Quiz 5-T

2017-11-16

Last name ....

First name ....

Student number ....

Email ....

Grade

For each computation of limits in this test, if the limit does not exist, indicate whether it diverges to $-\infty$ or $+\infty$.

1. Each part of this question is worth 1 mark.

(a) **(1pt)** Find the x-coordinates of the **local minimum** points of the function $f(x) = x^3 - 3x + 5$ defined on the whole real line.

Solution. We have $f'(x) = 3x^2 - 3 = 3(x-1)(x+1)$. Hence the critical points are 1 and -1. We also have no singular point. Using the second derivative test, we get: $f''(x) = 6x$, $f''(1) = 6 > 0$ and $f''(-1) = -6 < 0$. Hence $x = 1$ is the local minimum of $f(x)$.

(b) **(1pt)** Let $T_3(x)$ be the third degree Taylor polynomial about $x = 0$ of $g(x) = \frac{x}{1+x}$. Evaluate $T_3'(0)$.

Solution. We have $T_3'(0) = g'(0)$ so we just need to compute $g'(0)$. By direct calculations, we get

$$g'(x) = \frac{1}{(1+x)^2}$$

Hence $T_3'(0) = g'(0) = 1$.

2. You have to show all your work in order to get credit.

(a) **(2pt)** Find the x-coordinates of the **global maximum** points of $h(x) = x^5 - 5x + 5$ on $[0, 2]$.

Solution. By the Extreme Value Theorem, the candidates for the global maxima are:
+ end points: 0 and 2
+ critical points: $h'(c) = 5c^4 - 5 = 0$. Hence $c^4 = 1$ so $c = 1$ and -1. But -1 is not in the interval $[0, 2]$. So 1 is the only critical point in this case.
+ singular points: NONE
So we have three candidates for the global maximum: 0, 1 and 2. Also, $h(0) = 5; h(1) = 1$ and $h(2) = 32 - 10 + 5 = 27$. Hence the coordinate of the global maximum is $(2, 27)$.

(b) **(2pt)** Let $T_n(x)$ be the nth degree Taylor polynomial about $x = 0$ for the function $f(x) = \sin(x)$. Determine whether $T_{99}(0.1)$ gives an underestimate or overestimate of $\sin(0.1)$. Justify your answer.

Solution. $T_n(0.1)$ gives an **underestimate** of $\sin(0.1)$ when $R_n(0.1) = \sin(0.1) - T_n(0.1) > 0$. Similarly, $T_n(0.1)$ gives an **overestimate** of $\sin(0.1)$ when $R_n(0.1) = \sin(0.1) - T_n(0.1) < 0$.

By the Lagrange Remainder Theorem, we obtain $R_{99}(0.1) = \frac{f^{(100)}(c)}{100!}(0.1)^{100}$ for some c between 0 and 0.1.
We have that (note the patterns):

$$f^{(0)}(c) = f^{(4)}(c) = f^{(8)}(c) = ... = f^{(4j)}(c) = \sin(c)$$

$$f^{(1)}(c) = f^{(5)}(c) = f^{(9)}(c) = ... = f^{(4j+1)}(c) = \cos(c)$$

$$f^{(2)}(c) = f^{(6)}(c) = f^{(10)}(c) = ... = f^{(4j+2)}(c) = -\sin(c)$$

$$k^{(3)}(c) = k^{(7)}(c) = f^{(11)}(c) = ... = f^{(4j+3)}(c) = -\cos(c)$$

for $j = 0, 1, 2, ...$
Thus $f^{(100)}(c) = f^{(4*25)}(c) = \sin c > 0$ since c is between 0 and 0.1 (which means c is in the first quadrant). Hence $R_{99}(0.1) > 0$ which means that $T_{99}(0.1)$ gives an underestimate of $\sin(0.1)$

3. You have to show all your work in order to get credit.
Let $\ell(x) = x^4 + 6x^2 + 4x + 2$.

(a) **(2pt)** Prove that $\ell(x)$ has at least one **critical point**.

(b) **(2pt)** Prove that $\ell(x)$ has at most one **critical point**.

Solution. $\ell(x) = x^4 + 6x^2 + 4x + 2$ has exactly one critical point means that $f(x) = \ell'(x) = 4x^3 + 12x + 4 = 0$ has exactly one root.
Step 1: AT LEAST ONE root using IVT.
We have that the function $f(x)$ is continuous and differentiable everywhere. Also, $f(0) = 4$ and $f(-1) = -12$. So by the IVT, $f(x) = 0$ has at least one root c in $[-1, 0]$.

Step 2: AT MOST ONE root using MVT.
Suppose that there is another root d (that is $f(d) = 0$). Then by MVT (or Rolle's Theorem), there is some z between c and d such that $f'(z) = \frac{f(d)-f(c)}{d-c} = 0$. Compute $f'(x) = 12x^2 + 12$. Hence $12z^2 + 12 = 0$, that is $z^2 = -1$ which is impossible. So there is no other real root of $f(x)$.

