Lectures on Optimal Transport. May, 2022. KAIST

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Lecture 1	The Monge-Kantorovich problem and duality.
Lecture 2	Wasserstein geometry of the space of probability measures.
Lecture 3	Entropic regularization of optimal transport.
Lecture 4	Application of optimal transport to developmental processes.
Lecture 5	Multimarginal optimal transport. Wasserstein barycentre.
Seminar	Optimal Brownian stopping with free target and the supercooled Stefan problem.
Lecture 6	Optimal martingale transport **Today!**
Lecture 7	Optimal Brownian martingale transport

Some references for the lectures

- Lecture 1, 2, and 3:
  - Villani: Topics in Optimal Transport. Book
  - Villani: Optimal Transport. Old and New. Book
  - Cuturi & Payré: Computational Optimal Transport. Book
- Lecture 4
  - Schiebinger: https://broadinstitute.github.io/wot/tutorial/
  - Lavenant, Zhang, Kim, Schiebinger,: Towards a mathematical theory of trajectory inference. https://arxiv.org/abs/2102.09204
- Lecture 5
  - Cuturi & Payré: Computational Optimal Transport. Book
  - Kim & Pass: Wasserstein Barycenters over Riemannian manifolds. Adv. in Math. 2017.
- **Lecture 6**
  - Ghoussoub, Kim, & Lim: Structure of optimal martingale transport in general dimensions. Ann. Prob. 2019.
- Lecture 7
  - Ghoussoub, Kim, & Palmer: PDE Methods For Optimal Skorokhod Embeddings. Calc. Var. 2019.
  - Ghoussoub, Kim, & Palmer: A solution to the Monge transport problem for Brownian martingales. Ann. Prob. 2021.
  - I. Kim & Y. Kim.The Stefan problem and free targets of optimal Brownian martingale transport. Preprint. 2021

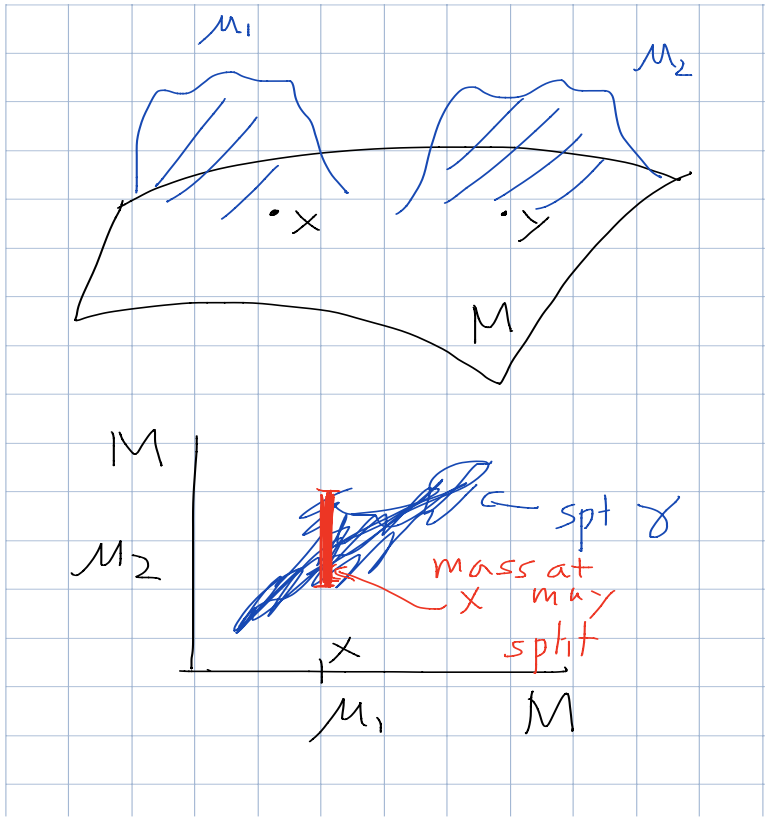
Optimal martingale transport has rich but hidden structures, especially in multi-dimensions.

Optimal Transport Problem

- $M_1 = M_2 = M$
  $\mathbf{R}^n$, domains in $\mathbf{R}^n$, ...
- μ_1, μ_2: probability measures on M.
- cost function $c : M_1 \times M_2 \to \mathbb{R}$.
- $\Pi(\mu_1, \mu_2)$: probability measures (transport plans) γ on $M_1 \times M_2$ which project to the marginals μ_1, μ_2.

What happens when we **minimize:**

$$\min_{\gamma \in \Pi(\mu_1, \mu_2)} \int_{M_1 \times M_2} c(x, y) d\gamma(x, y).$$



Motivation

- Most cost-effective matching between resources/agents, buyers/sellers, ... :
  [Monge 1791], [Kantorovich 1942], ...

Optimal Transport Problem

Reasonable assumptions (e.g. continuity of the cost c)
$\Rightarrow$ **Existence** of optimal transport plan γ from **weak compactness** for measures.

Question : When is an optimal measure γ **unique?**

Question What is the **structure?**
e.g. Is an optimal γ **graphical**, i.e. concentrated on the graph $\{(x, T(x))\}$ of a map $T : M_1 \to M_2$?

