# Lecture 5. Complex Numbers and Euler's Formula

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**Main purpose:**

To introduce some basic knowledge of complex numbers to students so that they are prepared to handle complex-valued roots when solving the characteristic polynomials for eigenvalues of a matrix.

**Eg:** In high school, students learned that the roots of a quadratic equation $ax^2 + bx + c = 0$ $(a \neq 0)$ are given by

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

where the sign of the discriminant $\Delta = b^2 - 4ac$ determines the following three outcomes

$$\text{If} \begin{cases} \Delta > 0, & \text{2 real roots;} \\ \Delta = 0, & \text{1 (repeated) real root;} \\ \Delta < 0, & \text{no real root.} \end{cases}$$

When complex-valued roots are allowed as in the case when solving eigenvalues, however, a polynomial of degree $n$ always has $n$ roots (Gauss' Fundamental Theorem of Algebra), of which some or all of them can be identical (repeated). Thus, a quadratic equation always has 2 roots irrespective of the sign of $\Delta$.

## 5.1 Definitions and basic concepts

**The imaginary number $i$:**

$$i \equiv \sqrt{-1} \qquad \Longleftrightarrow \qquad i^2 = -1. \tag{1}$$

Every *imaginary number* is expressed as a real-valued multiple of $i$:

$$\sqrt{-9} = \sqrt{9}\sqrt{-1} = \sqrt{9}i = 3i.$$

**A complex number:**

$$z = a + bi, \tag{2}$$

where $a,\ b$ are real, is the sum of a real and an imaginary number.

**The real part of z:** $Re\{z\} = a$ is a real number.

**The imaginary part of z:** $Im\{z\} = b$ is a also a real number.

**A complex number** represents a point $(a,\ b)$ in a 2D space, called the complex plane. Thus, it can be regarded as a 2D vector expressed in form of a number/scalar. Therefore, there exists a one-to-one correspondence between a 2D vectors and a complex numbers.

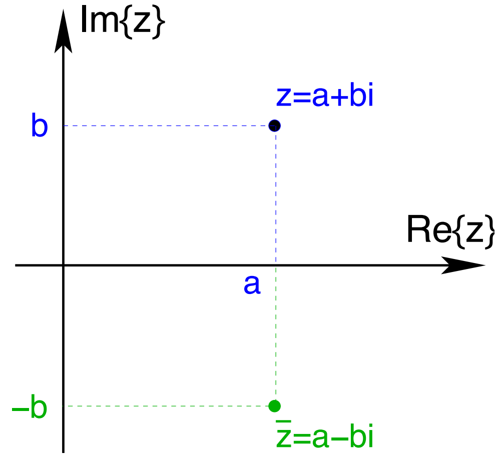


Figure 1: A complex number $z$ and its conjugate $\bar{z}$ in complex space. Horizontal axis contains all real numbers, vertical axis contains all imaginary numbers.

**The complex conjugate:**

$$\bar{z} = a - bi,$$

which is obtained by reversing the sign of $Im\{z\}$.

Notice that:

$$Re\{z\} = \frac{z + \bar{z}}{2} = a, \qquad Im\{z\} = \frac{z - \bar{z}}{2i} = b.$$

Therefore, both $Re\{z\}$ and $Im\{z\}$ are linear combinations of $z$ and $\bar{z}$.

## 5.2 Basic computations between complex numbers

### Addition/subtraction:

If $z_1 = a_1 + b_1 i$, $z_2 = a_2 + b_2 i$ $(a_1,\ a_2,\ b_1,\ b_2 \in \mathbb{R})$, then

$$z_1 \pm z_2 = (a_1 + b_1 i) \pm (a_2 + b_2 i) = (a_1 \pm a_2) + (b_1 \pm b_2)i.$$

Or, real parts plus/minus real parts, imaginary parts plus/minus imaginary parts.

### Multiplication by a real scalar $\alpha$:

$$\alpha z_1 = \alpha a_1 + \alpha b_1 i.$$

### Multiplication between complex numbers:

$$z_1 z_2 = (a_1 + b_1 i)(a_2 + b_2 i) = a_1 a_2 + a_1 b_2 i + a_2 b_1 i + b_1 b_2 i^2 = (a_1 a_2 - b_1 b_2) + (a_1 b_2 + a_2 b_1)i.$$

All rules are identical to those of multiplication between real numbers, just remember that $i^2 = -1$.

**Length/magnitude** of a complex number $z = a + bi$

$$|z| = \sqrt{z\bar{z}} = \sqrt{(a+bi)(a-bi)} = \sqrt{a^2+b^2},$$

which is identical to the length of a 2D vector $(a,\ b)$.

