Quiz 4

2017-11-02

Last name ....

First name ....

Student number ....

Email ....

Grade

For each computation of limits in this test, if the limit does not exist, indicate whether it diverges to $-\infty$ or $+\infty$.

1. (2pt) A turkey is taken out of a hot oven at 400 degrees Fahrenheit and placed on a table in the dining room which is at constant 70 degrees Fahrenheit. If after 25 minutes, the turkey has the temperature of 180 degrees Fahrenheit, then find the formula for the turkey's temperature with respect to time t.

Solution. By Newton's Law of Cooling, we have that the temperature of the turkey at time t (measured in minutes, starting from time $t = 0$ when the turkey is taken out of the oven) is given by a function $f(t)$ (measured in degrees Fahrenheit) satisfying the following equation:

$$f'(t) = k(f(t) - 70),$$

for some (negative) constant k. Then denoting $y(t) = f(t) - 70$, we get that $y'(t) = f'(t)$ and moreover,

$$y'(t) = ky(t),$$

which allows us to conclude that $y(t) = y(0)e^{kt}$. But $y(0) = f(0) - 70 = 400 - 70 = 330$, which yields that $y(t) = 330e^{kt}$. We compute the constant k using the information that $f(25) = 180$ which means that $y(25) = f(25) - 70 = 180 - 70 = 110$. So, using the formula $y(t) = 330e^{kt}$ along with the fact that $y(25) = 110$ yields

$$330e^{k\cdot 25} = 110,$$

which means that $e^{25k} = \frac{1}{3}$ and so, $25k = \ln(\frac{1}{3}) = -\ln(3)$. Therefore $k = -\frac{\ln(3)}{25}$ which means that $y(t) = 330e^{-\frac{\ln(3)}{25}\cdot t}$. Finally, using that $y(t) = f(t) - 70$, i.e., that $f(t) = 70 + y(t)$, we conclude that the function measuring the tempretaure of the turkey equals

$$f(t) = 70 + 330e^{-\frac{\ln(3)}{25}\cdot t}.$$

2. (a) **(2pt)** Estimate $\sqrt[3]{7}$ using a linear approximation.

Solution. For the function $f(x) = \sqrt[3]{x}$, we estimate $f(7)$ using the linear approximation based at $a = 8$ since $f(8) = \sqrt[3]{8} = 2$ and so, $\sqrt[3]{7}$ is approximated by

$$T_1(7) = f(8) + (7 - 8) \cdot f'(8).$$

We compute $f'(x) = \frac{1}{3\sqrt[3]{x^2}}$ and so, $f'(8) = \frac{1}{3 \cdot 2^2} = \frac{1}{12}$. Therefore the linear approximation to $\sqrt[3]{7}$ is

$$T_1(7) = 2 - \frac{1}{12} = \frac{23}{12}.$$

(b) **(2pt)** A particle is moving on the x-axis and its position at time t (measured in seconds) is given by $s(t) = t^2 - 4t$ (measured in meters). Find the total distance traveled by the particle in the first 3 seconds.

Solution. We determine the direction in which the particle is moving by computing its velocity and checking its sign; so, the velocity $v(t)$ equals $2t - 4$, which is positive for $t > 2$ and negative in the times interval 0 to 2 seconds. In other words, the particle traveled first backwards starting at $s(0) = 0$ until $s(2) = 2^2 - 4 \cdot 2 = -4$ and then traveled forward between $s(2) = -4$ and $s(3) = 3^2 - 4 \cdot 3 = -3$. Therefore, in total, the particle traveled

$$(0 - (-4)) + (-3 - (-4)) = 5 \text{ meters}.$$

3. (4pts) You have to show all your work in order to get credit.

Two particles A and B are moving in the xy-plane. Particle A starts at the point $(11, 0)$ and moves along the x-axis toward the origin with the constant speed v. Particle B starts at the point $(0, 6)$ and moves along the y-axis, away from the origin with the same speed v. The rate of change of the distance between the two particles is equal to 5 units per minute at the moment when the distance between the two particles is equal to 13 units. Find the common speed v of the two particles.