Two important cases:

For $c(x,y) = \frac{1}{2}|x-y|^2$:

- **[Brenier '87 and many many people], ..**
  If $d\mu_1 = f(x)dx$, $d\mu_2(x) = g(y)dy$,
  then, the optimal transport is uniquely given by an optimal **map** of the form
  - $T(x) = \nabla[\phi(x)]$, for μ_1-a.e. x;
  - $\phi : \mathbf{R}^n \to \mathbf{R}$ is **convex**.
  - $\psi = |x|^2/2 - \phi$ solves the **dual problem**:

$$\min_{\gamma \in \Pi(\mu_1,\mu_2)} \int c(x,y)d\gamma(x,y) = \max_{\psi+\psi^c \le c(x,y)} \int \psi(x)d\mu_1(x) + \int \psi^c(y)d\mu_2(y)$$

- the set $\{(x,y) \mid \psi(x) + \psi^c(y) = c(x,y)\}$ " = " $\operatorname{supp}\gamma$ " = " $\operatorname{Graph} T$
- Monge-Ampere equation:

$$\det(D^2\phi(x)) = \frac{f(x)}{g(\nabla\phi(x))} \quad \mu_1\text{-a.e. } x$$

- Applications to functional inequalities, nonlinear diffusions (gradient flows), geometry,..

Two important cases:

$c(x, y) = |x - y|$:

- [Sudakov '79, Evans-Gangbo, Ambrosio, Trudinger-Wang, Caffarelli-Feldman-McCann, ...]

No uniqueness of optimal γ. But, optimal **mappings** exist.

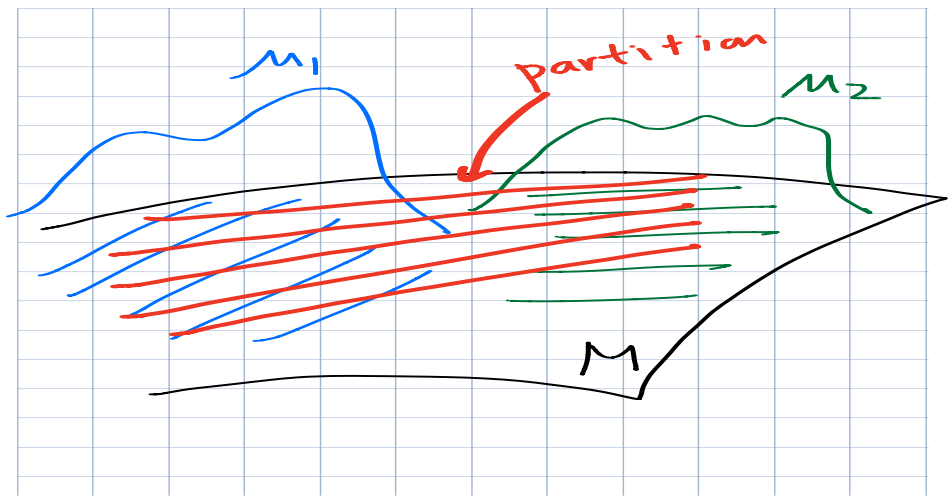
- duality

$$\min_{\gamma \in \Pi(\mu_1, \mu_2)} \int_{M_1 \times M_2} \mathrm{dist}(x, y) d\gamma(x, y)$$

$$= \max_{\psi(y) - \psi(x) \leq \mathrm{dist}(x,y)} \int \psi(y) d\mu_2(x) - \int \psi(x) d\mu_1(y)$$

- The set $\Gamma = \{(x, y) \mid \psi(y) - \psi(x) = \mathrm{dist}(x, y)\}$ gives a "good" 1-dimensional disintegration of the measure $\mu_1 - \mu_2$.

$\Rightarrow$ "localization technique" [Klatarg '14] [Ohta '15] [Cavalleti-Mondino '15] for Levi-Gromov isoperimetric inequality.



Some key points:

- **Optimality** of $\gamma \Rightarrow$ dual functions ψ, ψ^c.

To understand **the structure** for optimal γ, study the **dual functions** ψ, ψ^c.
For example, through some PDE (Monge-Ampére equation), one can study

- mapping properties
- regularity

Additional structures:

- **Localization (partition / disintegration)**

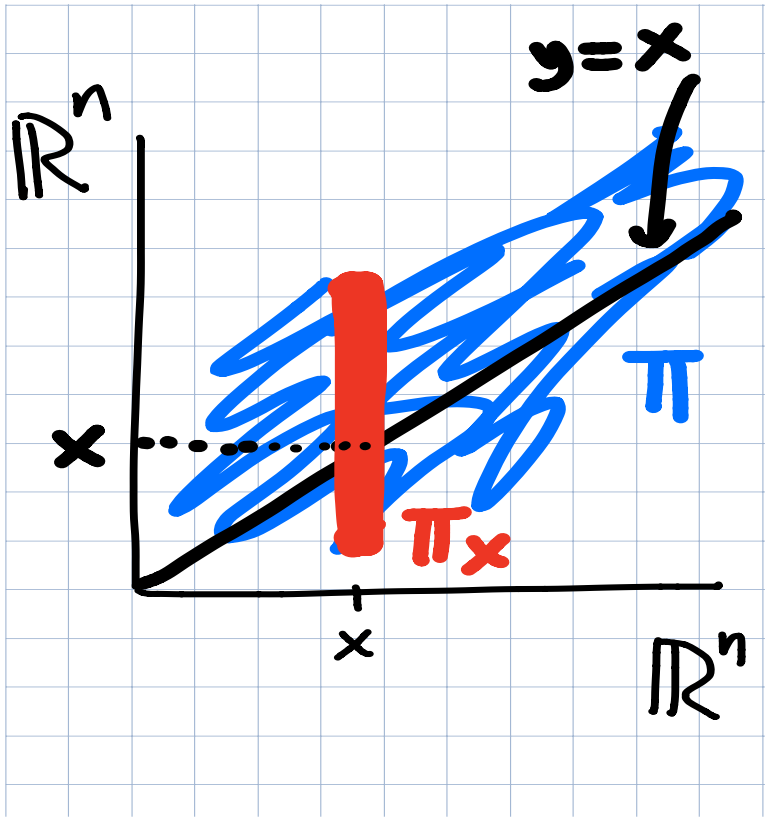
We see similar features in other types of optimal transport problems.

