Math 100. Quiz 4. Solutions 2017-11-02 (Thursday) **Time 25min**

Section . . . . Instructor name . . . .

Your email . . . .

- **For each computation of limits in this test, if the limit does not exist, indicate whether it diverges to $-\infty$ or $+\infty$.**
- Simplify all your answers as much as possible and express answers in terms of fractions or constants such as $\frac{1}{100}$, $\sqrt{e}$ or $\ln(4)$ rather than decimals.

1. This question is worth 2 mark. **You have to show all your work in order to get credit.**

(2pt) A man 6 feet tall is walking with a speed of 5 feet per second away from a 20 feet high lamp post. At what rate is the length of the man's shadow changing when the man is 15 feet away from the lamp post?

Solution. At each moment in time t, we denote by $\ell(t)$ the length of the man's shadow and we denote by $s(t)$ the distance between the man and the lamp post; both functions $\ell(t)$ and $s(t)$ are measured in feet, while the variable t is measured in seconds. Using similar triangles we obtain

$$\frac{\ell(t)}{\ell(t)+s(t)} = \frac{6}{20}.$$

So, $20\ell(t) = 6\ell(t) + 6s(t)$, which yields $14\ell(t) = 6s(t)$. Therefore $\ell(t) = \frac{6}{14} \cdot s(t)$ and so, its rate of change equals

$$\ell'(t) = \frac{6}{14} \cdot s'(t) = \frac{3}{7} \cdot 5 = \frac{15}{7} \text{ ft/s}$$

since $s'(t)$ represents the man's velocity, given to be equal to 5 (feet per second).

2. Each part of this question is worth 2 marks. **You have to show all your work in order to get credit.**

(a) **(2pt)** Estimate $\sqrt[4]{17}$ using a linear approximation.

Solution. For the function $f(x) = \sqrt[4]{x}$, we estimate $f(17)$ using the linear approximation based at $a = 16$ since $f(16) = \sqrt[4]{16} = 2$ and so, $\sqrt[4]{17}$ is approximated by

$$T_1(17) = f(16) + (17 - 16) \cdot f'(16).$$

We compute $f'(x) = \frac{1}{4\sqrt[4]{x^3}}$ and so, $f'(16) = \frac{1}{4 \cdot 2^3} = \frac{1}{32}$. Therefore the linear approximation to $\sqrt[4]{17}$ is

$$T_1(17) = 2 + \frac{1}{32} = \frac{65}{32}.$$

(b) **(2pt)** A curve $y = f(x)$ passes through the point $(2, 6)$ and has the property that at each point, the slope of the tangent line at the curve is three times the y-coordinate of that point. Find the function $f(x)$.

Solution. We are given that the equation of the curve satisfies $y'(x) = 3y(x)$ for each x and so, $y(x) = y(0)e^{3x}$. Since we are told that the point $(2, 6)$ is on the curve, i.e., $y(2) = 6$, then we get that

$$y(0)e^{3 \cdot 2} = 6,$$

which yields $y(0) = \frac{6}{e^6}$. Therefore, the equation of our curve is

$$y(x) = \frac{6}{e^6} \cdot e^{3x} = 6e^{3(x-2)}.$$

3. This question is worth 4 marks. **You have to show all your work in order to get credit.**

The function $f(x)$ satisfies the following equation (similar to Newton's law of cooling) for its derivative:

$$f'(x) = K(f(x) + C),$$

for some constants C and K. Find $f(x)$ if

$$\lim_{x\to+\infty} e^{-2x} f(x) = 5$$

and

$$\lim_{x\to-\infty} f(x) = 3.$$

Solution. Because the equation satisfied by f and its derivative is similar to the equation appearing in the Newton's Law of cooling, we will use the same strategy as in the Newton's Law of cooling. So, we let $y(x) = f(x) + C$; then $y'(x) = f'(x)$ and moreover, $y'(x) = K \cdot y(x)$. Therefore, $y(x) = y(0) \cdot e^{Kx}$. We let $y(0) = L$ and so, noting that $f(x) = y(x) - C$, we get that

$$f(x) = -C + Le^{Kx}.$$

Next we use the information regarding the limits at ∞ and at $-\infty$. So,

$$5 = \lim_{x\to\infty} e^{-2x} f(x) = \lim_{x\to\infty} e^{-2x} \cdot \left(-C + Le^{Kx}\right) = \lim_{x\to\infty} -Ce^{-2x} + Le^{Kx-2x}.$$

Regardless of the value of C, we have that $\lim_{x\to\infty} Ce^{-2x} = 0$, and so, we obtain that

$$5 = \lim_{x\to\infty} Le^{(K-2)x}.$$

This means that $K = 2$; indeed, otherwise if we would have $K > 2$, then $e^{(K-2)x}$ grows to infinity as $x \to \infty$, while if $K < 2$, then $e^{(K-2)x}$ converges to 0 as $x \to \infty$. So, if $K > 2$ then $Le^{(K-2)x}$ diverges (unless $L = 0$, in which case the limit equals 0, so it doesn't equal 5); thus K is not larger than 2. Also, if $K < 2$, then $Le^{(K-2)x}$ converges to 0 regardless of L, thus the limit is again not equal to 5.

