Quiz 3-T-p

2017-10-19

Last name ....

First name ....

Student number ....

Email ....

Grade

For each computation of limits in this test, if the limit does not exist, indicate whether it diverges to $-\infty$ or $+\infty$.

1. Each part of this question is worth 1 mark.

(a) **(1pt)** Suppose $f(x)$ and $g(x)$ are differentiable functions such that $f(1) = 3$, $f'(1) = -1$, $g(1) = 1$ and $g'(1) = -2$. If $h(x)$ is defined by

$$h(x) = \sqrt{f(x) + g(x)}$$

compute $h'(1)$.

Solution. We have $h'(x) = \frac{f'(x)+g'(x)}{2\sqrt{f(x)+g(x)}}$. Hence

$$h'(1) = \frac{f'(1) + g'(1)}{2\sqrt{f(1) + g(1)}} = \frac{-1 - 2}{2\sqrt{3 + 1}} = -\frac{3}{4}$$

(b) **(1pt)** Find all values α such that $f'(0) = 2$ where $f(x) = \arccos(\alpha x)$.
Solution. We have $f'(x) = -\frac{\alpha}{\sqrt{1-(\alpha x)^2}}$. Thus $f'(0) = -\alpha$. Hence $f'(0) = 2$ if and only if $\alpha = -2$.

2. Each part of this question is worth 2 marks. **You have to show all your work in order to get credit.**

(a) **(2pt)** Let $f(x)$ be a differentiable function such that $f(1) = -3$ and $f'(1) = 5$. Evaluate the limit by identifying it as a derivative

$$\lim_{x\to 1} \frac{x(f(x))^2 - 9}{x - 1}$$

Solution. Let $g(x) = x(f(x))^2$. We note that $g(1) = (-3)^2 = 9$. Hence

$$\lim_{x\to 1} \frac{x(f(x))^2 - 9}{x - 1} = \lim_{x\to 1} \frac{g(x) - g(1)}{x - 1} = g'(1).$$

We have $g'(x) = (f(x))^2 + 2xf(x)f'(x)$. Hence

$$g'(1) = (f(1))^2 + 2f(1)f'(1) = (-3)^2 + 2(-3)5 = -21.$$

Thus,

$$\lim_{x\to 1} \frac{x(f(x))^2 - 9}{x - 1} = -21.$$

(b) **(2pt)** Given the equation

$$\cos(xy) + \arctan(y) = 2y + x$$

compute $\dfrac{dy}{dx}$ at the point $(x, y) = (1, 0)$.

Solution. Using implicit differentiation, we have

$$-\sin(xy)(y + xy') + \frac{y'}{1 + y^2} = 2y' + 1.$$

At the point $(x, y) = (1, 0)$, we obtain $0 + y' = 2y' + 1$. That is $y' = -1$.

3. This question is worth 4 marks. **You have to show all your work in order to get credit.**

Find the equation of the tangent line to the graph

$$y = \frac{1}{2}x^{x^2}(1+x^2)^x$$

at the point $(x, y) = (1, 1)$.

Solution. The slope of the tangent line is y'. In order to compute y', we will use logarithmic differentiation.

$$\ln y = \ln \frac{1}{2}x^{x^2}(1+x^2)^x = \ln \frac{1}{2} + x^2 \ln x + x \ln(1+x^2).$$

Hence,

$$\frac{y'}{y} = 2x \ln x + x + \ln(1+x^2) + \frac{2x^2}{1+x^2}.$$

At the point $(x, y) = (1, 1)$, we obtain

$$y' = 2 \ln 1 + 1 + \ln 2 + 1 = 2 + \ln 2.$$

Hence, the equation of the tangent line is $y - 1 = (2 + \ln 2)(x - 1)$ or $y = (2 + \ln 2)(x - 1) + 1$.

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For each computation of limits in this test, if the limit does not exist, indicate whether it diverges to $-\infty$ or $+\infty$.

1. Each part of this question is worth 1 mark.

(a) **(1pt)** Suppose $f(x)$ and $g(x)$ are differentiable functions such that $f(1) = 3$, $f'(1) = -1$, $g(1) = 1$ and $g'(1) = -2$. If $h(x)$ is defined by

$$h(x) = (f(x) + g(x))^2$$

compute $h'(1)$.

Solution. We have $h'(x) = 2(f'(x) + g'(x))(f(x) + g(x))$. Hence $h'(1) = 2(f'(1) + g'(1))(f(1) + g(1)) = 2(-1 - 2)(3 + 1) = -24$

(b) **(1pt)** Find all values α such that $f'(0) = -3$ where $f(x) = \arcsin(\alpha x)$.

Solution. We have $f'(x) = \frac{\alpha}{\sqrt{1-(\alpha x)^2}}$. Thus $f'(0) = \alpha$. Hence $f'(0) = -3$ if and only if $\alpha = -3$.

2. Each part of this question is worth 2 marks. **You have to show all your work in order to get credit.**

(a) **(2pt)** Let $f(x)$ be a differentiable function such that $f(1) = 4$ and $f'(1) = 4$. Evaluate the limit by identifying it as a derivative

$$\lim_{x\to 1} \frac{x\sqrt{f(x)} - 2}{x - 1}$$

Solution. Let $g(x) = x\sqrt{f(x)}$. We note that $g(1) = \sqrt{4} = 2$. Hence

$$\lim_{x\to 1} \frac{x\sqrt{f(x)} - 2}{x - 1} = \lim_{x\to 1} \frac{g(x) - g(1)}{x - 1} = g'(1).$$

We have $g'(x) = \sqrt{f(x)} + x\frac{f'(x)}{2\sqrt{f(x)}}$. Hence

$$g'(1) = \sqrt{f(1)} + \frac{f'(1)}{2\sqrt{f(1)}} = \sqrt{4} + \frac{4}{2\sqrt{4}} = 3.$$

Thus,

$$\lim_{x\to 1} \frac{x\sqrt{f(x)} - 2}{x - 1} = 3.$$

(b) **(2pt)** Given the equation

$$\sin(xy) + \arccos(y) - \frac{\pi}{2} = 2y + x^2 - 1$$

compute $\frac{dy}{dx}$ at the point $(x, y) = (1, 0)$.

Solution. Using implicit differentiation, we have

$$\cos(xy)(y + xy') - \frac{y'}{\sqrt{1 - y^2}} = 2y' + 2x.$$

At the point $(x, y) = (1, 0)$, we obtain $0 + y' - y' = 2y' + 2$. That is $y' = -1$.

3. This question is worth 4 marks. **You have to show all your work in order to get credit.**

Find the equation of the tangent line to the graph

$$y = \frac{(1+x)^{x^2}}{2x^x}$$

at the point $(x, y) = (1, 1)$.

Solution. The slope of the tangent line is y'. In order to compute y', we will use logarithmic differentiation.

$$\ln y = \ln \frac{(1+x)^{x^2}}{2x^x} = x^2 \ln(1+x) - \ln 2 - x \ln x.$$

Hence,

$$\frac{y'}{y} = 2x \ln(1+x) + \frac{x^2}{1+x} - \ln x - x\frac{1}{x}.$$

At the point $(x, y) = (1, 1)$, we obtain

$$y' = 2\ln 2 + \frac{1}{2} - \ln 1 - 1 = 2\ln 2 - \frac{1}{2}.$$

Hence, the equation of the tangent line is $y - 1 = (2\ln 2 - \frac{1}{2})(x - 1)$ or $y = (2\ln 2 - \frac{1}{2})(x - 1) + 1$.