**Division between complex numbers:**

$$\frac{z_1}{z_2} = \frac{z_1\bar{z}_2}{z_2\bar{z}_2} = \frac{(a_1+b_1i)(a_2-b_2i)}{|z_2|^2} = \frac{(a_1a_2+b_1b_2)+(a_2b_1-a_1b_2)i}{a_2^2+b_2^2}.$$

**Eg 5.2.1** Given that $z_1 = 3 + 4i$, $z_2 = 1 - 2i$, calculate

1. $z_1 - z_2$;
2. $\frac{z_1}{2}$;
3. $|z_1|$;
4. $\frac{z_2}{z_1}$.

**Ans:**

1. $z_1 - z_2 = (3-1) + (4-(-2))i = 2 + 6i$;
2. $\frac{z_1}{2} = \frac{3}{2} + \frac{4}{2}i = 1.5 + 2i$;
3. $|z_1| = \sqrt{z_1\bar{z}_1} = \sqrt{3^2+4^2} = \sqrt{25} = 5$;
4. $\frac{z_2}{z_1} = \frac{z_2\bar{z}_1}{z_1\bar{z}_1} = \frac{(1-2i)(3-4i)}{5^2} = \frac{-5-10i}{25} = -\frac{1}{5} - \frac{2}{5}i$.

## 5.3 Complex-valued exponential and Euler's formula

**Euler's formula:**

$$e^{it} = \cos t + i \sin t. \tag{3}$$

Based on this formula and that $e^{-it} = \cos(-t) + i\sin(-t) = \cos t - i \sin t$:

$$\cos t = \frac{e^{it} + e^{-it}}{2}, \qquad \sin t = \frac{e^{it} - e^{-it}}{2i}. \tag{4}$$

**Why?** Here is a way to gain insight into this formula.

Recall the Taylor series of $e^t$:

$$e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!}.$$

Suppose that this series holds when the exponent is imaginary.

$$e^{it} = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} = \sum_{n \text{ even}}^{\infty} \frac{(it)^n}{n!} + \sum_{n \text{ odd}}^{\infty} \frac{(it)^n}{n!} = \sum_{m=0}^{\infty} \frac{(it)^{2m}}{(2m)!} + \sum_{m=0}^{\infty} \frac{(it)^{2m+1}}{(2m+1)!}$$

$$= \sum_{m=0}^{\infty} \frac{i^{2m} t^{2m}}{(2m)!} + \sum_{m=0}^{\infty} \frac{i^{2m+1} t^{2m+1}}{(2m+1)!} = \sum_{m=0}^{\infty} \frac{(i^2)^m t^{2m}}{(2m)!} + \sum_{m=0}^{\infty} \frac{i(i^2)^m t^{2m+1}}{(2m+1)!}$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m t^{2m}}{(2m)!} + i \sum_{m=0}^{\infty} \frac{(-1)^m t^{2m+1}}{(2m+1)!} = \cos t + i \sin t.$$

**Remarks:**

- Sine and cosine functions are actually linear combinations of exponential functions with imaginary exponents.
- Similarly, hyperbolic sine and cosine functions are linear combinations of exponential functions with real exponents.

$$\sinh(t) = \frac{e^t - e^{-t}}{2}, \qquad \cosh(t) = \frac{e^t + e^{-t}}{2}.$$

## 5.4 Polar representation of complex numbers

For any complex number $z = x + iy$ ($\neq 0$), its length and angle w.r.t. the horizontal axis are both uniquely defined.

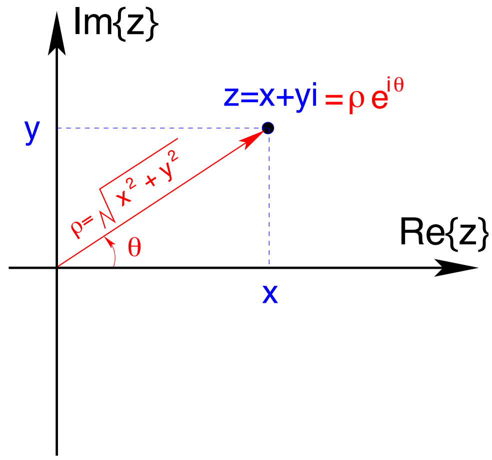


Figure 2: A complex number $z = x + iy$ can be expressed in the polar form $z = \rho e^{i\theta}$, where $\rho = \sqrt{x^2 + y^2}$ is its length and $\theta$ the angle between the vector and the horizontal axis. The fact $x = \rho\cos\theta,\ y = \rho\sin\theta$ are consistent with Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$.

One can convert a complex number from one form to the other by using the Euler's formula:

$$z = x + iy \qquad \Leftrightarrow \qquad z = \rho e^{i\theta}, \qquad \text{where}$$
$$x = \rho\cos\theta, \quad y = \rho\sin\theta; \qquad \rho = \sqrt{x^2 + y^2}, \quad \theta = \tan^{-1}\frac{y}{x};$$

where we often restrict $0 \leq \theta \leq 2\pi$ or $-\pi \leq \theta \leq \pi$. Otherwise, the conversion from Cartesian to polar coordinates is not unique, $\theta$ can differ by an integer multiple of $2\pi$.

