Queen Victoria Annex Math Workshop
Problems #1

Brian Wetton

January 2, 2005

The Characters

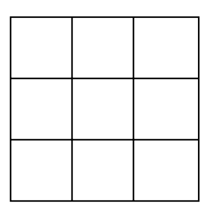
The main characters in the problems this week are Farmers Fred and Louise, their baby daughter, Serpentia, and their grown son, Bobo the clown.

Problem Set A

1. Farmer Fred has 2 apple trees. He picks 5 apples from one tree, 12 from the other, eats 2 and gives 3 to his son, Bobo. How many apples does he have left?

2. A movie is listed as 170 minutes long. How long is this in hours and minutes? Is it closer to two hours or three hours long? *Note:* an hour has 60 minutes in it.

3. How many squares can you find in the following picture? 1 by 1, 2 by 2, and 3 by 3 squares all count.



4. Farmer Louise has 3 more apples than Farmer Fred. Together they have 7 apples. How many do each have?

Problem Set B

1. Your family rents two movies: "Terror of Planet X" (105 minutes) and "Soft Love Puppies" (45 minutes). It is 1:00 and you know you have to leave for soccer practice at 3:00. Do you have time to watch both movies before you go?

2. Farmer Fred bakes a pumpkin pie for his family. He divides it into quarters for Farmer Louise, Bobo, Serpentia and himself. Bobo eats all of his piece of pie. Fred and Louise both eat two-thirds of theirs. Serpentia eats only one third of her piece. They put the left-overs together. How much of the pie is left? Is it more or less than a quarter of the original pie?

3. Twelve people purchased supplies for a ten day camping trip with the understanding that each of the twelve will get equal daily shares. They are joined by three more people, but no additional food is purchased. How long will the supplies now last if the original daily share for each person is not changed?

4. Bobo, the clown, drives from his home in the city of Abacus at 30 km per hour to the circus which is 20 km away. On the return trip, there is heavy traffic and he averages only 12 km per hour. How much time does it take Bobo to drive to and from the circus?

Problem Set C

1. Baby Serpentia opens her mathematics book. She notices that the page number on the left times the page number on the right is 1806. What pages is she looking at?

2. When Fred, Louise and Bobo compared the amount of money each had, they discovered that Fred and Louise together had $12, Louise and Bobo together had $18 and Fred and Bobo together had $10. Who had the least amount of money? How much did each have?

3. Three girls (Cindy, Charlene and Fredrika) and three boys (Gilbert, Dirk and Bob) go to a party. There is a small platform on which only two people can dance at a time. How many different couples can dance

on the stage if only a girl and a boy can dance together? How many different couples if 2 boys or 2 girls can also dance together?

4. 2004 was a leap year, which has 366 days. How many full weeks in a leap year and how many extra days? If your birthday was on Sunday in 2004 before February 29 (when the extra day occurs), what day will it be on this year?

Different Types of Mathematics

There are different types of mathematics, just like other subjects. In this problem set, three types were seen:

Arithmetic: Using the standard operations $\times$, $\div$, $+$ and $-$ on given numbers or fractions. This is something a calculator can do. You've also learned how to do this by hand. First, you memorized addition and multiplication of single digit numbers. Then, you learned short-cuts to do multi-digit addition, subtraction and multiplication. Finally, you learned a short-hand method for division (long division). Arithmetic is the oldest branch of mathematics, used even in primitive times to work out things like: does the tribe have enough food stored to last the winter? It is used in all other branches of the subject.

Algebra: The use of arithmetic rules to solve for unknown quantities. Simple algebra is not that different from arithmetic.

Combinatorics: Counting the number of ways something can happen.

As an additional exercise, go back to the problems you did and see if you can identify what type of mathematics you used to solve them.

Often, more than one type of mathematics is needed to solve a particular problem. You will see other types of mathematics in school in the years to come, such as geometry, trigonometry, probability and calculus.

Elementary Algebra

Remember that Algebra is a technique to find unknown quantities using the rules of Arithmetic. Often, letters are used to represent unknown numbers. The letter x is a particular favourite to use for an unknown number, but you can use a, b, or any letter you like. Sometimes in more advanced mathematics, greek letters are used (like θ or ω). This may seem confusing, but often particular letters are used for particular quantities (like θ is often used for angles) and this can help you remember what you're doing.