Optimal transport problem with martingale constraint

Optimal Martingale Transport Problem

- cost function $c : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$,
- Probability measures μ, ν on $\mathbf{R}^n$.
- $MT(\mu, \nu)$: probability measures π on $\mathbf{R}^n \times \mathbf{R}^n$ with the marginals μ, ν,
  and its disintegration $(\pi_x)_{x \in \mathbf{R}^n}$ has barycenter at x (martingale constraint):

$$\int y d\pi_x(y) = x.$$



Study the optimal solutions of

$$\max / \min_{\pi \in MT(\mu,\nu)} \int_{\mathbf{R}^n \times \mathbf{R}^n} c(x, y) d\pi(x, y).$$

Remark: [Strassen '65]

- $MT(\mu, \nu) \neq \emptyset$

$\Leftrightarrow$ μ and ν are in convex order;
$\mu \leq_c \nu$, i.e. $\int \xi d\mu \leq \int \xi d\nu$, $\forall$ convex $\xi : \mathbf{R}^n \to \mathbf{R}$.

Convex order

μ and ν are in convex order;
$\mu \leq_c \nu$, i.e. $\int \xi d\mu \leq \int \xi d\nu$, $\forall$ convex $\xi : \mathbf{R}^n \to \mathbf{R}$.

Exercise:
If $\mu \leq_c \nu$ then $\mathrm{supp}\mu \subset \mathrm{conv}[\mathrm{supp}\nu]$.
The converse does not hold.
e.g. $\mu = \delta_{1/3}$, $\nu = \frac{1}{2}(\delta_0 + \delta_1)$. Then, $\mu \not\leq_c \nu$.

Probabilistic formulation

- $(\Omega, \mathcal{F}, \mathbb{P})$: probability space
- $X, Y : \Omega \to \mathbf{R}^n$: random variables
- cost function $c : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$,
- $\mathrm{Law}(X) = \mu$, $\mathrm{Law}(Y) = \nu$,
- $E(Y|X) = X$. (the martingale condition)

$$\max / \min_{X \sim \mu, Y \sim \nu, E(Y|X)=X} E_{\mathbb{P}} c(X, Y).$$

Optimal martingale transport problem

$$\max / \min_{\pi \in MT(\mu,\nu)} \int_{\mathbf{R}^n \times \mathbf{R}^n} c(x,y) d\pi(x,y).$$

$$\max / \min_{X \sim \mu, Y \sim \nu, E(Y|X)=X} E_{\mathbb{P}} c(X,Y).$$

Motivation:

- [Finance] The maximum / minimum price of an option, given the information observed from the market (the marginals).

It is recently very active:

- **Beiglböck, Cox, Juillet, Galichon, Huesman, Hobson, Neuberger, Dolinsky, Henry-Labordere, Penkner, Schachermayer, Soner, Touzi, ...**

Remark

The quadratic cost $c = |x - y|^2$ is NOT interesting for martingale problems:
All martingale transport $\pi \in MT(\nu, \mu)$ have the same transport cost for $c = |x - y|^2$!

Proof.

For $\pi \in MT(\mu, \nu)$,

$$\begin{aligned}
&\int_{\mathbf{R}^n \times \mathbf{R}^n} |x - y|^2 d\pi(x, y) \\
&= \int_{\mathbf{R}^n \times \mathbf{R}^n} |x|^2 d\pi(x, y) + \int_{\mathbf{R}^n \times \mathbf{R}^n} |y|^2 d\pi(x, y) + 2 \int_{\mathbf{R}^n \times \mathbf{R}^n} x \cdot y d\pi(x, y) \\
&= \int_{\mathbf{R}^n} |x|^2 d\mu(x) + \int_{\mathbf{R}^n} |y|^2 d\nu(y) + 2 \int_{\mathbf{R}^n} |x|^2 d\mu(x)
\end{aligned}$$

since from martingale condition, we get

$$\begin{aligned}
\int_{\mathbf{R}^n \times \mathbf{R}^n} x \cdot y d\pi(x, y) &= \int_{\mathbf{R}^n} x \cdot \int_{\mathbf{R}^n} y d\pi_x(y) dx \\
&= \int_{\mathbf{R}^n} x \cdot x dx
\end{aligned}$$

□

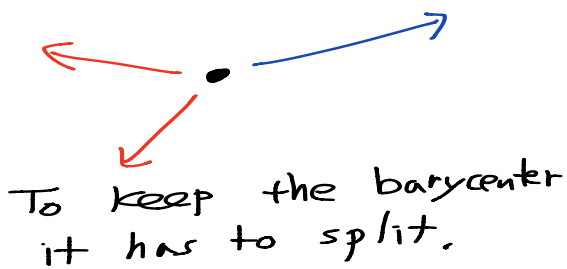
From now on, we will focus on the case:

$$c(x, y) = |x - y|$$

Optimal Martingale Transport Problem

Existence of optimal π again follows from **weak compactness.**

[*Graphical solution (mapping solution) not available*]
π is martingale $\int y d\pi_x(y) = x$
$\Rightarrow$
for π to move a unit mass, it has to **split the mass**!



