phi\rangle = 0$ for every $\Phi \in \mathcal{S}$ so since $U \in \mathcal{S}$ we have $\langle U, U\rangle = \|U\|_{H^1}^2 = 0$, the desired contradiction.

7.5 Convergence Proof, Linear Element Case

7.5.1 Aside: a simple Sobolev embedding

Theorem 1. *If $u \in H^1$ in the 1D interval [0,1], then $u \in C$ and $\|u\|_\infty < \sqrt{2}\|u\|_{H^1}$.*

We will consider $u \in C_\infty$ but we will only use arguments that rely on the H^1 norm, so the result will pass to all H^1 functions in the limit. We can find the average value u_{ave} and bound it by the L_2 norm of u using the Cauchy Schwartz (CS) inequality as follows:

$$|u_{\text{ave}}| = \left|\int_0^1 1 \cdot u\right| \leq 1 \cdot \|u\|_2.$$

By the Intermediate Value Theorem, the value u_{ave} is attained at some point x_* in the interval. Now

$$u(x) = u_{\text{ave}} + \int_{x_*}^x 1 \cdot u' \tag{7.5}$$

and again using the L_2 CS inequality, this time with the characteristic function on the interval $[x_*, x]$, we have

$$|u(x)| \le |u_{\text{ave}}| + \sqrt{|x_* - x|}\|u'\|_2 \le \|u\|_2 + \|u'\|_2 \le \sqrt{2}\|u\|_{H^1}$$

which is the desired bound. Returning to the idea in (7.5) we have for every x_1 and x_2

$$|u(x_1) - u(x_2)| = \left| \int_{x_1}^{x_2} u' \right| \le \sqrt{|x_1 - x_2|}\|u'\|_2$$

which shows the (uniform) continuity of u.

Note 2. *In fact, we have shown that a set of functions with a uniform bound in H^1 is equicontinuous. This leads to the first compactness theorem in a functional analysis course.*

7.5.2 Optimality of $U \in \mathcal{S}$

We have for the exact solution

$$\langle u, \phi \rangle = (f, \phi) \quad \text{for all } \phi \in H_1$$

and the approximate solution

$$\langle U, \phi \rangle = (f, \phi) \quad \text{for all } \phi \in \mathcal{S} \subset H_1.$$

Subtracting we have

$$\langle E, \phi \rangle = 0 \quad \text{for all } \phi \in \mathcal{S}. \tag{7.6}$$

where $E = U - u$ is the error in the approximate solution. Consider now the size of $V - u$ for any element $V \in \mathcal{S}$:

$$\begin{aligned} \|V - u\|_{H^1}^2 &= \|V - U + E\|_{H_1}^2 \\ &= \langle V - U + E, V - U + E \rangle \quad \text{now note that } E \perp (V - U) \text{ from (7.6)} \\ &= \|E\|_{H^1}^2 + \|V - U\|_{H^1}^2. \end{aligned}$$

Thus

$$\|E\|_{H^1} \le \|V - u\|_{H^1} \text{for all } V \in \mathcal{S} \tag{7.7}$$

and we have the remarkable result that the computed solution U is the closest element of $\mathcal{S}$ to the exact solution u in the norm induced by the bilinear form that defines the weak solution, the H^1 norm in this convenient case.

Note 3. *If you consider the system in the right way, you can see that U is the projection (in the inner product that defines the weak form of the problem) of the exact solution u onto the subspace $\mathcal{S}$ and then this result is "obvious".*

Considering (7.7) we see that to proceed with the convergence proof, we just need to show that $\|V_h - u\|_{H^1}$ goes to zero with "h" as we consider a sequence of subspaces $\mathcal{S}_h$ with dimension $N \to \infty$. We will take V_h to be a suitable interpolant of u onto $\mathcal{S}_h$ in the next section.

Note 4. *In the language of the Lax Equivalence Theorem, (7.7) is the stability result, and the interpolation results below are the consistency.*

7.5.3 Approximation in H^1

Here we restrict ourselves to piecewise linear elements as anticipated by the section title. Let $V(x)$ be the linear interpolation of the exact solution u on the discrete grid points (remember that to do this, we need u to have values that are defined pointwise, but this is guaranteed by the Theorem in Section 7.5.1). Now consider $w = u - V$. This is what we want to bound and use in (7.7). It satisfies $w(x_j) = 0$ at all grid points.