Look back at question A5. You probably obtained the right answer by guessing and checking (the oldest algebra solution method ever, and a very good one). However, some problems are just too hard to "guess and check". Let's use some algebra rules to solve A5. First, let's call the number of apples that Farmer Fred has "x", which we don't know yet. If we read the question we know that Farmer Louise has 3 more apples than Fred, or "$x+3$" apples. Together, they have 7 apples. That is,

$$x + x + 3 = 7.$$

This is an equation we can solve for x using algebra. First, notice that $x+x$ is $2 \times x$, which you can write as $2x$. *Note:* when you use x as a variable you have to make sure it doesn't look like a multiplication sign $\times$. So now,

$$2x + 3 = 7.$$

This says that $2 \times x$ plus 3 is the number 7. We want to isolate x to figure out what it is. We are allowed to subtract 3 from both sides:

$$2x = 7 - 3 = 4.$$

This is allowed because if we start with two numbers that are the same ($2x+3$ and 7 are the same) and we subtract 3 from both, they'll still be the same. We're allowed to divide both sides by 2:

$$x = 4 \div 2 = 2.$$

So Fred has 2 apples and Louise has 2+3 = 5 apples. We can check our answer ($2 + 5 = 7$ check). Many people (even grown-ups!) have trouble with algebra, so don't worry if it takes you a few tries to get the hang of it.

Try to find an equation for the left hand page number in question C1. *Don't* try to solve the equation! It's what's called a quadratic equation. These aren't so hard, but you won't see them in school for a while.

Pick one of the people in question C2 and let the amount of money they have be x. Use algebra to find an equation for x. This is a bit tricky, you might need some help to do this one.

Problem Hints

B2: Imagine Fred had cut the pie into 12 pieces and given 3 to each person.

B3: Start by working out the number of days the food would feed one person.

B4: 30 km per hour is the same as $\frac{1}{2}$ km per minute and 12 km per hour is the same as $\frac{1}{5}$ km per minute.

C1: If the pages she were looking at were 40 and 41, what would the number be? What about 49 and 50?

C2: Use the information to determine the difference in the amount of money each has.

Problem Answers

A1: 12

A2: 2 hours and 50 minutes, closer to three hours than two.

A3: 14

A4: Fred has 2, Louise has 5.

B1: Total movie time is 150 minutes (2 hours, 30 minutes) so there would not be time to watch both movies.

B2: $\frac{4}{12}$ or $\frac{1}{3}$ of the pie remains. This is more than a quarter (one quarter is $\frac{3}{12}$).

B3: 8 days.

B4: 140 minutes, or two hours and 20 minutes.

C1: pages 42 and 43.

C2: Fred has the least ($2). Louise has $10 and Bobo has $8.

C3: 9 couples if a couple can only have one boy and one girl, 15 otherwise.

C4: A leap year has 52 full weeks and 2 days. The birthday would be on Tuesday in 2005.

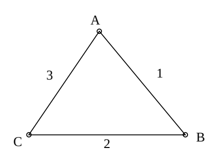
Queen Victoria Annex Math Workshop
Problems #2

Brian Wetton

January 12, 2005

Graph Theory

In this problem set we will be using a new kind of mathematics called *Graph Theory*. The "graphs" in Graph Theory are not the kind of graphs you may have seen already, that are a visual aid to presenting numerical data (but these are really useful, too). In this kind of mathematics, graphs are dots (vertices, the plural of vertex) connected by lines (edges) as shown in the diagram below:



In this graph the vertices are the dots A, B and C and the edges are the lines labelled 1, 2, and 3. Note that it is allowed to connect the same two vertices with more than one edge and to connect a vertex with an edge to itself. Below are listed some important concepts about graphs. It is not necessary to understand these completely right now. They will be made more clear in the problems below.

connectivity: A graph is said to be *connected* if you can move from any vertex to any other vertex on a path of edges. The graph above is connected.

completeness: A graph is said to be *complete* if every vertex is connected to every other vertex by a single edge. Complete graphs are connected.

The picture above shows the complete graph with three vertices. It gets the short-hand name, K_3. The complete graph with four vertices is named K_4, etc.

planar: A planar graph is one that you can draw on a piece of paper without any edges crossing each other. The graph K_3 shown above is planar.

Some of the graph theory questions below will use *combinatorics*.

Our Story This Week

This week, we'll consider the island nation of Dedlos. Dedlos has four cities: Abacus, Beatific, Continuous, and Dreamy. Travel between the cities is difficult and the government is considering building a road network to make travel easier.

Problem Set A

1. If roads are built between each pair of the four cities, how many roads will need to be built? (In the mathematical language of graph theory, this is asking how many edges there are in K_4).

2. Show that if a road is built between each pair of cities, this can be done so that the roads do not cross (show that K_4 is planar).

3. What is the minimum number of roads that need to be built so that one can travel from any city to any other city? (what is the minimum number of edges in connected graphs with four vertices?).

4. How many different road networks are there that have the minimum number of roads found above and that connect all four cities?

