Quiz 1

2017-09-22

Last name ....

First name ....

Student number ....

Email ....

Grade

For each computation of limits in this test, if the limit does not exist, indicate whether it diverges to $-\infty$ or $+\infty$.

1. Each part of this question is worth 1 mark.

(a) **(1 pt)** Compute

$$\lim_{x \to -2} \sqrt{1 - x^3}$$

Solution.

$$\lim_{x \to -2} \sqrt{1 - x^3} = \sqrt{1 - (-2)^3} = \sqrt{9} = 3.$$

(b) **(1 pt)** Compute

$$\lim_{x \to +\infty} \frac{5x^2 + x - 6}{3x^2 - 7x + 2}$$

Solution. We divide by x^2 both numerator and denominator and obtain:

$$\lim_{x \to +\infty} \frac{\frac{5x^2+x-6}{x^2}}{\frac{3x^2-7x+2}{x^2}} = \lim_{x \to +\infty} \frac{5 + \frac{1}{x} - \frac{6}{x}}{3 - \frac{7}{x} + \frac{2}{x^2}} = \frac{5}{3}.$$

2. Each part of this question is worth 2 marks. **You have to show all your work in order to get credit.**

(a) **(2 pt)** Compute

$$\lim_{x\to -2} \frac{|x+2|}{x^2-4}$$

Solution. We need to compute left and right limits and then compare our answers. So,

$$\lim_{x\to -2^-} \frac{|x+2|}{x^2-4} = \lim_{x\to -2^-} \frac{-(x+2)}{(x-2)(x+2)} = \lim_{x\to -2^-} \frac{-1}{x-2} = \frac{1}{4}$$

and

$$\lim_{x\to -2^+} \frac{|x+2|}{x^2-4} = \lim_{x\to -2^+} \frac{x+2}{(x-2)(x+2)} = \lim_{x\to -2^+} \frac{1}{x-2} = -\frac{1}{4}.$$

Since the lateral limits don't match, we conclude that

$$\lim_{x\to -2} \frac{|x+2|}{x^2-4} \text{ does not exist.}$$

(b) **(2 pt)** Compute

$$\lim_{x\to 1} \frac{\sqrt{x+3}-2}{x-1}$$

Solution. We apply the conjugate:

$$\lim_{x\to 1} \frac{\sqrt{x+3}-2}{x-1} = \lim_{x\to 1} \frac{(\sqrt{x+3}-2)(\sqrt{x+3}+2)}{(x-1)(\sqrt{x+3}+2)}$$

and so, we obtain

$$\lim_{x\to 1} \frac{x+3-4}{(x-1)(\sqrt{x+3}+2)} = \lim_{x\to 1} \frac{x-1}{(x-1)(\sqrt{x+3}+2)} = \lim_{x\to 1} \frac{1}{\sqrt{x+3}+2} = \frac{1}{\sqrt{1+3}+2} = \frac{1}{4}$$

3. This question is worth 4 marks. **You have to show all your work in order to get credit.**

Find the real number a such that $\lim_{x\to 1} f(x)$ exists for the function

$$f(x) = \begin{cases} (x-1)\cdot \sin\left(\frac{1}{x-1}\right) & \text{if} \quad x > 1 \\ x^2 + ax + 1 & \text{if} \quad x < 1. \end{cases}$$

Solution. First of all, the limit exists if

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x).$$

On the other hand, $\lim_{x\to 1^-} x^2 + ax + 1 = a + 2$ and so, all we need to ensure now is that

$$\lim_{x\to 1^+} (x-1)\sin\left(\frac{1}{x-1}\right) = a + 2.$$

We compute $\lim_{x\to 1^+}(x-1)\sin\left(\frac{1}{x-1}\right) = 0$ using Squeeze Theorem. Indeed, as $x \to 1^+$, we have that

$$-1 \le \sin\left(\frac{1}{x-1}\right) \le 1$$

and after multiplying by $x - 1$ (which is positive, when $x \to 1^+$), we get

$$-(x-1) \le (x-1)\cdot \sin\left(\frac{1}{x-1}\right) \le (x-1).$$

Letting $x \to 1^+$, we get that both the left-most and the right-most functions ($x-1$ and $1-x$) converge to 0, and this forces (by Squeeze Theorem) that also $\lim_{x\to 1^+}(x-1)\cdot \sin\left(\frac{1}{x-1}\right) = 0$.