Quiz 5-T-p

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For each computation of limits in this test, if the limit does not exist, indicate whether it diverges to $-\infty$ or $+\infty$.

1. Each part of this question is worth 1 mark.

(a) **(1pt)** Find the x-coordinates of the **local maximum** points of the function $f(x) = x^3 - 12x - 1$ defined on the whole real line.

Solution. We have $f'(x) = 3x^2 - 12 = 3(x^2 - 4) = 3(x-2)(x+2)$. Hence the critical points are 2 and -2. We also have no singular point. Using the second derivative test, we get: $f''(x) = 6x$, $f''(2) = 12 > 0$ and $f''(-2) = -12 < 0$. Hence $x = -2$ is the x-coordinate of the local maximum of $f(x)$.

(b) **(1pt)** Let $T_3(x)$ be the third degree Taylor polynomial about $x = 1$ of $g(x) = x^2 e^x$. Evaluate $T_3'(1)$.

Solution. We have $T_3'(1) = g'(1)$ so we just need to compute $g'(1)$. By direct calculations, we get

$$g'(x) = 2xe^x + x^2 e^x$$

Hence $T_3'(1) = g'(1) = 3e$.

2. You have to show all your work in order to get credit.

(a) **(2pt)** Find the x-coordinates of the **global minimum** points of $h(x) = 3x^4 - 8x^3 + 6x^2 + 1$ on $[-1, 1]$.

Solution. By the Extreme Value Theorem, the candidates for the global minima are:
+ end points: -1 and 1
+ critical points: $h'(c) = 12c^3 - 24c^2 + 12c = 12c(c-1)^2 = 0$. Hence $c = 0$ and $c = 1$ (already an endpoint).
+ singular points: NONE
We have three candidates for the global maximum: -1, 0 and 1. Compute, $h(-1) = 18$, $h(0) = 1$ and $h(1) = 2$. The x-coordinate of the global minimum is $x = 0$.

(b) **(2pt)** Let $T_n(x)$ be the nth degree Taylor polynomial about $x = 0$ for the function $f(x) = \sin(x)$. Determine whether $T_{101}(0.1)$ gives an underestimate or overestimate of $\sin(0.1)$. Justify your answer.

Solution. $T_n(0.1)$ gives an **underestimate** of $\sin(0.1)$ when

$$R_n(0.1) = \sin(0.1) - T_n(0.1) > 0$$

Similarly, $T_n(0.1)$ gives an **overestimate** of $\sin(0.1)$ when

$$R_n(0.1) = \sin(0.1) - T_n(0.1) < 0$$

By the Lagrange Remainder Theorem, we obtain

$$R_{101}(0.1) = \frac{f^{(102)}(c)}{102!}(0.1)^{102}$$

for some c between 0 and 0.1. Compute:

$$f^{(0)}(c) = f^{(4)}(c) = f^{(8)}(c) = ... = f^{(4j)}(c) = \sin(c)$$
$$f^{(1)}(c) = f^{(5)}(c) = f^{(9)}(c) = ... = f^{(4j+1)}(c) = \cos(c)$$
$$f^{(2)}(c) = f^{(6)}(c) = f^{(10)}(c) = ... = f^{(4j+2)}(c) = -\sin(c)$$
$$f^{(3)}(c) = f^{(7)}(c) = f^{(11)}(c) = ... = f^{(4j+3)}(c) = -\cos(c)$$

for $j = 0, 1, 2, ...$
Thus $f^{(102)}(c) = f^{(4*25+2)}(c) = -\sin c < 0$ since c is between 0 and 0.1 (which means c is in the first quadrant). Hence $R_{101}(0.1) < 0$ which means that $T_{101}(0.1)$ gives an overestimate of $\sin(0.1)$.

3. You have to show all your work in order to get credit.
Let $\ell(x) = x^6 + 4x^2 + x + 2$.

(a) **(2pt)** Prove that $\ell(x)$ has at least one **critical point**.

(b) **(2pt)** Prove that $\ell(x)$ has at most one **critical point**.

Solution. $\ell(x) = x^6 + 4x^2 + x + 2$ has exactly one critical point means that $f(x) = \ell'(x) = 6x^5 + 8x + 1 = 0$ has exactly one root.
Step 1: AT LEAST ONE root using IVT.
We have that the function $f(x)$ is continuous and differentiable everywhere. Also, $f(0) = 1$ and $f(-1) = -13$. So by the IVT, $f(x) = 0$ has at least one root c in $[-1, 0]$.

Step 2: AT MOST ONE root using MVT.
Suppose that there is another root d (that is $f(d) = 0$). Then by MVT (or Rolle's Theorem), there is some z between c and d such that $f'(z) = \frac{f(d)-f(c)}{d-c} = 0$. Compute $f'(x) = 30x^4 + 8$. Since $x^4 \geq 0$, $f'(x) > 0$ for all x. There is no other real root of $f(x)$.