Problem Set B

1. Dedlos sometimes has severe rain that leads to road flooding. What is the minimum number of roads that are needed between the cities so that even if one cannot be used, it is still possible to travel between each city on the remaining road network?

2. How many different road networks are there that have the number of roads above and their redundancy property?

3. What are the minimum number of roads needed in the network so that even if *two* roads are flooded out the cities are still connected?

Problem Set C

1. Edge is a small village on Dedlos. When diamonds are discovered on the beaches nearby it develops quickly and becomes a fifth city. How many roads are needed in a network that connects each city to each other city now? (that is, how many edges in K_5?)

2. Show that it is not possible to build the roads in the network above that do not cross (that is, K_5 is not planar).

Formulas

We have seen that K_3 has 3 edges, K_4 has 6 edges and K_5 has 10 edges. Let E_n be the number of edges in K_n. So far, we know that $E_3 = 3$, $E_4 = 6$, and $E_5 = 10$. There are two formulas that can be used to find the number of edges in complete graphs with more vertices. The first is

$$E_{n+1} = E_n + n.$$

This is true for every value of n. When $n = 3$ it says $E_4 = E_3 + 3$ ($6 = 3+3$ check). When $n = 4$ it says that $E_5 = E_4 + 4$ ($10 = 6 + 4$ check). When $n = 5$ it says that $E_6 = E_5 + 5$, so E_6 which we don't know yet has the value $10+ 5 = 15$. We could use the formula over and over to get the value of E_7, E_8, etc. Note that this is not an *algebra* equation since n is not an unknown. Since it's true for all n it's called a formula. See if you can show why this formula is true.

The formula above is true, but it would be a lot of work to figure out E_{100}, the number of edges in a complete graph of 100 vertices. There is another formula we can use:

$$E_n = \frac{n \times (n-1)}{2}.$$

We can check this formula for small values of n:

$$E_3 = 3 \times 2 \div 2 = 3 \text{ (check)}$$

$$
\begin{aligned}
E_4 &= 4 \times 3 \div 2 = 6 \text{ (check)} \\
E_5 &= 5 \times 4 \div 2 = 10 \text{ (check)} \\
E_6 &= 6 \times 5 \div 2 = 15 \text{ (check)}.
\end{aligned}
$$

Of course, showing it works for some values of n doesn't make it true for all n. It is possible to show it's true starting with the first formula using something called *mathematical induction*. There is an easier way to see that it is true, however. See if you can come up with a convincing argument.

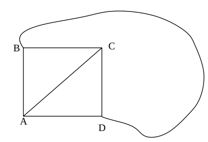
Abstraction

It was easier to work out the answers to the Dedlos road network problem when we represented the cities and network as dots with lines connecting them (a graph). The process of taking a real world problem and looking at its underlying structure in a simple way is called *abstraction*. Abstraction can help solve problems because many distracting but irrelevant details are brushed away. Also, since many different real world problems are the same kind of abstract problem (like many problems are graph theory problems), it is worth while to spend the effort showing really difficult things about the abstract problems that then can be used again and again.

Problem Answers

A1: 6, I give this away in the later discussion, I hope you didn't read ahead.

A2: See the figure below:



A3: 3

A4: 16

B1: 4

B2: 6

B3: 6, you need a road between every city for this property.

C1: 10

C2: Imagine that the complete road network has to start with the complete road network between the four original cities and then four new roads have to be built between Edge and the four original cities. The original road network must look like the one in the figure above (maybe with the city names changed around). The new city has to sit somewhere. If it's outside the original network, then the road to C has to cross an existing road. If it's inside the triangle ABC then the road to D must cross an existing road. If it's inside the triangle ACD then the road to B must cross an existing road. Finally, if Edge is in the curved region by C, then the road must cross an existing road to get to A. So, wherever Edge is, at least one road from Edge to the four existing cities must cross an existing road. In other words, K_5 is not planar.

Queen Victoria Annex Math Workshop Problems #3

Brian Wetton

January 21, 2005

Theorems

In real life, it is hard to make statements that are always true, like "Buddy never lies" (Buddy, whoever he is, has probably lied at least once). However, Mathematics allows for definite statements about abstract concepts that really are true in every situation. These true statements are called "Theorems". The exercises today are based on two theorems of graph theory which are stated below. Just stating a theorem does not make it true. Showing that the theorem applies to a few, or even a great many specific cases does not make it true. These theorems were proved to be true for all cases using mathematics beyond what we can do in this class. However, we can still use the results.