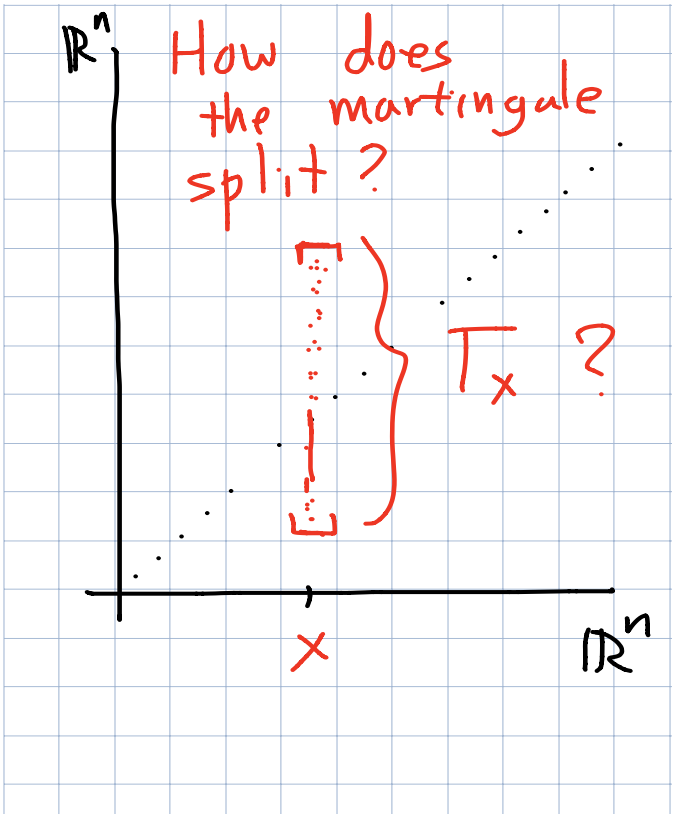
So, π **cannot** be supported on the graph $\{(x, T(x))\}$ of a map $T : \mathbf{R}^n \to \mathbf{R}^n$, **unless** the trivial case $\mu = \nu$.

Optimal Martingale Transport Problem

Question: How does it split?

Let

- $\pi \in MT(\mu, \nu)$ optimal solution
- $\Gamma \subset \mathbf{R}^n \times \mathbf{R}^n$: concentration set of π, i.e. $\pi(\Gamma) = 1$
- $\Gamma_x = \Gamma \cap (\{x\} \times \mathbf{R}^n)$ the vertical slice at x (the **"Splitting set"**)



Question :

- What is the structure of π, or the set Γ, especially Γ_x?
- When is π unique?

1-dimensional results

Many papers in 1-dim:
Beiglböck, Cox, Juillet, Galichon, Huesman, Hobson, Neuberger, Dolinsky, Henry-Labordere, Penkner, Schachermayer, Soner, Touzi, ...

Theorem (Hobson-Neuberger '13, Beiglböck-Juillet '13)

Suppose $n = 1$ *and*

- $c(x, y) = |x - y|$
- $\mu \leq_C \nu$ *on* $\mathbf{R}$ *and* $\mu << \mathcal{L}^1$.
- $\pi \in MT(\mu, \nu)$ *optimal solution (for max / min).*
- *Assume* $\mu \wedge \nu = 0$ *for the minimization problem.*

Then

- *There exists* $\Gamma \subset \mathbf{R} \times \mathbf{R}$*: concentration set of* π*, i.e.* $\pi(\Gamma) = 1$*,*

  such that for a.e. $x \in \mathbf{R}$*,*

  $$\#(\Gamma_x) \leq 2, \quad \text{for } \Gamma_x = \Gamma \cap (\{x\} \times \mathbf{R}),$$

  that is, the disintegration (conditional probability) π_x *is concentrated on at most two points.*
- *In particular, the optimal solution* π *is unique.*

Higher dimensions?

Theorem (Dimension reduction. Ghousshoub, K. & Lim)

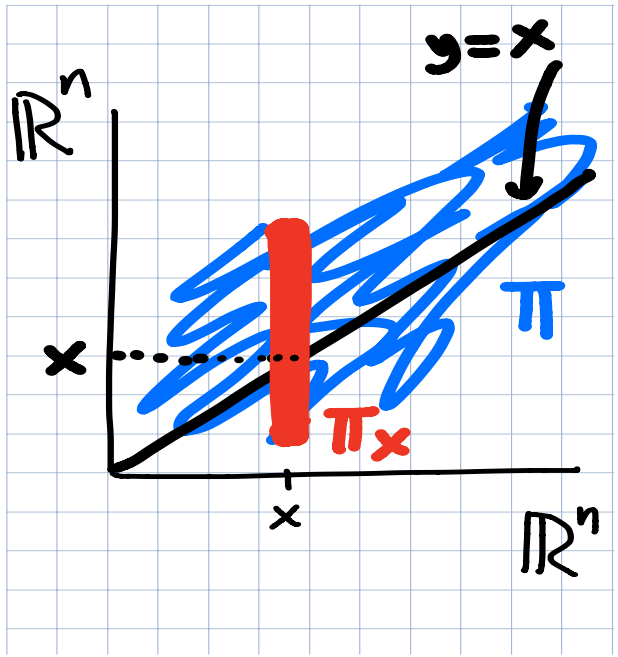
Assume:

- $c(x, y) = |x - y|$
- $\mu << \mathcal{L}^n$
- $\pi \in MT(\mu, \nu)$ *be optimal.*

Then the following holds:

- *There is concentration set of* π, $\Gamma \subset \mathbf{R}^n \times \mathbf{R}^n$ *such that*

$$\dim(\Gamma_x) \leq n - 1 \quad \textit{for } \mu\textit{-almost every } x,$$



A conjecture in higher dimensions. [Ghousshoub, K. & Lim]

Assume:

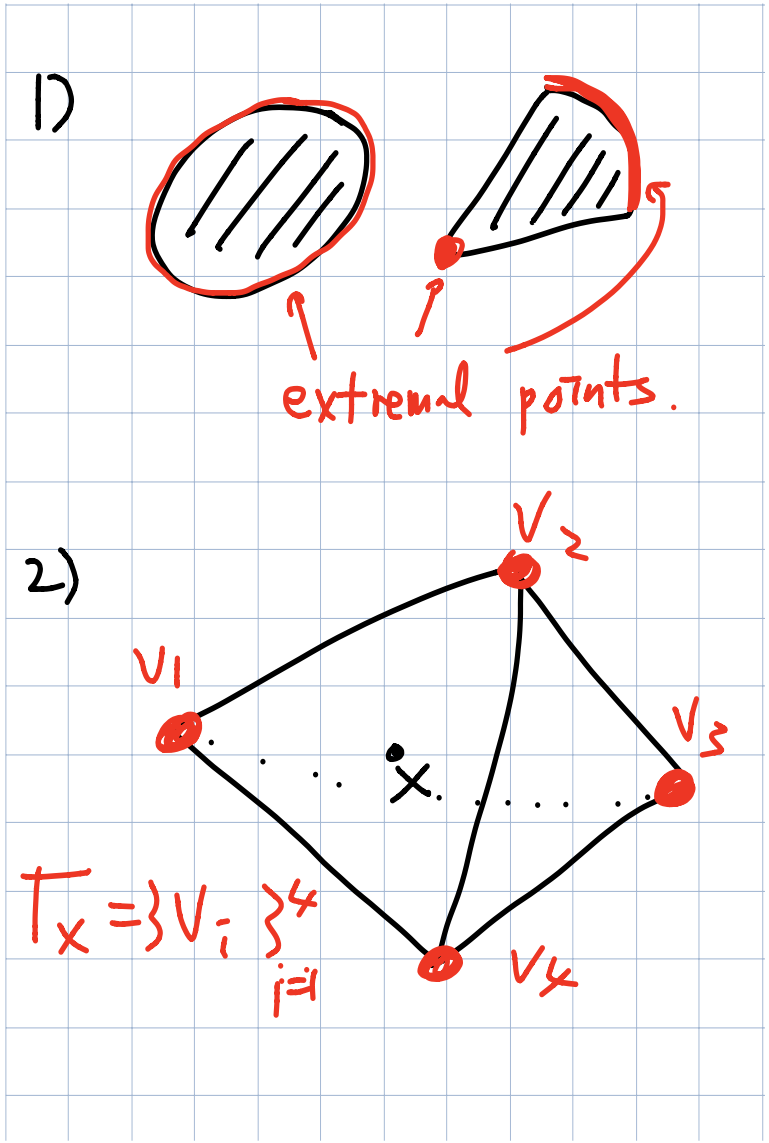
- $c(x, y) = |x - y|$
- $\mu << \mathcal{L}^n$
- $\pi \in MT(\mu, \nu)$ be optimal.

Conjecture: Then, $\exists$ concentration set Γ, such that for μ almost every x,

$$\overline{\Gamma}_x = \text{Ext}\left(\text{conv}(\overline{\Gamma}_x)\right).$$

Definition (Extremal points)

We say $p \in Ext(S)$
if $p \in [q_0, q_1]$ with $q_0, q_1 \in S, \Rightarrow p = q_0 = q_1$.



Progress towards the conjecture

Assume:

- $c(x, y) = |x - y|$
- $\mu << \mathcal{L}^n$
- $\pi \in MT(\mu, \nu)$ be optimal.

Conjecture: Then, $\exists$ concentration set Γ, such that for μ almost every x,

$$\overline{\Gamma}_x = \text{Ext}\left(\text{conv}(\overline{\Gamma}_x)\right).$$

Theorem (Ghoussuob, K. & Lim)

The conjecture holds in the following cases:

- $n = 2$, *or*
- ν *is obtained from* μ *by diffusion with respect to a time-dependent elliptic operator. More generally, if there is a stopping time* $T > 0$ *of a Brownian motion with* $B_0 \sim \mu$ *and* $B_T \sim \nu$.

Key principles

- Duality
- Localization (partition / disintegration)

Duality

- Duality (e.g, [Beiglböck-Juillet '13])
  - $$\inf_{\pi \in MT(\mu,\nu)} \int c(x,y) d\pi(x,y)$$
    $$= \sup\{ \int \beta(y) d\nu(y) - \int \alpha(x) d\mu(x) \ : \ \beta(y) \leq c(x,y) + \alpha(x) + \gamma(x) \cdot (y-x), \quad \forall x, y\}$$
  - $$\sup_{\pi \in MT(\mu,\nu)} \int c(x,y) d\pi(x,y)$$
    $$= \inf\{ \int \beta(y) d\nu(y) - \int \alpha(x) d\mu(x) \ : \ \beta(y) \geq c(x,y) + \alpha(x) + \gamma(x) \cdot (y-x), \quad \forall x, y\}$$
- If the maximizer/minimizer (α, β, γ) exists, then the set

$$\Gamma = \{(x,y) \mid \beta(y) = c(x,y) + \alpha(x) + \gamma(x) \cdot (y-x)\}$$

gives a concentration set of an optimal π.
In this case, we say "π **admits a dual.**"

Does an optimal martingale π attain dual functions?

Question Can one always have a dual (α, β, γ) for an optimal π?

Answer

No! [Beiglböck-Juillet '13]

Counterexample:

The maximization problem with $\mu = \nu$ on $[0, 1]$ does not admit dual functions.

Proof.

Exercise. Some details:

- Suppose the dual α, β, γ exist.
- $\mu = \nu \Rightarrow$ the plan π is the identity $x \mapsto x$, thus, $\Gamma = \{(x, x)\}$.
- $\beta(y) \geq |x - y| + \alpha(x) + \gamma(x)(y - x), \quad \forall x, y;$
  $\beta(x) = \alpha(x), \quad \forall x.$
- So, $\beta(y) \geq |x - y| + \beta(x) + \gamma(x)(y - x), \quad \forall x, y.$
- WLOG, let $\beta(0) = 0$ and $\gamma(0) = 0$.
- Continuous use of the above inequality will induce $\beta(x) = +\infty$, $\gamma(x) = +\infty$ at any point $x > 0$.