Consider $w(x)$ on each subinterval $[x_j, x_{j+1}]$ and let $h = \max_j |x_{j+1} - x_j|$. Since w vanishes at the end points we can use Rollé's theorem to find a point θ in the interval for which $w'(\theta) = 0$. We will use arguments similar to those in Section 7.5.1 and like there we can assume more smoothness to u as long as we can pass the limit in H^2. Now

$$\begin{aligned}
w'(x) &= \int_\theta^x 1 \cdot w'' = \int_\theta^x 1 \cdot u'' \quad \text{since } V \text{ is linear} \\
|w'(x)|^2 &\leq h \int_{x_j}^{x_{j+1}} |u''|^2 \\
\int_{x_j}^{x_{j+1}} |w'(x)|^2 &\leq h^2 \int_{x_j}^{x_{j+1}} |u''|^2
\end{aligned} \tag{7.8}$$

Summing over all subintervals we have

$$\int_0^1 |w'(x)|^2 \leq h^2 \int_0^1 |u''(x)|^2 \tag{7.9}$$

We can get an L_2 bound on w using similar arguments. On each subinterval

$$\begin{aligned}
w(x) &= \int_{x_j}^x w' \\
|w(x)|^2 &\leq h \int_{x_j}^{x_{j+1}} |w'|^2 \\
\int_{x_j}^{x_{j+1}} |w(x)|^2 &\leq h^4 \int_{x_j}^{x_{j+1}} |u''|^2
\end{aligned}$$

where in the last line above we have used (7.8). Summing over all subintervals we have

$$\|w\|_2^2 \leq h^4 \|u\|_{H^2}^2. \tag{7.10}$$

Combining (7.9), (7.10), and the a priori bound (7.2) we have

$$\|w\|_{H_1} \leq Ch\|f\|_2$$

which then in (7.7) shows the first order convergence of the discrete solution in the H^1 norm. Note that since the derivatives are approximated by constant value on subintervals, this is the best convergence rate we can expect in H^1. However we will observe second order convergence in L_2, as shown below.

7.5.4 Approximation in L_2

Let v solve the weak form of

$$-v'' + v = E.$$

Since $E \in H^1$, $v \in H^3$ with $\|v\|_{H^2} \leq C\|E\|_2$. Note we do not actually need to solve for v, its existence is just a theoretical tool to get the L_2 convergence result. We have

$$\langle v, \phi \rangle = (E, \phi)$$

for all $\phi \in H^1$. Use $\phi = E$ in the relationship above and recall that $\langle E, \phi \rangle = 0$ for all $\phi \in \mathcal{S}$. The expression above becomes (reversing the sides)

$$\|E\|_2^2 = (E, E) = \langle v, E \rangle = \langle (v - V), E \rangle$$

for any $V \in \mathcal{S}$. Letting V interpolate v as in the section above, we have

$$\|E\|_2^2 \leq \|E\|_{H^1} Ch\|v\|_{H^2} \leq Ch\|E\|_{H^1} C\|E\|_2$$

Dividing by $\|E\|_2$ and using $\|E\|_{H^1} \leq Ch\|f\|_2$ from the previous section we have

$$\|E\| \leq Ch^2\|f\|_2$$

(different C), so second order convergence in L_2.

7.6 Application of Boundary Conditions

7.6.1 Dirichlet

Consider first homogeneous Dirichlet conditions at $x = 0$ and $x = 1$, that is $u(0) = 0$ and $u(1) = 0$. The weak form of this problem is the same as the periodic case (7.1) except that the space of functions for the test functions and solutions is taken to be H_0^1, the completion of C_∞ functions with zero boundary conditions in the H^1 norm.

If a non-homogeneous condition $u(0) = \beta \neq 0$ is given the consider a function $b(x) \in H^1$ with $b(1) = \beta$ and $b(0) = 0$. Now consider $u = b + v$ with $v \in H_0^1$ that solves

$$\langle v, \phi \rangle = (f, \phi) - \langle b, \phi \rangle \quad \text{for all } \phi \in H_0^1.$$

Although v depends on the choice of b, the resulting u does not. You can implement this by adding an element Φ_0 that would correspond to the value at $x = 0$ and taking $b(x) = \beta\Phi_0$. This is equivalent to introducing Φ_0 into the element basis but setting its coefficient to β rather than using Φ_0 as a test function in the weak form.

