**Math 100. Quiz 5** 2017-11-17 (Friday) **Time 25min**

Section ......... Instructor name ......................................

Your email ...............................................................

- **For each computation of limits in this test, if the limit does not exist, indicate whether it diverges to $-\infty$ or $+\infty$.**
- Simplify all your answers as much as possible and express answers in terms of fractions or constants such as $\frac{1}{100}$, $\sqrt{e}$ or $\ln(4)$ rather than decimals.

**1.** Each part of this question is worth 1 mark, and the correct answer will get the full mark.

(a) **(1pt)** Let $f(x) = x^4 + 3x^2 + 8$, and let $T_3(x)$ be its third-degree Taylor polynomial about $x = 1$. Find $T_3''(1)$.

**Solution.** The third-degree Taylor polynomial about 1 satisfies

$$T_3''(1) = f''(1).$$

We have that $f'(x) = 4x^3 + 6x$, $f''(x) = 12x^2 + 6$, then $T_3''(1) = 18$ and therefore

$$T_3''(1) = 18.$$

(b) **(1pt)** Find the smallest value for the parameter $a$ such that the function

$$f(x) = (x + a)e^x$$

is increasing on the interval $(-1, \infty)$.

**Solution.** The function $f(x)$ is increasing at $x$ if and only if $f'(x) > 0$. We have

$$f'(x) = (x + a + 1)e^x.$$

Note that $e^x > 0$ for all $x$, then $f'(x) > 0$ if and only if $x + a + 1 > 0$. It follows that $f$ is increasing for $x > -a - 1$, that is the interval $(-a - 1, \infty)$. Now, $a = 0$ the smallest value such that $f$ is increasing on $(-1, \infty)$.

**2.** Each part of this question is worth 2 marks. **You have to show all your work in order to get credit.**

(a) **(2pt)** Find the $x$-coordinates of the global minimum points for

$$f(x) = \frac{1}{\sqrt{x}} + \sqrt{x}$$

on the interval $[\frac{1}{4}, 4]$.

**Solution.** The function $f(x)$ is differentiable in $(\frac{1}{4}, 4)$, so there is no singular point, and we only need to compare the values of $f(x)$ at critical points and endpoints. First we find the critical points:

$$f'(x) = -\frac{1}{2}x^{-3/2} + \frac{1}{2}x^{-\frac{1}{2}},$$

then $f'(x) = 0$ when $-x^{-3/2} + x^{-\frac{1}{2}} = 0$. By multiplying $x^{3/2}$ we see that the latter implies $-1 + x = 0$ and hence $x = 1$. Therefore $x = 1$ is a critical point. (By plugging-in, we can check $f'(1) = 0$.) Next, we compare $f(1)$ with the values at the endpoints $x = \frac{1}{4}$ and $x = 4$. We have

$$f(\frac{1}{4}) = 2 + \frac{1}{2} \qquad f(4) = \frac{1}{2} + 2 \qquad f(1) = 1 + 1 = 2.$$

Therefore, <u>global minimum is at $x = 1$.</u>

(b) **(2pt)** Consider the function

$$f(t) = t^2 + \cos(t).$$

defined for all real values $t$. Prove that it has at most one **critical point**.

**Solution** Suppose that there are two critical points $t_0, t_1$. Then Rolle's Theorem implies that there exists $t_2$ between $t_0$ and $t_1$ such that $f'(t_2) = 0$. However we have

$$f''(t) = 2 - \cos(t)$$

Since $2 - \cos(t) > 0$ for all $t$, then $t_2$ and hence two critical points cannot exist.

**3. You have to show all your work in order to get credit.**
Let $f(x) = \ln(1+3x)$.

(a) **(1pt)** Use the 2nd degree Taylor polynomial to estimate $f(1/9)$.

(b) **(2pt)** Show that the error (in absolute value) of your estimate is smaller than $3^{-4}$.

(c) **(1pt)** Determine whether your estimate is an overestimate or underestimate. You have to justify your answer.

**Solution.**

Denote by $T_2(x)$ the 2nd degree Taylor polynomial of $f$ about $x = 0$ and let $R_2(x)$ be the remainder. Compute

$$f'(x) = \frac{3}{1+3x}, \quad f''(x) = -\frac{3^2}{(1+3x)^2}, \quad f^{(3)}(x) = \frac{2\cdot 3^3}{(1+3x)^3}.$$

Then $f(0) = 0$, $f'(0) = 3$ and $f''(0) = -9$ so the Taylor polynomial is

$$T_2(x) = 3x - \frac{9}{2}x^2.$$

Then the approximation value is $\underline{T_2\left(\frac{1}{9}\right) = \frac{1}{3} - \frac{1}{18}}.$

Write the remainder in terms of the Lagrange remainder formula:

$$R_2(1/9) = \frac{f^{(3)}(y)}{3!}(1/9)^3,$$

for some $y$ between 0 and 1/9. Note that $f^{(3)}(y) = 2\cdot 3^3/(1+3y)^3$ is a decreasing function and positive for $y > 0$, then

$$f^{(3)}(y) < f^{(3)}(0) = 2\cdot 3^3, \qquad \text{for all } 0 < y < \frac{1}{9}.$$

Use this bound for the Lagrange remainder formula above, and we get

$$\underline{0 < R_2\left(1/9\right) < \frac{2\cdot 3^3}{3!}\left(\frac{1}{3^2}\right)^3 = \frac{1}{3^4}}$$

This shows that the error is less than $3^{-4}$. As the remainder $R_2(1/9)$ is positive, this is <u>an underestimate.</u>

**Math 100. Quiz 5** 2017-11-17 (Friday-p) **Time 25min**

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- **For each computation of limits in this test, if the limit does not exist, indicate whether it diverges to $-\infty$ or $+\infty$.**
- Simplify all your answers as much as possible and express answers in terms of fractions or constants such as $\frac{1}{100}$, $\sqrt{e}$ or $\ln(4)$ rather than decimals.