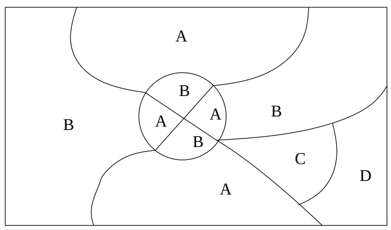
Problem Set A

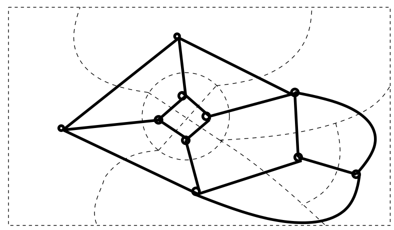
This problem set uses the **Four Colour Theorem**:

Theorem 1 *Any map can be coloured using four colours (or less) in such a way that adjacent regions (that is, those sharing a common boundary segment, not just a point) receive different colours.*

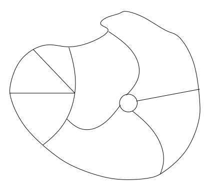
This seems like it might be an easy thing to show, but it took more than 100 years for a proof to be found and it involved exhaustively checking many cases using a computer. You can read more about the theorem and its history by entering "four colour theorem" in google.

You can see an example of a map coloured with four colours in the figure below on the left. The theorem says that no matter how weird a map you make, you can always colour it with only four colours (notice that it doesn't say how you find the colour scheme and this can be a hard problem for large and complicated maps). On the right part of the figure, you can see how the areas and borders of the map can be encoded as a graph. The Four Colour Theorem is really a theorem about planar graphs.

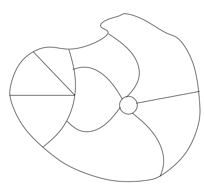




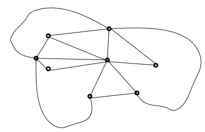
1. Colour the following map using only three colours.

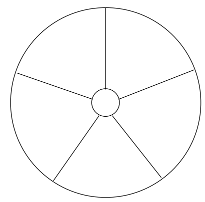


2. Colour the following map using only four colours.



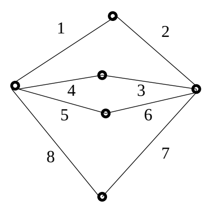
3. Convert the areas and borders above in to a graph.
4. Draw a map that corresponds to the graph below:
5. Consider the game wheel in the picture below: Show that the wheel cannot be coloured with only three colours. How many different ways can the wheel be coloured with four colours? In this problem, when you count, patterns that are the same if you turn the wheel don't count extra.





Problem Set B

An circuit in a graph is a series of adjacent edges that start and end at the same vertex. An Euler Tour is a circuit which includes every edge exactly once. An example of an Euler tour in a graph is shown in the figure below, the series of edges given by numbers.



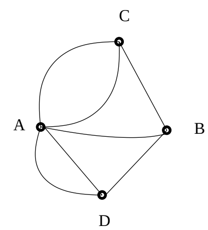
Not every graph has an Euler Tour. In fact, the kind of graphs that have them are exactly characterized by the following:

Theorem 2 *If a graph has an Euler Tour, then every vertex must have an even number of edges attached to it. Also, if every vertex in a connected graph has an even number of edges attached to it, then the graph has an Euler Tour.*

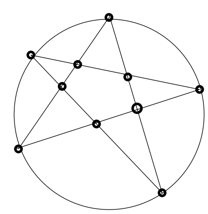
Note that the theorem does not say how to find a Tour even if you know there is one. This can be a hard problem for large and complicated graphs.

Euler was a famous historical mathematician, who first studied graphs and proved this theorem to show that there was no way to tour the town of Konigsberg crossing the city's seven bridges only once. For more information on this problem, enter "Bridges of Konigsberg" in google. The proof of this theorem is much easier than the proof of the Four Colour Theorem, but still too hard for us.

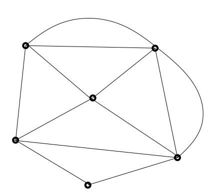
1. Why does the following graph not have an Euler Tour? (This is the graph that corresponds to the bridges of Konigsberg, with A and B islands in a river and C and D the river shores). You get to apply the theorem above, so see if there are any vertices with an odd number of adjacent edges.



2. Find an Euler Tour for the following graph:



3. Decide whether the following graph has an Euler Tour or not. Find one if you know it has one.



4. See if you can determine how many different tours there are in the graph of question 2 above. For this question, tours don't count as different if they follow the same path, just start at different vertices.

Queen Victoria Annex Math Workshop Problems #4

Brian Wetton

January 28, 2005

Problem Set A

In this problem set, there are some problems using the kinds of mathematics we have already talked about: Arithmetic, Algebra, Combinatorics and Graph Theory.

1. A train leaves Vancouver on Monday afternoon at 3:00 and arrives in Toronto on Thursday morning at 7:30. How long did the trip take?