7.6.2 Neumann

Consider now homogeneous Neumann conditions $u'(0) = 0$ and $u'(1) = 0$. If we examine the integration by parts that led to the weak form, we have for $\phi \in H^1$ (no restrictions)

$$\langle u, \phi\rangle - u'\phi\,\big|_0^1 = (f, \phi). \tag{7.11}$$

and the middle term vanishes with the given boundary conditions. Thus, Neumann conditions are "natural" in the weak formulation. The values at $u(0)$ and $u(1)$ are arbitrary and must have degrees of freedom associated with them. Because the mass and stiffness matrices will have one sided integrals for the boundary elements, the coefficients will be modified there to account for the boundary condition.

7.6.3 Robin

We consider the physically well-posed conditions

$$\begin{aligned} u(0) - \alpha u'(0) &= 0 \\ u(1) + \alpha u'(1) &= 0 \end{aligned}$$

with parameter $\alpha > 0$. We consider the integration by parts formula (7.11) again and incorporate the boundary conditions

$$\langle u, \phi\rangle + \alpha\left(u(0)\phi(0) + u(1)\phi(1)\right) = (f, \phi)$$

Note that the left hand side above defines a new inner product whose induced norm is equivalent to the H^1 norm. We can implement and analyze the FEM for this problem in the same way as before, adding some additional terms (near the boundary) to the mass and stiffness matrices when generating the matrix $\mathcal{A}$.

7.7 Assembly and Quadrature

The integrals

$$F_j = \int_0^1 \Phi_j(x) f(x) \tag{7.12}$$

$$K_{i,j} = \int_0^1 \Phi_i' \Phi_j' \tag{7.13}$$

$$M_{i,j} = \int_0^1 \Phi_i \Phi_j \tag{7.14}$$

$$\|E\|_{H^1}^2 = \int_0^1 \left\{|U' - u'|^2 + |U - u|^2\right\} \tag{7.15}$$

need to be evaluated to complete the implementation of the method. Note that (7.15) is needed only if we want to verify the convergence to an exact solution in the norm we used for the analysis in the previous sections. Also note that (7.13) and (7.14) involve integrals of piecewise polynomials, and so these can be done analytically. With the ideas below it will be shown that after transforming to the reference intervals there are only a small number of integrals to evaluate, and their contributions added together with the appropriate scaling of the subinterval lengths. Thus, we will concentrate the discussion on (7.12). In most textbooks on the FEM the discussion of quadrature is quite limited, but there are some interesting questions here.

Consider a part of the integral of (7.12) over one subinterval, transformed to the reference interval [-1,1]. On a given subinterval, a basis function $\Phi_j(x)$ is either identically zero or one of a finite number of shape functions Ψ_l when transformed to the reference interval. For example, for piecewise linear elements, it must be either $\Psi_1 = (1-y)/2$ or $\Psi_2 = (1+y)/2$. Thus we can write

$$\int_{x_m}^{x_{m+1}} f(x)\Phi_j(x)dx = \frac{h_m}{2}\int_{-1}^{1} f(x(y))\Psi_l(y)dy.$$

Consider now the integral on the left approximated with Gaussian quadrature with n points and so with error proportional to the $2n$'th derivative (in y) of the integrand, $g(y) = f(x(y))\Psi_l(y)$. Let us consider the case of linear elements for which Ψ_l are linear polynomials and the midpoint rule ($n=1$) approximation for which the error is proportional to the second derivative. We evaluate the second derivative of g below, remembering that $dx/dy = h_m/2$.

$$\begin{aligned}\frac{dg}{dy} &= \frac{h_m}{2}\frac{df}{dx} + f(x(y))\Psi_l' \\ \frac{d^2}{dy^2} &= \frac{h_m^2}{4}\frac{d^2 f}{dx^2} + h_m\frac{df}{dx}\Psi_l'\end{aligned}$$

where the Ψ'' term you expect is not there because Ψ_l is linear on the subinterval. Note that the quadrature error is now $O(h)$ rather that the $O(h^2)$ normally expected from midpoint rule. This is because the derivatives of Ψ are not scaled by h. The same argument can use used to show that with n point Gaussian quadrature and elements with q order polynomials, the error in

$$\int_{-1}^{1} f(x(y))\Psi_l(y)dy$$

is $O(h^{2n-q})$.