□

- In the same example, for the minimization problem, the optimal π^* admits the dual functions $\alpha = 0, \beta = 0, \gamma = 0$.

We do not know for the minimization problem in general.

There are cases where optimal dual functions exist

Theorem (Ghoussoub, K., & Lim)

Suppose

- $\mu << Leb$, *compactly supported and*
- ν *is obtained from* μ *by diffusion with respect to a time-dependent elliptic operator. More generally, if there is a stopping time* $T > 0$ *of a Brownian motion with* $B_0 \sim \mu$ *and* $B_T \sim \nu$.

Then *an optimal* $\pi \in MT(\mu, \nu)$ *admits dual functions.*

It is good to have those dual functions....

$\Gamma \subset \mathbf{R}^n \times \mathbf{R}^n$ is said to **admit a dual** if there exist functions α, β, γ such that

$$\Gamma \subset \{(x, y) \in \mathbf{R}^n \times \mathbf{R}^n | \ \beta(y) = c(x, y) + \alpha(x) + \gamma(x) \cdot (y - x)\}$$

and

$$\beta(y) \leq c(x, y) + \alpha(x) + \gamma(x) \cdot (y - x), \quad \forall (x, y) \in \mathbf{R}^n \times \mathbf{R}^n$$

(For maximization problem, use the reverse inequality.)

Dual functions are a.e. differentiable.

Lemma (Ghousshoub, K. & Lim)

Suppose

- $c(x, y)$ *is Lipschitz (e.g.* $|x - y|$*)*
- Γ *admits a dual* $\{\alpha, \beta, \gamma\}$

Then,

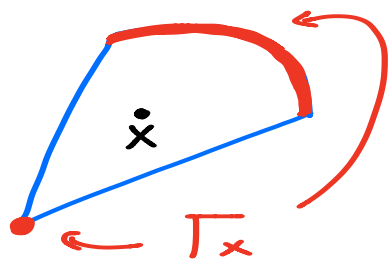
- α *is* **locally Lipschitz**.
- *Furthermore,* γ *is* **differentiable** *at points where* α *is differentiable.*

If differentiable dual functions are attained

Lemma (Ghousshoub, K. & Lim)

Let $c = |x - y|$. *Suppose* $\Gamma \subset \mathbf{R}^n \times \mathbf{R}^n$ **admits** *a dual* (α, β, γ).
Then *for every* x *where* α, γ *are differentiable,*

$$\overline{\Gamma}_x = \mathrm{Ext}\left(\mathrm{conv}(\overline{\Gamma}_x)\right).$$



Proof.

“ Differentiate the duality relation to get information!"

□

Proof continued.

- **duality relation** (for the minimization problem)

$$\beta(y) \leq c(x,y) + \alpha(x) + \gamma(x)\cdot(y-x)\ \forall x \in X_\Gamma, y \in Y_\Gamma,$$
$$\beta(y) = c(x,y) + \alpha(x) + \gamma(x)\cdot(y-x)\ \forall (x,y) \in \Gamma.$$

- If $(x,y) \in \Gamma$,

$$|x-y| + \gamma(x)\cdot(y-x) + \alpha(x) \leq |x'-y| + \gamma(x')\cdot(y-x') + \alpha(x')\ \ \forall x'$$
$$\Rightarrow \nabla_x(|x-y| + \gamma(x)\cdot(y-x) + \alpha(x))$$
$$= \frac{x-y}{|x-y|} + \nabla\gamma(x)\cdot(y-x) - \gamma(x) + \nabla\alpha(x) = 0.$$

- Now suppose that we can find $\{y, y_0, ..., y_s\} \subset \overline{\Gamma}_x$ with $y = \Sigma_{i=0}^s p_i y_i$, $\Sigma_{i=0}^s p_i = 1$, $p_i > 0$. Then out of $\frac{x-y}{|x-y|} + \nabla\gamma(x)\cdot(y-x) - \gamma(x) + \nabla\alpha(x) = 0 =$ $\sum_i p_i \left[\frac{x-y_i}{|x-y_i|} + \nabla\gamma(x)\cdot(y_i-x) - \gamma(x) + \nabla\alpha(x)\right]$ we get

$$\frac{x-y}{|x-y|} = \sum_{i=0}^{s} p_i \frac{x-y_i}{|x-y_i|}.$$

- But this can hold only if all y_i lie on the same ray emanated from x. Hence...

Summary: the conclusion under dual attainment

If there are dual functions α, β, γ for Γ

$\Rightarrow$ Differentiability (a.e.) of α, γ

$\Rightarrow$ Structure of Γ

Theorem (Ghousshoub, K. & Lim)

Let

- $c(x, y) = |x - y|$,
- $\mu << \mathcal{L}^n$,
- $\pi \in MT(\mu, \nu)$*: optimal solution for martingale transport problem.*

Suppose *that* π *admits a dual* (α, β, γ)*. Let*

$$\Gamma = \{(x, y) \in \mathbf{R}^d \times \mathbf{R}^d | \ \beta(y) = c(x, y) + \alpha(x) + \gamma(x) \cdot (y - x)\}.$$

Then Γ *is a concentration set of* π*, (i.e.* $\pi(\Gamma) = 1$*), and for* μ *a.e.* x*,*

$$\overline{\Gamma}_x = \mathrm{Ext}\left(\mathrm{conv}(\overline{\Gamma}_x)\right).$$

...The dual may not be attained. What shall we do?

Partition:
Make partition into duality attainable components!

For general cases first consider a weaker form of dual admitability:

Theorem (Beiglböck-Juillet '13)

Suppose

- $c : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ *continuous.*
- $\pi \in MT(\mu, \nu)$*: an optimal solution for martingale transport problem.*

Then *there exists a concentration set* $\Gamma \subset \mathbf{R}^n \times \mathbf{R}^n$*, (i.e.* $\pi(\Gamma) = 1$*)*
such that Γ *is* **monotone***,*
that is, any **finite subset** $H \subset \Gamma$ *admits a dual.*

Partition into dual attainable components.