2. Andrew, Bob, Claudia, Devon, Eliza, Fritz and Gisela are a group of friends, but not everyone likes everyone else. Andrew, Bob and Claudia all like each other, Gisela and Edward like each other, Fred and Gisela like each other, Devon and Fred like each other, Claudia and Fred like each other and Devon and Edward like each other. What one friendship would break the group apart if it were to fail? *Hint:* Make a graph of the friendships.

3. Bobo the clown wants to paint the four walls of his square room using the three colours of paint he bought at the store. Each wall will be painted with one of the colours. How many different ways can he paint his room if each colour has to be used on at least one wall and no two walls next to each other can be painted the same colour? *Hint:* Figure out how many different patterns he can use and them multiply by the number of ways the three colours can be assigned to the patterns.

4. Farmer Fred has twice as much money as Farmer Louise. Together they have \$15. How much money to each have? *Hint:* You could "guess and

check" or you could use algebra. This same hint applies to the next problem.

5. Farmer Fred asks his daughter, Baby Serpentia, to go out and count the number of chickens and the number of pigs on the farm. She comes back in and says she counted 12 animal heads and 38 animal legs and then went to have her nap before Farmer Fred could remind her of what he actually wanted to know. Help farmer Fred work out the number of chickens and the number of pigs from the information Baby Serpentia gave him.

Multiple Solutions and Optimization

Many mathematical problems have only one solution: if you have 3 apples and are give 4 more apples, then you have 7 apples. Other problems may have many solutions, such as "how do I get home from here?" (which we could make into a mathematical problem if we made a graph of locations and edges representing possible paths between adjacent locations). We would expect this problem to have many possible answers (solutions). In some cases, we might want to refine our question to "what is the fastest way home?". The fastest time would be called the optimal time in mathematical language, and the route that gives the optimal time is the optimal route (there might be more than one optimal route, of course). Finding optimal strategies for things can be very important:

- How can our company make the most money?
- What is the most earthquake design for a school?
- What is the safest design for a parachute?
- How can we cut down a given amount of forest making the least impact on the environment?

These kinds of questions involve the process of "optimization" which can be used with many other types of mathematics. Mathematical optimization is a powerful tool. However, for real life questions some care must be used in evaluating the results of an optimization study. For example, the way a company might be able to make the most money might also involve destroying the environment or ruining the health of its workers. Mathematical optimization doesn't lie, but it will only optimize what you put in the problem. Optimization of engineering questions is often clear-cut, but in mathematical optimization of questions involving Society or Politics, it is very difficult to say that all important aspects of the problem are included.

Finding optimal strategies can lead to very hard mathematical problems. It is often not necessary to find the "optimal" strategy and a "good" strategy is often good enough. For example, if the optimal route home would take 4 minutes but you can easily figure out a route that only takes 5 minutes, you might be happy to take this route.

The story this week

The citizens of Abacus has been victimized by a counterfeit coin ring. Their $5 coin (the fooney) has been counterfeited and the fake coins exchanged in bags for paper money at the bank. The problem is discovered and a large number of bank workers are put to the task of finding which bags of coins are full of fake foonies. The real coins are known to weigh 10g and it is discovered that the fake coins weigh only 9g. Each worker is given 10 bags at a time and it is known that one of the 10 bags has fake coins in it. There is only one digital scale in the bank and workers stand in line while the person in front of them weighs one coin from each bag until they find a fake one. This continued until they received a letter, written in crayon:

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Dear Bank Manager,

I have heard of your strategy for identifying the bags of
fake coins. Your method requires that each person use the
scales up to 9 times. If you had your people weigh the coins
two at a time to find the pair of coins that had the fake coin
in it and then weighed one of the pair, you could determine
the bag that had the fake coins in it with at most 5 uses of the
scale.

This is an improved strategy but not the optimal one. There is a
way to determine which of the bags has the fake coins in it
with just one use of the scales. However, I must leave you to
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`discover how to do this on your own, since it is time for my nap.`

`Sincerely, Serpentia`

Problem Set B

1. Explain why weighing one coin at a time requires at most 9 uses of the scale, and not 10.

2. Give more detailed instructions on Serpentia's strategy of weighing two coins at a time, showing that at most 5 uses of the scale are needed.

3. Give a strategy for determining which bag has the fake coins in it using only one use of the scales, as Serpentia claims is possible.

4. If it is known that the fake coins weigh less than the real ones, but their exact weight is not known, give a strategy for determining the fake bag with only two uses of the scale.

5. Suppose you only had a balance scale. What is a strategy for identifying the bag of fake coins? You can assume that you have bag of known real coins also.

6. What is a strategy to identify the bag of fake coins above that you can guarantee uses the scales the least number of times?