In conclusion, $K = 2$ and then $5 = \lim_{x\to\infty} Le^{(K-2)x} = \lim_{x\to\infty} Le^0 = L$ yields $L = 5$. So, $f(x) = -C + 5e^{2x}$; we compute C using that

$$3 = \lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} -C + 5e^{2x}.$$

Because $\lim_{x\to-\infty} e^{2x} = 0$, we conclude that $C = -3$. Thus $f(x) = 3 + 5e^{2x}$.

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1. This question is worth 2 mark. **You have to show all your work in order to get credit.**

(2pt) A spot light is on the ground 12 ft away from a wall and a 6 ft tall person is walking towards the wall at a rate of 2 ft/sec. How fast is the height of the shadow changing when the person is 6 ft from the wall?

Solution. At each moment in time t, we denote by $h(t)$ the height of the person's shadow and we denote by $s(t)$ the distance between the person and the wall; both functions $h(t)$ and $s(t)$ are measured in feet, while the variable t is measured in seconds. We know that $s'(t) = -2$ ft/sec. Note also that the distance between the person and the spot light is $12 - s(t)$.

Using similar triangles we obtain

$$\frac{12 - s(t)}{12} = \frac{6}{h(t)}.$$

Thus $h(t) = \frac{72}{12-s(t)}$ and its rate of change equals $h'(t) = \frac{72}{(12-s(t))^2} \cdot s'(t)$. Therefore, at the instant T when $s(T) = 6$, the rate of change of the height of the shadow is

$$h'(T) = \frac{72}{(12-6)^2} \cdot (-2) = -4 \text{ ft/s}.$$

2. Each part of this question is worth 2 marks. **You have to show all your work in order to get credit.**

(a) **(2pt)** Estimate $\sqrt[4]{15}$ using a linear approximation.

Solution. For the function $f(x) = \sqrt[4]{x}$, we estimate $f(17)$ using the linear approximation based at $a = 16$ since $f(16) = \sqrt[4]{16} = 2$ and so, $\sqrt[4]{15}$ is approximated by

$$T_1(15) = f(16) + (15 - 16) \cdot f'(16).$$

We compute $f'(x) = \frac{1}{4\sqrt[4]{x^3}}$ and so, $f'(16) = \frac{1}{4 \cdot 2^3} = \frac{1}{32}$. Therefore the linear approximation to $\sqrt[4]{15}$ is

$$T_1(15) = 2 - \frac{1}{32} = \frac{63}{32}.$$

(b) **(2pt)** A curve $y = f(x)$ passes through the point $(1, 2)$ and has the property that at each point, the slope of the tangent line at the curve is four times the y-coordinate of that point. Find the function $f(x)$.

Solution. We are given that the equation of the curve satisfies $y'(x) = 4y(x)$ for each x and so, $y(x) = y(0)e^{4x}$. Since we are told that the point $(1, 2)$ is on the curve, i.e., $y(1) = 2$, then we get that

$$y(0)e^{4 \cdot 1} = 2,$$

which yields $y(0) = \frac{2}{e^4}$. Therefore, the equation of our curve is

$$y(x) = \frac{2}{e^4} \cdot e^{4x} = 2e^{4(x-1)}.$$

3. This question is worth 4 marks. **You have to show all your work in order to get credit.**

The function $f(x)$ satisfies the following equation (similar to Newton's law of cooling) for its derivative:

$$f'(x) = K(f(x) + C),$$

for some constants C and K. Find $f(x)$ if

$$\lim_{x\to-\infty} e^x f(x) = 2$$

and

$$\lim_{x\to+\infty} f(x) = 4.$$

Solution. Because the equation satisfied by f and its derivative is similar to the equation appearing in the Newton's Law of cooling, we will use the same strategy as in the Newton's Law of cooling. So, we let $y(x) = f(x) + C$; then $y'(x) = f'(x)$ and moreover, $y'(x) = K \cdot y(x)$. Therefore, $y(x) = y(0) \cdot e^{Kx}$. We let $y(0) = L$ and so, noting that $f(x) = y(x) - C$, we get that

$$f(x) = -C + Le^{Kx}.$$

Next we use the information regarding the limits at ∞ and at $-\infty$. So,

$$2 = \lim_{x\to-\infty} e^x f(x) = \lim_{x\to-\infty} e^x \cdot \left(-C + Le^{Kx}\right) = \lim_{x\to-\infty} -Ce^x + Le^{Kx+x}.$$

Regardless of the value of C, we have that $\lim_{x\to-\infty} Ce^x = 0$, and so, we obtain that

$$2 = \lim_{x\to-\infty} Le^{(K+1)x}.$$

This means that $K = -1$; indeed, otherwise if we would have $K < -1$, then $e^{(K+1)x}$ grows to infinity as $x \to -\infty$, while if $K > -1$, then $e^{(K+1)x}$ converges to 0 as $x \to -\infty$. So, if $K < -1$ then $Le^{(K+1)x}$ diverges (unless $L = 0$, in which case the limit equals 0, so it doesn't equal 2); thus K is not smaller than -1. Also, if $K > -1$, then $Le^{(K+1)x}$ converges to 0 regardless of L, thus the limit is again not equal to 2.

In conclusion, $K = -1$ and then $2 = \lim_{x\to-\infty} Le^{(K+1)x} = \lim_{x\to-\infty} Le^0 = L$ yields $L = 2$. So, $f(x) = -C + 2e^{-x}$; we compute C using that

$$4 = \lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} -C + 2e^{-x}.$$

Because $\lim_{x\to\infty} e^{-x} = 0$, we conclude that $C = -4$. Thus $f(x) = 4 + 2e^{-x}$.