Solution. We denote by $x(t)$ the position of particle A on the x-axis at time t and we denote by $y(t)$ the position of particle B on the y-axis at time t. Since both particles travel with the same constant speed v and also, we know where they both start and know their direction, we conclude that

$$x(t) = 11 - v \cdot t \text{ and } y(t) = 6 + v \cdot t.$$

First we find the position of the two particles when the distance between them is 13 units; this happens at some time t_0, which must be positive. We let $\alpha = v \cdot t_0$; then also α must be positive. From the above equations we have that

$$x(t_0) = 11 - \alpha \text{ and } y(t_0) = 6 + \alpha.$$

Since the distance between the two particles at time t_0 is 13 units, then using Pythagoras' theorem, we conclude that

$$x(t_0)^2 + y(t_0)^2 = 13^2.$$

So, $(11-\alpha)^2+(6+\alpha)^2 = 169$ and then after expanding and then collecting the terms, we obtain

$$(121 + 36 - 169) + (-22\alpha + 12\alpha) + 2\alpha^2 = 0,$$

i.e., $-12 - 10\alpha + 2\alpha^2 = 0$, which after dividing by 2 yields the equation

$$\alpha^2 - 5\alpha - 6 = 0 \text{ i.e. } (\alpha - 6)(\alpha + 1) = 0,$$

with solutions $\alpha_1 = 6$ and $\alpha_2 = -1$. However, only the positive solution makes sense for this problem (because the time t_0 is positive and $\alpha = v \cdot t_0$); thus the distance between the two particles is 13 units when particle A is at $x(t_0) = 11 - 6 = 5$ and particle B is at $y(t_0) = 6 + 6 = 12$.

Now, we denote by $D(t)$ the distance between the two particles at any time t (≥ 0); then from Pythagoras' theorem we obtain that

$$D(t)^2 = x(t)^2 + y(t)^2$$

at any time t. Then differentiating with respect to t yields

$$2D(t)D'(t) = 2x(t)x'(t) + 2y(t)y'(t).$$

Now, we use the information that $x'(t) = -v$ while $y'(t) = v$ (for any t), while we are given that $D'(t_0) = 5$. Also, we know that $D(t_0) = 13$, $x(t_0) = 5$ and $y(t_0) = 12$. Using all of these in the equation

$$2D(t_0)D'(t_0) = 2x(t_0)x'(t_0) + 2y(t_0)y'(t_0)$$

yields

$$2 \cdot 13 \cdot 5 = 2 \cdot 5 \cdot (-v) + 2 \cdot 12 \cdot v$$

and so (after dividing by 2 and simplifying the above expression), we have $65 = 7v$ and so, $v = \frac{65}{7}$ units per minute.

Quiz 4 (Friday-s)

2017-11-02

Last name ....

First name ....

Student number ....

Email ....

Grade

For each computation of limits in this test, if the limit does not exist, indicate whether it diverges to $-\infty$ or $+\infty$.

1. **(2pt)** A lasagna is taken out of a hot oven at 170 degrees Fahrenheit and placed on a table in the dining room which is at constant 70 degrees Fahrenheit. After 20 minutes, the lasagna has the temperature of 120 degrees Fahrenheit Find the formula for the lasagna's temperature with respect to time t. **You have to show all your work in order to get credit.**

Solution. By Newton's Law of Cooling, we have that the temperature of the lasagna at time t (measured in minutes, starting from time $t = 0$ when the lasagna is taken out of the oven) is given by a function $f(t)$ (measured in degrees Fahrenheit) satisfying the following equation:

$$f'(t) = k(f(t) - 70),$$

for some (negative) constant k. Then denoting $y(t) = f(t) - 70$, we get that $y'(t) = f'(t)$ and moreover,

$$y'(t) = ky(t),$$

which allows us to conclude that $y(t) = y(0)e^{kt}$. But $y(0) = f(0) - 70 = 170 - 70 = 100$, which yields that $y(t) = 100e^{kt}$. We compute the constant k using the information that $f(20) = 120$ which means that $y(20) = f(20) - 70 = 120 - 70 = 50$. So, using the formula $y(t) = 100e^{kt}$ along with the fact that $y(20) = 50$ yields

$$100e^{k \cdot 20} = 50,$$

which means that $e^{20k} = \frac{1}{2}$ and so, $20k = \ln(\frac{1}{2}) = -\ln(2)$. Therefore $k = -\frac{\ln(2)}{20}$ which means that $y(t) = 100e^{-\frac{\ln(2)}{20} \cdot t}$. Finally, using that $y(t) = f(t) - 70$, i.e., that $f(t) = 70 + y(t)$, we conclude that the function measuring the temperature of the lasagna equals

$$f(t) = 70 + 100e^{-\frac{\ln(2)}{20} \cdot t}.$$

2. (a) **(2pt)** Estimate $\sqrt[3]{9}$ using a linear approximation.