7. Answer the question above when it is not known for sure that only one of the 10 bags you have is full of fake foonies, but that two, one or none of the bags might be counterfeit?

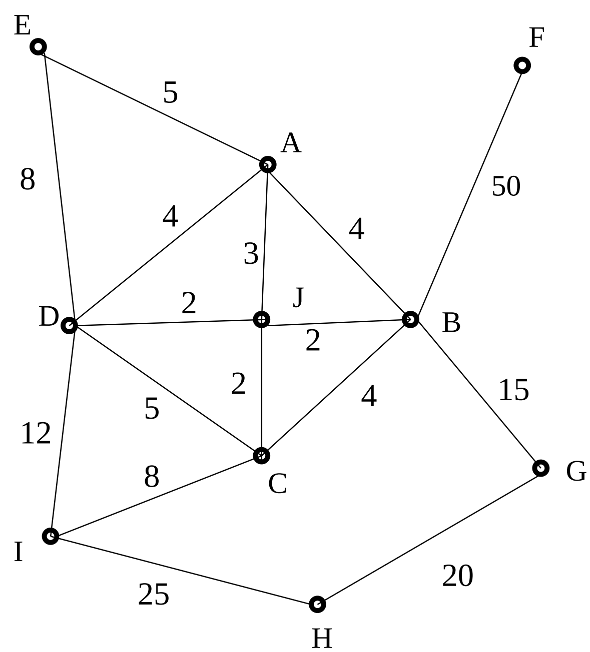
Queen Victoria Annex Math Workshop Problems #5

Brian Wetton

February 11, 2005

The story this week

The island nation of Dedlos has grown and prospered. There are now 10 major cities: Abacus, Beatific, Continuous, Dreamy, Edge, Fortuna, Gargantua, Helical, Iota and Junction City. The road network consists of 16 major roads as shown below:



Note that the diagram above is a graph and the numbers above each edge are the length of each road in km. Such a graph and edge numbering combination is known as a *weighted graph*. National Highway Department planners are doing a study on the road network and ways to improve it. You can help them by answering the questions below.

Problem Set A

1. The planners discover that one road only gets local traffic (that is, only people that live or work at places along the road use it). Can you guess which road this is?
2. Work out the shortest route from Helical to Abacus.
3. Work out the shortest route from Helical to Dreamy.
4. Work out the shortest route from Helical to Edge. *Note:* if you use the information from the last two questions, this question is much easier to answer.
5. If you always take the shortest route between cities, which two cities are furthest apart?
6. Say you want to drive from Dreamy to Junction City taking the *longest* route possible that never takes you to the same city twice. What route would you take?

Problem Set B

Highway planners have five proposals to improve the national road network:

freeway: Build a faster road from Beatific to Junction City that is only 1 km long.

beach road: Build a road from Iota to Edge that is 15 km long.

tunnel: Build a road from Fortuna to Abacus that is 30km long.

mountain pass: Build a road from Gargantua to Continuous that is 15 km long.

swamp trail: Build a road from Helical to Continuous that is 30 km long.

Pretend that you are the people described below. In each case, decide what proposal you would vote for.