To examine the influence of the quadrature error on the solution, consider modifying the right hand side function f in each subinterval to account for the error. This is not something to implement, just a theoretical tool. From the discussion above, we know that if $G_{m,j}$ is the Gaussian quadrature approximation of

$$I_{m,j} = \int_{x_m}^{x_{m+1}} f(x)\Phi_j(x)dx$$

then

$$G_{m,j} = I_{m,j} + h\tau_{m,l}$$

where $\tau_{m,l} = O(h^{2n-q})$. On each subinterval there will be $q+1$ different quadrature errors, each associated with one Ψ_l, $l = 1, 2, \dots q+1$. We construct the function $r(y)$ on each reference interval as follows

$$r(y) = \sum_{l=1}^{q+1} a_l \Psi_l(y)$$

where we have omitted the interval index m with $\mathcal{B}\mathbf{a} = \tau$ where

$$B_{ij} = \int_{-1}^{1} \Psi_i \Psi_j.$$

Since $\{\Psi_l\}$ are linearly independent, $\mathcal{B}$ is invertible. We have that a_l are $O(h^{2n-q})$, the same size as τ. We assemble the r's on each subinterval into a global function r which will then have $\|r\|_2 = O(2n-q)$. Even I get confused at the details, but it works out that the finite element solution with Gaussian quadrature for the integrals with f is the same as the finite element solution with exact integrals with RHS $f - r$ instead of f. Since $\|r\|_2 = O(2n-q)$ and we used q order elements which give $O(h^q)$ convergence in H^1, we have a total error of

$$O(h^q) + O(h^{2n-q})$$

so taking Gaussian quadrature with $n = q$ points is enough to retain the H^1 convergence order.

Note 5. *When I try to continue the argument to the L_2 norm convergence, the quadrature error remains at $O(h^{2n-q})$ which suggests that taking $n = q + 1$ could be "safer". I do not know whether this is a real reduction or just a limitation of my analysis. Numerical tests are hard to design since if the grid size is uniform or smoothly varying, the dominant error term for piecewise linear elements cancels for adjacent subintervals.*

7.8 Elements and Quadrature in Higher Dimensions

There are only two differences to the implementation and analysis of the FEM in higher dimensions. First, there are errors associated with the approximation of the domain boundary by the elements. Secondly, quadrature rules on reference elements become even more interesting. A minor additional issue is that the norm embeddings change with dimension.

7.8.1 Elements

Some examples of conforming elements in 2D are given below. They are presented as a list of the following:

$\hat{K}$**:** the reference element

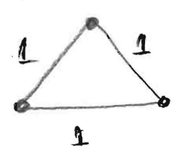
$\hat{P}$**:** the finite dimensional set of functions considered on $\hat{K}$

$\hat{\Sigma}$**:** the basis $\{\Psi_l\}$ for $\hat{P}$, often given as the points at which shape functions are prescribed.

F**:** allowable maps from $\hat{K}$ to an element K in the mesh.

Note that if $F = \hat{P} \times \hat{P}$ then the element is called *isoparametric*.

Piecewise linear, conforming elements on triangles



$\hat{K}$**:** unit equilateral triangle

$\hat{P}$**:** first order polynomials, $P_1 = \text{span}\{1, y_1, y_2\}$

$\hat{\Sigma}$**:** shape functions at the three vertices.

F**:** isoparamteric

If the mesh is composed of triangles with no hanging nodes, then the values along each edge will be linear between the vertex values shared between adjacent triangles, making the resulting approximation continuous and conforming in H^1. For elliptic problems, first order convergence in H^1 and second in L_2 results.

Piecewise quadratic, conforming elements on triangles

$\hat{K}$**:** unit equilateral triangle

$\hat{P}$**:** first order polynomials, $P_2 = \text{span}\{1, y_1, y_2, y_1^2, y_2^2, y_1 y_2\}$

$\hat{\Sigma}$**:** shape functions at the three vertices and edge mid-points.

F**:** $P_1 \times P_1$.