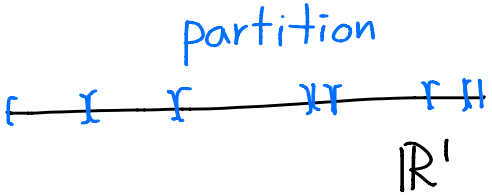
Theorem (Beiglböck-Juillet '13 for 1-dim, Ghousshoub, K. & Lim for general dim)

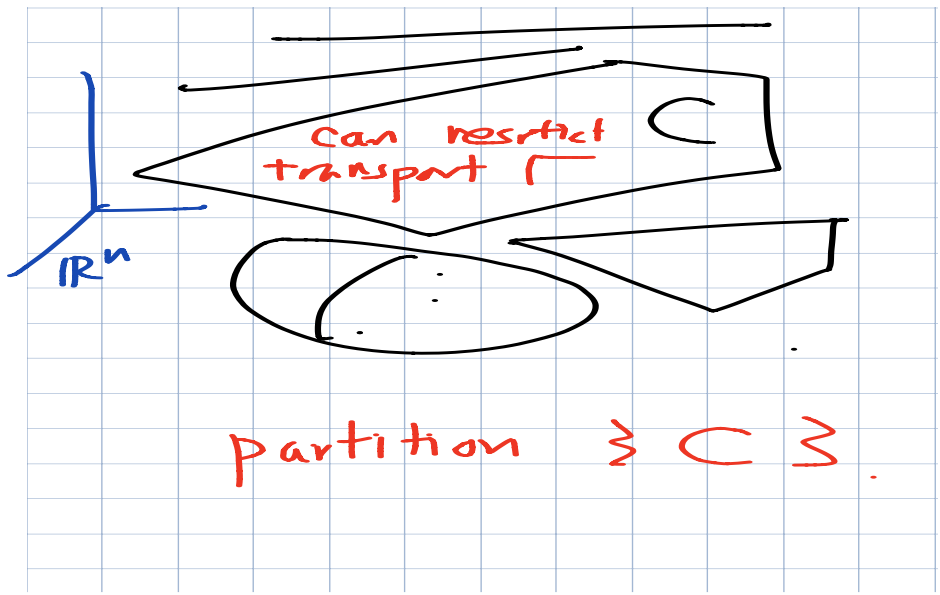
Suppose

- $c : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ *continuous.*
- $\pi \in MT(\mu, \nu)$*: an optimal solution for martingale transport problem.*

Then *there exists a* **monotone** *concentration set* $\Gamma \subset \mathbf{R}^n \times \mathbf{R}^n$*, (i.e.* $\pi(\Gamma) = 1$*):*

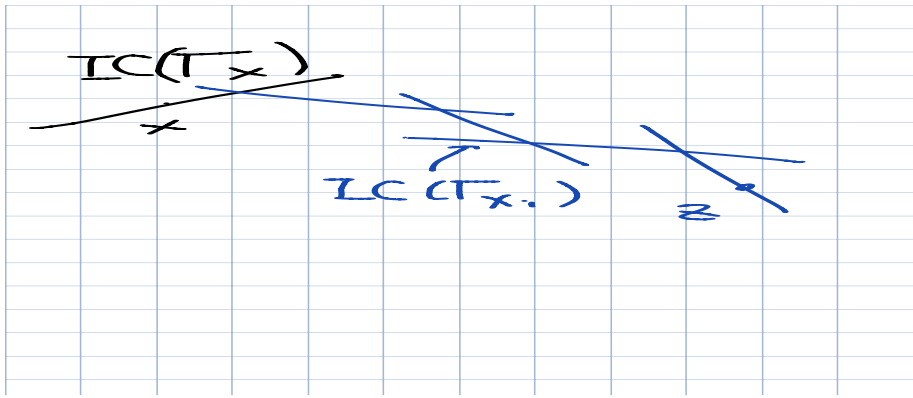
- *One can define mutually disjoint convex sets* $\{C\}$
- *such that "transport"* Γ *is partitioned on C's,*
- *and on each such component C, the set* Γ *admits a dual.*





Convex Partition in *n*-dimensions

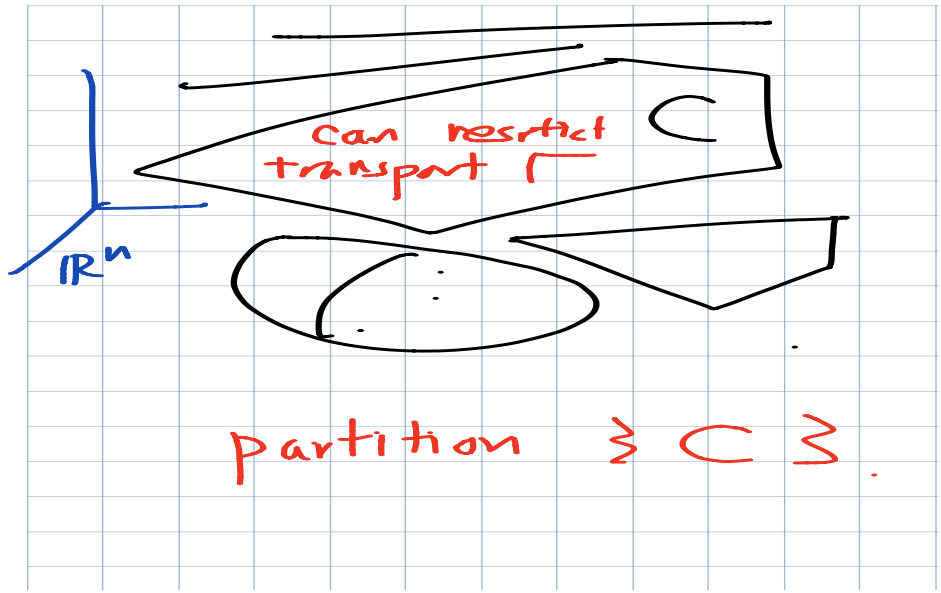
- $x \sim_1 z$ equivalence relation if there is a chain of $IC(\Gamma_x) := int(conv\Gamma_{x_i})$'s