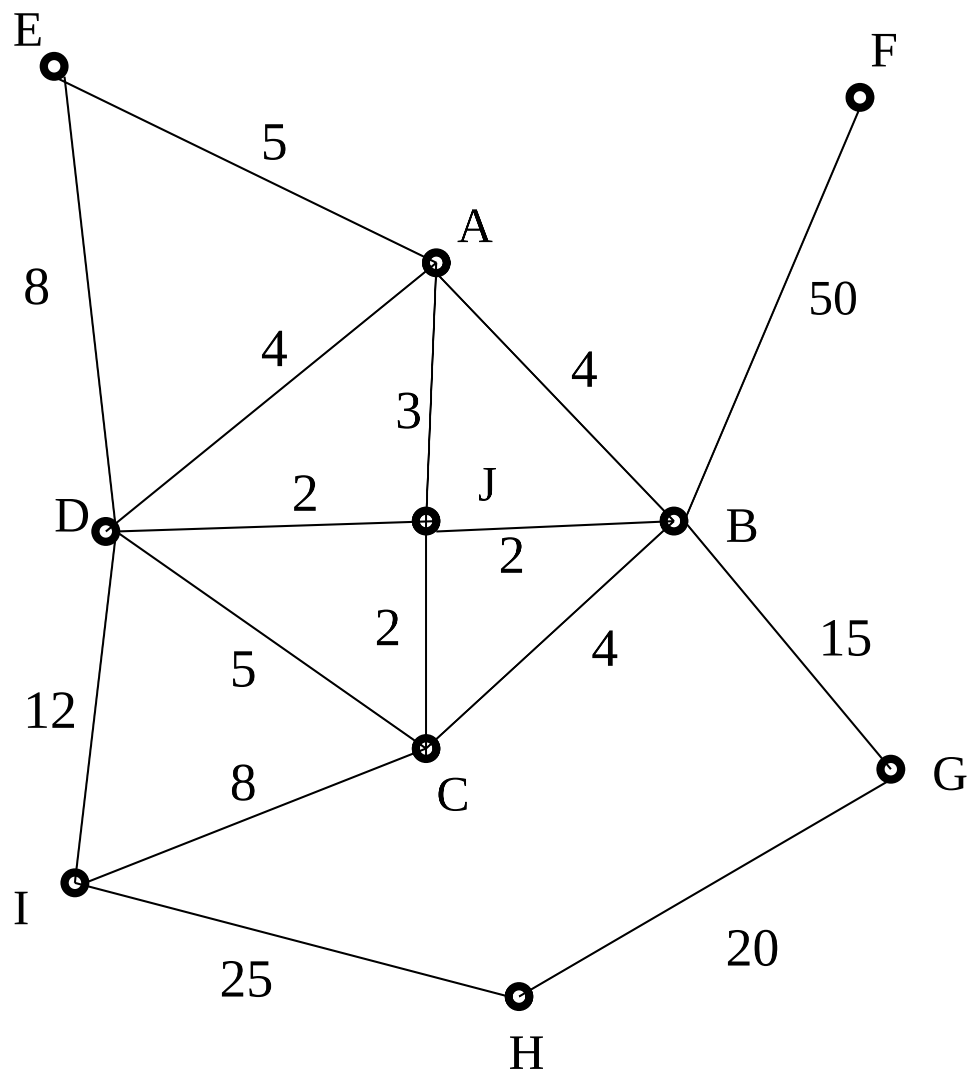
1. You only drive between Dreamy and Beatific.
2. You only drive between Helical and Edge.
3. You only drive on the shortest routes between any two cities and you want to have the largest distance between any two cities reduced as much as possible.
4. You think it is wasteful if the new road makes any of the old roads obsolete (such that only local traffic would use them). There are three possible choices if you feel this way.
5. You like to drive from Dreamy to Junction City taking the longest route possible that does not visit any two cities twice.

Algorithms

An algorithm is a step by step description of how to solve a problem. You already know some algorithms (how to do multiple digit multiplication and long division, for example). Algorithms are especially useful since computers were invented: computers are able to follow step by step instructions accurately even if there are billions of steps.

An algorithm that is very useful for the shortest path problems in section A (# 2-4) is called Dijkstra's Algorithm (try searching in google if you would like more information on the algorithm and its history as well as some animated versions). Dijkstra's algorithm will find the shortest paths between a given vertex and all the other vertices in a connected, weighted graph. How it works is that it adds vertices one after the other to the given vertex, where the vertex that is added is closest to the set of vertices already chosen. Since Dijkstra's algorithm can be written as a computer program, it can find shortest routes in graphs with millions of vertices.

See if you can apply Dijkstra's algorithm to our graph to find the shortest paths between Helical and all the other cities. It can be shown that the shortest paths only branch out and never cross back over again: they form what is called a "sub-tree" of the original graph.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Queen Victoria Annex Math Workshop Problems #6

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Probability

If you flip a coin, it will come up "heads" half the time and "tails" (the side of the coin that doesn't have a face on it) half of the time. We say that the probability of flipping a coin and getting heads is 1/2 and the probability of getting tails is 1/2. Only these two outcomes are possible and the probabilities add up to 1 (1/2+1/2 = 1). If you flip a coin a lot of times, approximately half of the time it will come up heads.

If you flip the coin twice, there are four possible outcomes: HH, TT, HT, and TH, all equally likely. The probability of getting two heads is 1/4, two tails is 1/4 and one head and one tail is 1/2 (1/4 for HT and 1/4 for TH). Only these three outcomes are possible and the probabilities add up to 1 (1/4 + 1/2 + 1/4 = 1). If you flipped two coins a bunch of times, approximately one quarter of the time they would both come up heads. The probability of flipping three coins and having them all come up heads is 1/8. Flipping four coins and having them all come up heads happens 1/16 of the time, and so on.

If you roll a die, there are six outcomes (1, 2, 3, 4, 5, and 6) all equally likely. The probability of rolling a 1 (or any other given number) is 1/6.

To describe the mathematical rules of probability is a bit technical, but hopefully this introduction gives you enough intuition to solve the following problems:

Problem Set A

1. If you roll two dice and add up the numbers, what is the probability you will get 12?
2. If you roll two dice and add up the numbers, what is the probability that you will get 11?
3. If you roll two dice and add up the numbers, what is the probability that you will get 10?