**1.** Each part of this question is worth 1 mark, and the correct answer will get the full mark.

(a) **(1pt)** Let $f(x) = x^4 - 4x^2 + x + 2$, and let $T_3(x)$ be its third-degree Taylor polynomial about $x = 1$. Evaluate $T_3''(1)$.

**Solution.** The third-degree Taylor polynomial about 1 satisfies

$$T_3''(1) = f''(1).$$

We have that $f'(x) = 4x^3 - 8x + 1$, $f''(x) = 12x - 8$, then $T_3''(1) = 4$.

(b) **(1pt)** Find the largest value for the parameter $a$ such that the function

$$f(x) = (x - a)e^{-x}$$

is decreasing on the interval $(-1, \infty)$.

**Solution.** The function $f(x)$ is decreasing at $x$ if and only if $f'(x) < 0$. We have

$$f'(x) = e^{-x} - (x - a)e^{-x} = e^{-x}(1 - x + a).$$

Note that $e^{-x} > 0$ for all $x$, then $f'(x) < 0$ if and only if $1 - x + a < 0$. It follows that $f(x)$ is decreasing for $1 + a < x$, that is the interval $(1 + a, \infty)$. Therefore $a = -2$ is the largest value such that $f(x)$ is decreasing on $(-1, \infty)$.

**2.** Each part of this question is worth 2 marks. **You have to show all your work in order to get credit.**

(a) **(2pt)** Find the $x$-coordinates of the global minimum points for $f(x) = \dfrac{2x}{1+x^2}$ on the interval $[-2, 2]$.

**Solution.** Compute the derivative:

$$f'(x) = \frac{(1+x^2)(2) - 2x(2x)}{(1+x^2)^2} = \frac{2(1-x^2)}{(1+x^2)^2},$$

Note that $f'(x)$ is defined everywhere therefore there are no singular points. Then $f'(x) = 0$ when $1 - x^2 = 0$ and so $x = \pm 1$. Therefore $x = 1, -1$ are critical points in $[-2, 2]$.

Compare values at critical points and end points:

$$f(-2) = -\frac{4}{5} \qquad f(-1) = -1 \qquad f(1) = 1 \qquad f(2) = \frac{4}{5}.$$

Therefore, global minimum is at $x = -1.$

(b) **(2pt)** Consider the function $f(t) = \cos(t) - t^2 + 1$ defined for all real values $t$. Prove that it has at most one **critical point**.

**Solution** Suppose that there are two critical points $t_0, t_1$, ie. $f'(t_0) = f'(t_1) = 0$. Rolle's Theorem implies that there exists $t_2$ between $t_0$ and $t_1$ such that $f''(t_2) = 0$. However we have

$$f''(t) = -\cos(t) - 2$$

Since $-\cos(t) - 2 < 0$ for all $t$, then $t_2$ and hence two critical points cannot exist.

**3. You have to show all your work in order to get credit.**
Let $f(x) = \ln(1+2x)$.

(a) **(1pt)** Use the 2nd degree Taylor polynomial to estimate $f(1/8)$.

(b) **(2pt)** Show that the error (in absolute value) of your estimate is smaller than $\dfrac{1}{3(2)^6}$.

(c) **(1pt)** Determine whether your estimate is an overestimate or underestimate. You have to justify your answer.

**Solution.** Denote by $T_2(x)$ the 2nd degree Taylor polynomial of $f$ about $x=0$ and let $R_2(x)$ be the remainder. Compute

$$f'(x) = \frac{2}{1+2x}, \quad f''(x) = -\frac{2^2}{(1+2x)^2}, \quad f^{(3)}(x) = \frac{2^4}{(1+2x)^3}.$$

Then $f(0)=0$, $f'(0)=2$ and $f''(0)=-4$ so the Taylor polynomial is

$$T_2(x) = 2x - 2x^2.$$

Then the approximation value is $\underline{T_2\left(\frac{1}{8}\right) = \frac{1}{4} - \frac{1}{32} = \frac{7}{32}}$. Write the remainder in terms of the Lagrange remainder formula:

$$R_2(1/8) = \frac{f^{(3)}(c)}{3!}(1/8)^3,$$

for some $c$ between 0 and 1/8. Note that $f^{(3)}(c) = 2^4/(1+2c)^3$ is a decreasing function and positive for $c>0$, then

$$f^{(3)}(c) < f^{(3)}(0) = 2^4, \qquad \text{for all } 0 < c < \frac{1}{8}.$$

Use this bound for the Lagrange remainder formula above, and we get

$$\underline{0 < R_2\left(1/8\right) < \frac{2^4}{3!}\left(\frac{1}{8}\right)^3 = \frac{1}{3 \cdot 2^6}}$$

This shows that the error is less than $\frac{1}{3 \cdot 2^6}$. As the remainder $R_2(1/8)$ is positive, this is an underestimate.