Solution. For the function $f(x) = \sqrt[3]{x}$, we estimate $f(9)$ using the linear approximation based at $a = 8$ since $f(8) = \sqrt[3]{8} = 2$ and so, $\sqrt[3]{9}$ is approximated by

$$T_1(9) = f(8) + (9 - 8) \cdot f'(8).$$

We compute $f'(x) = \frac{1}{3\sqrt[3]{x^2}}$ and so, $f'(8) = \frac{1}{3 \cdot 2^2} = \frac{1}{12}$. Therefore the linear approximation to $\sqrt[3]{9}$ is

$$T_1(9) = 2 + \frac{1}{12} = \frac{25}{12}.$$

(b) **(2pt)** A particle is moving on the x-axis and its position at time t (measured in seconds) is given by $s(t) = 12t - 3t^2$ (measured in meters). Find the **total distance traveled** by the particle in the first 3 seconds.

Solution. We determine the direction in which the particle is moving by computing its velocity and checking its sign; so, the velocity $v(t)$ equals $12 - 6t$, which is positive for $t < 2$ and negative in the time interval 2 to 3 seconds. In other words, the particle traveled first forward starting at $s(0) = 0$ until $s(2) = 12(2) - 3(2)^2 = 12$ and then traveled backward between $s(2) = 12$ and $s(3) = 12(3) - 3(3)^2 = 9$. Therefore, in total, the particle traveled

$$(12 - 0) + (12 - 9) = 15 \text{ meters.}$$

3. (4pts) Two particles A and B are moving in the xy-plane. Particle A starts at the point $(10, 0)$ and moves along the x-axis away from the origin with the constant speed v. Particle B starts at the point $(0, 7)$ and moves along the y-axis, toward the origin with the same speed v. The rate of change of the distance between the two particles is equal to 3 units per minute at the moment when the distance between the two particles is equal to 13 units. Find the common speed v of the two particles.

Solution. We denote by $x(t)$ the position of particle A on the x-axis at time t and we denote by $y(t)$ the position of particle B on the y-axis at time t. Since both particles travel with the same constant speed v and also, we know where they both start and know their direction, we conclude that

$$x(t) = 10 + v \cdot t \text{ and } y(t) = 7 - v \cdot t.$$

First we find the position of the two particles when the distance between them is 13 units; this happens at some time t_0, which must be positive. We let $\alpha = v \cdot t_0$; then also α must be positive. From the above equations we have that

$$x(t_0) = 10 + \alpha \text{ and } y(t_0) = 7 - \alpha.$$

Since the distance between the two particles at time t_0 is 13 units, then using Pythagoras' theorem, we conclude that

$$x(t_0)^2 + y(t_0)^2 = 13^2.$$

So, $(10+\alpha)^2+(7-\alpha)^2 = 169$ and then after expanding and then collecting the terms, we obtain

$$(100 + 49 - 169) + (20\alpha - 14\alpha) + 2\alpha^2 = 0,$$

i.e., $-20 + 6\alpha + 2\alpha^2 = 0$, which after dividing by 2 yields the equation

$$\alpha^2 + 3\alpha - 10 = 0 \text{ i.e. } (\alpha - 2)(\alpha + 5) = 0,$$

with solutions $\alpha_1 = 2$ and $\alpha_2 = -5$. However, only the positive solution makes sense for this problem (because the time t_0 is positive and $\alpha = v \cdot t_0$); thus the distance between the two particles is 13 units when particle A is at $x(t_0) = 10 + 2 = 12$ and particle B is at $y(t_0) = 7 - 2 = 5$.

Now, we denote by $D(t)$ the distance between the two particles at any time t (≥ 0); then from Pythagoras' theorem we obtain that

$$D(t)^2 = x(t)^2 + y(t)^2$$

at any time t. Then differentiating with respect to t yields

$$2D(t)D'(t) = 2x(t)x'(t) + 2y(t)y'(t).$$

Now, we use the information that $x'(t) = v$ while $y'(t) = -v$ (for any t), while we are given that $D'(t_0) = 3$. Also, we know that $D(t_0) = 13$, $x(t_0) = 12$ and $y(t_0) = 5$. Using all of these in the equation

$$2D(t_0)D'(t_0) = 2x(t_0)x'(t_0) + 2y(t_0)y'(t_0)$$

yields

$$2 \cdot 13 \cdot 3 = 2 \cdot 12 \cdot (v) + 2 \cdot 5 \cdot (-v)$$

and so (after dividing by 2 and simplifying the above expression), we have $39 = 7v$ and so, $v = \frac{39}{7}$ units per minute.