- Get partition for $\sim_1$.
  Rmk: In 1-dim, we can stop here.
- Take convex hull for each component of $\sim_1$.
- Define equivalence relation $\sim_2$ using chains of those convex hulls
- Iterate this procedure on and on,
- to get equivalence relation $\sim$ and corresponding "convex" partition $\{C\}$ generated by Γ.
- It can be shown (highly nontrivial) that each such component C attains dual!

Disintegration along the convex partition

- Now, for each such component, dual is attained.



- The method of [Ghousoub, K. & Lim '15]:

  Disintegrate μ and ν into partition $\{C\}$, each of which attains dual.
  If the disintegration of μ on each C is **absolutely continuous**,
  then apply a.e. differentiability of the dual functions on each C,
  to get the structural result for μ-a.e. x.

Partition can be useful **only if** we know good disintegration of μ along it.

But unfortunately, getting such a **good disintegration** is NOT clear in general.

Nikodym set [Ambrosio, Kirchheim, and Pratelli '04]
There is a Nikodym set in $\mathbf{R}^3$,

having full measure in the unit cube,

intersecting each element of a family of pairwise disjoint open lines

only at one point.

This means, the point where we have differentiability of dual may not, in general, belong to the set we want.

Still can handle dimension question even without good disintegration:

Corollary (Ghousshoub, K. & Lim)

Suppose

- $c(x, y) = |x - y|$
- $\pi \in MT(\mu, \nu)$ *optimal*
- $\mu << \mathcal{L}^n$.

Then, *there is a concentration set* Γ *of* π*, such that for* μ*-almost every* x*,*

$$\dim \Gamma_x \leq n - 1.$$

Proof.

- If $\dim C = n$, then C is open, thus, μ can be restricted on C, so absolutely continuous on C! Apply previous results.
- For other components with $\dim C \leq n - 1$, but, in this case already the dimension is $\leq n - 1$.

□

A case with good disintegration: two dimensions

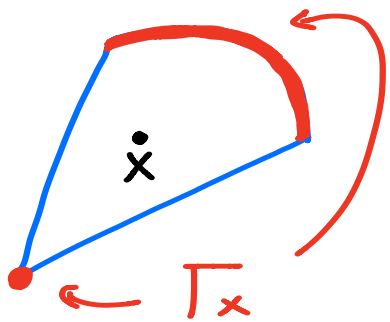
Theorem (Ghousshoub, K. & Lim $n = 2$)

Suppose

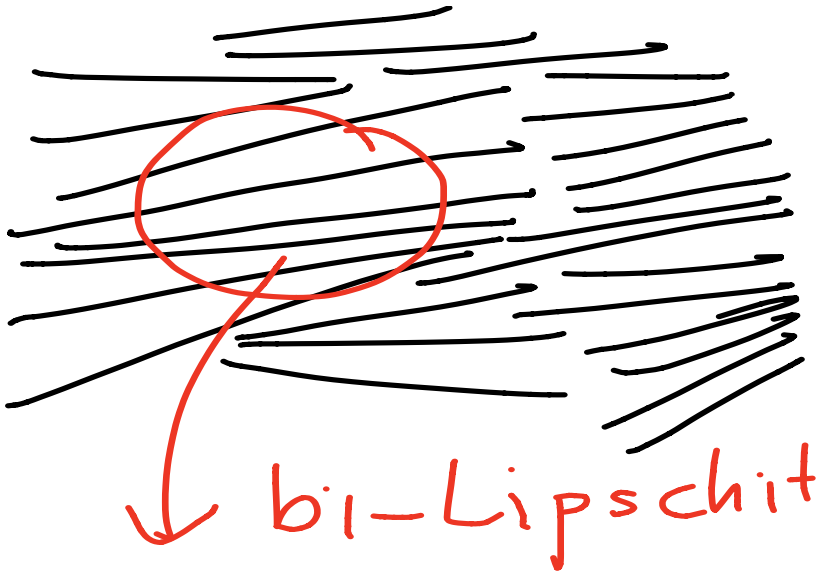
- $c(x, y) = |x - y|$,
- $\pi \in MT(\mu, \nu)$ *optimal,*
- $\mu << \mathcal{L}^n$,
- $n = 2$,

Then, *there is a concentration set* Γ *of* π*, such that for* μ*-almost every* x*,*

$$\overline{\Gamma}_x = \mathrm{Ext}(\mathrm{conv}(\overline{\Gamma}_x)).$$



Codimension ≤ 1 case. **Idea:** Flattening!



Summary:

- To study the structure of optimal martingale transport in $MT(\mu, \nu)$ with $\mu << \mathcal{L}^n$ in **general dimensions** n:
  - Find **optimal** martingale plan $\pi \in MT(\mu, \nu)$ using compactness.
  - Get a suitable **monotone set** Γ.
  - Apply the **partition** of Γ into **duality** attainable components C.
  - Get **dual functions** α, β, γ for Γ in C.
  - **Almost everywhere differentiability** of α, γ on C.
- If μ disintegrates into an **absolutely continuous measure** μ_C on each component C,
  - Get the **structure** of Γ (of Γ_x for μ_C a.e. x) in each C from **almost everywhere differentiability**,
  - thus finally, get the **structure** of Γ (of Γ_x for μ-a.e. x)!