In conclusion, the value of a which makes $\lim_{x\to 1} f(x)$ exist is $a = -2$ since then $a + 2 = 0$.

1. Each part of this question is worth 1 mark.

(a) **(1 pt)** Compute

$$\lim_{x\to 5} \sqrt[3]{x^2 - 17}$$

Solution.

$$\lim_{x\to 5} \sqrt[3]{x^2 - 17} = \lim_{x\to 5} \sqrt[3]{5^2 - 17} = \sqrt[3]{8} = 2.$$

(b) **(1 pt)** Compute

$$\lim_{x\to -\infty} \frac{-5x^2 + x - 2}{-3x^2 - 7x + 3}$$

Solution. We divide by x^2 both numerator and denominator and obtain:

$$\lim_{x\to -\infty} \frac{\frac{-5x^2+x-2}{x^2}}{\frac{-3x^2-7x+3}{x^2}} = \lim_{x\to -\infty} \frac{-5 + \frac{1}{x} - \frac{2}{x^2}}{-3 - \frac{7}{x} + \frac{3}{x^2}} = \frac{-5}{-3} = \frac{5}{3}.$$

2. Each part of this question is worth 2 marks. **You have to show all your work in order to get credit.**

(a) **(2 pt)** Compute

$$\lim_{x\to 2} \frac{|x-2|}{x^2-3x+2}$$

Solution. We need to compute left and right limits and then compare our answers. So,

$$\lim_{x\to 2^-} \frac{|x-2|}{x^2-3x+2} = \lim_{x\to 2^-} \frac{-(x-2)}{(x-2)(x-1)} = \frac{-1}{x-1} = -1$$

and

$$\lim_{x\to 2^+} \frac{|x-2|}{x^2-3x+2} = \lim_{x\to 2^+} \frac{x-2}{(x-2)(x-1)} = \frac{1}{x-1} = 1.$$

Since the lateral limits don't match, we conclude that

$$\lim_{x\to 2} \frac{|x-2|}{x^2-3x+2} \text{ does not exist.}$$

(b) **(2 pt)** Compute

$$\lim_{x\to 1} \frac{\sqrt{x+8}-3}{1-x}$$

Solution. We apply the conjugate:

$$\lim_{x\to 1} \frac{\sqrt{x+8}-3}{1-x} = \lim_{x\to 1} \frac{(\sqrt{x+8}-3)(\sqrt{x+8}+3)}{(1-x)(\sqrt{x+8}+3)}$$

and so, we obtain

$$\lim_{x\to 1} \frac{x+8-9}{(1-x)(\sqrt{x+8}+3)} = \lim_{x\to 1} \frac{x-1}{(1-x)(\sqrt{x+8}+3)} = \lim_{x\to 1} \frac{-1}{\sqrt{x+8}+3} = \frac{-1}{\sqrt{1+8}+3} = \frac{-1}{6}$$

3. This question is worth 4 marks. **You have to show all your work in order to get credit.**

Find the real number a such that $\lim_{x \to 1} f(x)$ exists for the function

$$f(x) = \begin{cases} (x-1) \cdot \cos\left(\frac{1}{x-1}\right) & \text{if} \quad x > 1 \\ x^2 - 1 + a & \text{if} \quad x < 1. \end{cases}$$

Solution. First of all, the limit exists if

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x).$$

On the other hand, $\lim_{x \to 1^-} x^2 - 1 + a = a$ and so, all we need to ensure now is that

$$\lim_{x \to 1^-} (x-1) \cos\left(\frac{1}{x-1}\right) = a.$$

We compute $\lim_{x \to 1^+} (x-1) \cos\left(\frac{1}{x-1}\right) = 0$ using Squeeze Theorem. Indeed, as $x \to 1^+$, we have that

$$-1 \leq \cos\left(\frac{1}{x-1}\right) \leq 1$$

and after multiplying by $x - 1$ (which is positive, when $x \to 1^+$), we get

$$-(x-1) \leq (x-1) \cdot \cos\left(\frac{1}{x-1}\right) \leq (x-1).$$

Letting $x \to 1^+$, we get that both the left-most and the right-most functions ($x-1$ and $1-x$) converge to 0, and this forces (by Squeeze Theorem) that also $\lim_{x \to 1^+} (x-1) \cdot \cos\left(\frac{1}{x-1}\right) = 0$.

In conclusion, the value of a which makes $\lim_{x \to 1} f(x)$ exist is $a = 0$.