The values along each edge will be the same between adjacent triangles, since they are quadratic along the edge length and there are three shared nodes. For elliptic problems, second order convergence in H^1 and third in L_2 results.

Bilinear conforming elements on rectangles

$\hat{K}$**:** unit square

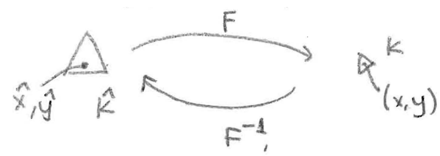
$\hat{P}$**:** $Q_1 = \text{span}\{1, y_1, y_2, y_1 y_2\}$

$\hat{\Sigma}$**:** shape functions at the four vertices.

F**:** $P_1 \times P_1$.

The values along each edge will be the same between adjacent triangles, since they are linear (because of the particular quadratic term that is added) along the edge length and there are two shared nodes. For elliptic problems, first order convergence in H^1 and second in L_2 results.

7.8.2 Quadrature



Quadrature is done on the reference element. Consider part of the calculation of an F_i value, doing the integral of one element K:

$$I := \int_K f(\mathbf{x})\Phi_i(\mathbf{x})d\mathbf{x} = \int_K f(\mathbf{x})\Psi_l\left(F^{-1}(\mathbf{x})\right)d\mathbf{x}$$

for one of the shape functions on the element $\Psi_l \in \hat{\Sigma}$. We now change the integral to one over the reference element

$$I = \int_{\hat{K}} f\left(F(\mathbf{y})\right)\Psi_l(\mathbf{y})\det \mathcal{J} d\mathbf{y}$$

where $\mathcal{J}$ is the Jacobian of F with respect to $\mathbf{y}$. Note that if F is in $P_1 \times P_1$ then $\mathcal{J}$ is a constant, the ratio of the area of K to the area of $\hat{K}$. In general this will be a polynomial in $\mathbf{y}$ and the order of this polynomial, plus the order of the polynomials in $\hat{P}$, must be taken into account when choosing the required order of quadrature.

Quadrature points for the square can be taken to be the tensor product of 1D quadrature points. This is convenient, but not optimal for higher order quadrature. For the equilateral triangle, evaluating at the midpoint with weight 1/2 (the area of the triangle) is second order accurate (exact for functions in P_1). Finding higher order quadrature rules is one of the Lecture problems below.

7.9 Lecture #7 Problems

Problem 1. *Code up the finite element method with basic linear elements for the problem (7.1) above. Use fourth order (two point Gauss-Legendre) quadrature for the right hand sides. Implement the following numerical convergence test: start with a regular grid of N points with spacing $h = 1/N$. Then move the points randomly with a uniform distribution on the interval $[-h/3, h/3]$. This should give points randomly between $h/3$ and $5h/3$ apart. Compute errors to a known solution on several resolutions with $h \to 0$, making the grids random in this way at every resolution. Report the test example you picked and the errors you observed.* Note: *this should give you the flavour of what convergence on unstructured grids is like. Compare the results when second order quadrature (midpoint rule) is used.*

Problem 2. *Consider the problem*

$$-(b(x)u')' + u = f(x)$$

where b is a given, smooth function with positive values. Write the weak formulation of this problem and consider its approximation with the basic conforming, linear finite element method. Identify the subinterval quadrature problem for the stiffness matrix in this case. What order quadrature should be used here to maintain the optimal convergence rate?

Problem 3. *In the lectures we considered elements in 1D that are piecewise cubic and C_1 across subinterval boundaries. Consider C_1 elements in 2D using a triangular mesh. What is the minimum polynomial degree needed for such elements? (use a counting argument).*

Problem 4. *Find the dimension of P_4 (polynomials of degree 4) in 2 dimensions (2D). Describe an element on triangles that is conforming (continuous between elements if linear mappings from the reference element are used) and spans P_4. Give points so that a general fourth order polynomial is determined by values at these points (there is a matrix invertibility question to be checked here).*

Problem 5. *Find a third order quadrature method on the unit equilateral triangle. That is, find n points (x_i, y_i) and weights w_i such that*

$$\int_\Omega p(x,y) = \sum_{i=1}^{n} w_i p(x_i, y_i)$$

for all polynomials $p(x)$ of degree 2. Your weights should all be positive. See if you can also get quadrature of fourth and higher orders.