Averages

Consider a coin toss, where you count heads as 1 and tails as 2. If you flipped the coin a lot of times and added up the numbers you get, then divided by the number of times you flip, you will get a number that is about 1 and 1/2. This is the average number you get from the coin flip. You can work out averages by adding up the numbers times the probability they come up: $1/2 \times 1 + 1/2 \times 2 = 1\frac{1}{2}$ in this case.

If you flipped two coins and counted heads as 1 and tails as 2 and added up the numbers, the average would be 3. You can work this out in three different ways:

- One coin has average 1 and 1/2 so two coins will have average twice as much ($2 \times 1\frac{1}{2} = 3$).
- Two heads (counts 2) come up 1/4 of the time, two tails (counts 4) come up 1/4 of the time and one head and one tail (counts 3) come up 1/2 of the time, so the average is

$$2 \times \frac{1}{4} + 4 \times \frac{1}{4} + 3 \times \frac{1}{2} = 3$$

- HH (2), TT (4), HT (3), TH (3) are all equally likely, so the average is

$$(2 + 4 + 3 + 3)/4 = 12/4 = 3.$$

Problem Set B

1. What is the average of a dice roll?
2. What is the average of two dice rolled and added together?

The story this week

Ydnew the sorcerer and his apprentice, Xavier, have a magical duel as part of a circus act in the city of Abacus. They take turns casting spells at each other. The spells don't always work, but when they do, the opponent is knocked unconscious. Xavier gets to go first in the duel but his spells only work one third of the time. Ydnew can cast his spells one half of the time. When they practise, they find that each win half the time, which seems fair. However, sometimes the duels go on for a long time and the audience gets restless. The circus manager decided to limit the number of rounds that the duel would go on before declaring Ydnew the winner by default. The sorcerers found that in this situation, Ydnew won more often than Xavier.

1. What is the probability that the duel would last past two spell attempts? (one by Xavier and one by Ydnew).

2. What is the probability that the duel would last past four spell attempts? (XYXY).

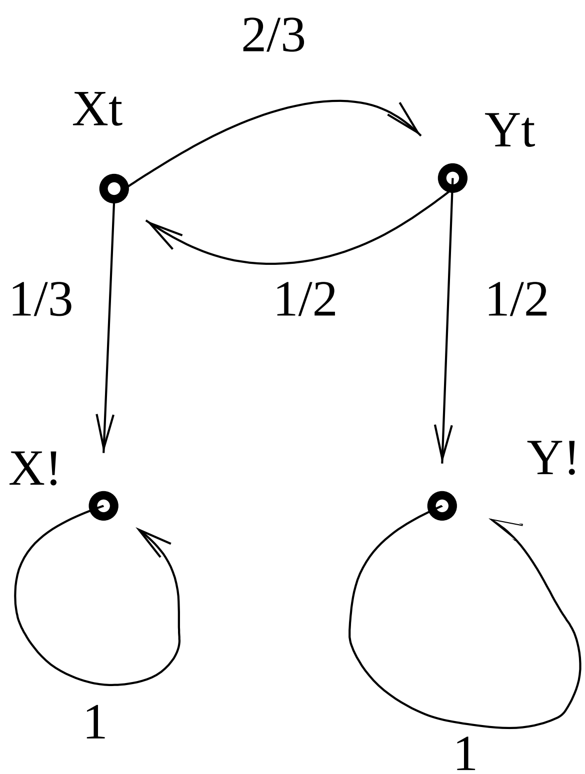
3. What is the probability that Xavier will win if he is limited to two spell attempts before being declared the loser by default?

4. What is the probability that Xavier will win if he is limited to three spell attempts before being declared the loser by default?

5. What is the probability that Xavier will win if he is limited to four spell attempts before being declared the loser by default? See if you can see why if no limit is placed on the number of rounds the duel can last, Xavier will win half the time.

State Diagrams

In the case where the duel has no time limit, the duel can be described by a state diagram as shown below. A state diagram is nothing but a particular kind of directed, weighted graph (a directed graph just means the edges have arrows attached to them). The vertices are the states of the duel. There are four possible states: either Xavier has won (X!) or Ydnew has won (Y!) or there is no winner yet and it's Xavier's turn (Xt) or there is no winner and it's Ydnew's turn (Yt). The arrows from a vertex are possible outcomes and the weights are the probabilities (so the weights from a vertex must add up to 1). It is possible to use the state diagram to help work out the questions from the previous section. Actually, it's even easier to use techniques from matrix algebra to work out these questions, but that is another story.



Consider the case where the sorcerers duel but have armour that allows them to be hit once and continue the duel but the second hit will knock them out. Describe this situation with a state diagram (*hint:* there are 10 states to consider here).