1. Each part of this question is worth 1 mark.

(a) **(1 pt)** Compute

$$\lim_{x \to 2} \frac{1 - x^2}{\sqrt{1 + x^3}}$$

Solution.

$$\lim_{x \to 2} \frac{1 - x^2}{\sqrt{1 + x^3}} = \lim_{x \to 2} \frac{1 - 2^2}{\sqrt{1 + 2^3}} = -1$$

(b) **(1 pt)** Compute

$$\lim_{x \to +\infty} \frac{1 - 5x + 2x^3}{2 + 4x^2 - x^3}$$

Solution. We divide by x^3 both numerator and denominator and obtain:

$$\lim_{x \to +\infty} \frac{\frac{1-5x+2x^3}{x^3}}{\frac{2+4x^2-x^3}{x^3}} = \lim_{x \to +\infty} \frac{\frac{1}{x^3} - \frac{5}{x^2} + 2}{\frac{2}{x^3} + \frac{4}{x} - 1} = -2.$$

2. Each part of this question is worth 2 marks. **You have to show all your work in order to get credit.**

(a) **(2 pt)** Compute

$$\lim_{x\to -1} \frac{x^2+3x+2}{|x+1|}$$

Solution. We need to compute left and right limits and then compare our answers. So,

$$\lim_{x\to -1^-} \frac{x^2+3x+2}{|x+1|} = \lim_{x\to -1^-} \frac{(x+1)(x+2)}{-(x+1)} = \lim_{x\to -1^-} -(x+2) = -1$$

and

$$\lim_{x\to -1^+} \frac{x^2+3x+2}{|x+1|} = \lim_{x\to -1^+} \frac{(x+1)(x+2)}{(x+1)} = \lim_{x\to -1^+} (x+2) = 1.$$

Since the lateral limits don't match, we conclude that

$$\lim_{x\to -1} \frac{x^2+3x+2}{|x+1|} \text{ does not exist.}$$

(b) **(2 pt)** Compute

$$\lim_{x\to -2} \frac{1-\sqrt{x+3}}{x+2}$$

Solution. We apply the conjugate:

$$\lim_{x\to -2} \frac{1-\sqrt{x+3}}{x+2} = \lim_{x\to -2} \frac{(1-\sqrt{x+3})(1+\sqrt{x+3})}{(x+2)(1+\sqrt{x+3})}$$

and so, we obtain

$$\lim_{x\to -2} \frac{1-(x+3)}{(x+2)(1+\sqrt{x+3})} = \lim_{x\to -2} \frac{-x-2}{(x+2)(1+\sqrt{x+3})} = \lim_{x\to -2} \frac{-1}{1+\sqrt{x+3}} = -\frac{1}{2}$$

3. This question is worth 4 marks. **You have to show all your work in order to get credit.**

Find the real number a such that $\lim_{x\to 0} f(x)$ exists for the function

$$f(x) = \begin{cases} a(x+1)^2 - 1 & \text{if} \quad x > 0 \\ 1 + x^2 \cos\left(\frac{1}{x}\right) & \text{if} \quad x < 0. \end{cases}$$

Solution. First of all, the limit exists if

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x).$$

On the other hand, $\lim_{x\to 0^+} a(x+1)^2 - 1 = a - 1$ and so, all we need to ensure now is that

$$\lim_{x\to 0^-} \left(1 + x^2 \cos\left(\frac{1}{x}\right)\right) = a - 1.$$

We compute $\lim_{x\to 0^-} x^2 \cos\left(\frac{1}{x}\right) = 0$ using Squeeze Theorem. Indeed, as $x \to 0^-$, we have that

$$-1 \leq \cos\left(\frac{1}{x}\right) \leq 1$$

and after multiplying by x^2, we get

$$-x^2 \leq x^2 \cos\left(\frac{1}{x}\right) \leq x^2.$$

Letting $x \to 0^-$, we get that both the left-most and the right-most functions ($-x^2$ and x^2) converge to 0, and this forces (by Squeeze Theorem) that also $\lim_{x\to 0^-} x^2 \cos\left(\frac{1}{x}\right) = 0$. Therefore

$$\lim_{x\to 0^-} \left(1 + x^2 \cos\left(\frac{1}{x}\right)\right) = 1$$

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In conclusion, the value of a which makes $\lim_{x\to 0} f(x)$ exist is $a = 2$ since then $a - 1 = 1$.