**Eg 5.4.1** Convert the following complex numbers from one form to the other.

1. $z = 3i$;
2. $z = 1$;
3. $z = 1 + i\sqrt{3}$;
4. $z = -2 - 2i$;
5. $z = e^{-i\frac{\pi}{6}}$;
6. $z = 5e^{i\frac{\pi}{4}}$;
7. $z = -5e^{-i\frac{\pi}{3}}$.

**Ans:**

1. $z = 3i = 3e^{i\frac{\pi}{2}}$;
2. $z = 1 = e^{i0} = 1$, (for a positive real number, there is no change!);
3. $z = 1 + i\sqrt{3} = \sqrt{1^2 + (\sqrt{3})^2} e^{i\tan^{-1}\frac{\sqrt{3}}{1}} = 2e^{i\frac{\pi}{3}}$;
4. $z = -2 - 2i = \sqrt{(-2)^2 + (-2)^2} e^{i\tan^{-1}(\frac{-2}{-2})} = 2\sqrt{2}e^{i\frac{5\pi}{4}} = 2\sqrt{2}e^{-i\frac{3\pi}{4}}$;
5. $z = e^{-i\frac{\pi}{6}} = \cos(-\frac{\pi}{6}) + i\sin(-\frac{\pi}{6}) = \frac{\sqrt{3}}{2} - i\frac{1}{2}$;
6. $z = 6e^{i\frac{\pi}{4}} = 6\cos\frac{\pi}{4} + i6\sin\frac{\pi}{4} = 3\sqrt{2} + 3\sqrt{2}i = 3\sqrt{2}(1+i)$;
7. $z = -4e^{-i\frac{\pi}{3}} = (-4)\left[\cos(-\frac{\pi}{3}) + i\sin(-\frac{\pi}{3})\right] = (-4)\left[\frac{1}{2} - i\frac{\sqrt{2}}{2}\right] = -2 + 2\sqrt{2}i$.

**Remark:** It is important to know that the collection of all complex numbers of the form $z = e^{i\theta}$ form a circle of radius one (unit circle) in the complex plane centered at the origin. In other words, the equation for a unit circle centered at the origin in complex plane is $z = e^{i\theta}$ (see figure).

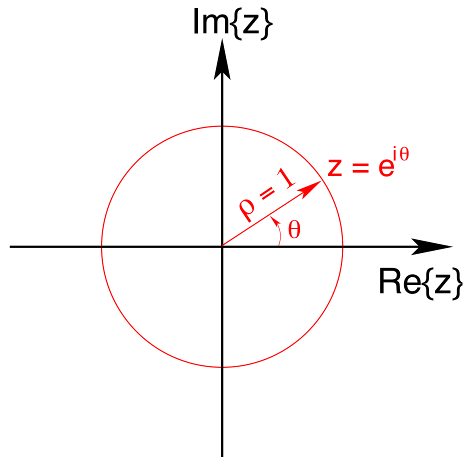


Figure 3: The collection of all complex numbers of the form $z = e^{i\theta}$ form a unit circle centered at the origin in the complex plane.

**Remark:** Rotation of a vector represented by a complex number $z = \rho e^{i\theta}$ counter-clockwise by angle $\phi$ is achieved by multiplying $e^{i\phi}$ to it:

$$e^{i\phi} z = e^{i\phi} \rho e^{i\theta} = \rho e^{i(\theta+\phi)}.$$

**Remark:** The product between $z_1 = \rho_1 e^{i\theta_1}$ and $z_2 = \rho_2 e^{i\theta_2}$ yields

$$z_1 z_2 = \rho_1 e^{i\theta_1} \rho_2 e^{i\theta_2} = (\rho_1 \rho_2) e^{i(\theta_1+\theta_2)}$$

which is a vector of length $\rho_1 \rho_2$ and an angle $\theta_1 + \theta_2$.

**Eg 5.4.2** Find all roots of $\sqrt[3]{1}$, complex and real.

**Ans:**

$$1 = e^0 = e^{i(0)} = e^{(2\pi n)i}, \qquad \text{for any integer n.}$$

Therefore,

$$\sqrt[3]{1} = 1^{\frac{1}{3}} = \left(e^{(2\pi n)i}\right)^{\frac{1}{3}} = e^{\frac{2\pi n}{3}i}, \qquad \text{for any integer n.}$$

Often, the angle $\theta$ for a complex number expressed in form of $e^{\theta i}$ is restricted in the range $0 \leq \theta < 2\pi$.

If so, $\sqrt[3]{1}$ has only three roots in this range: $\sqrt[3]{1} = 1, \quad e^{\frac{2\pi}{3}i}, \quad e^{\frac{4\pi}{3}i}$.

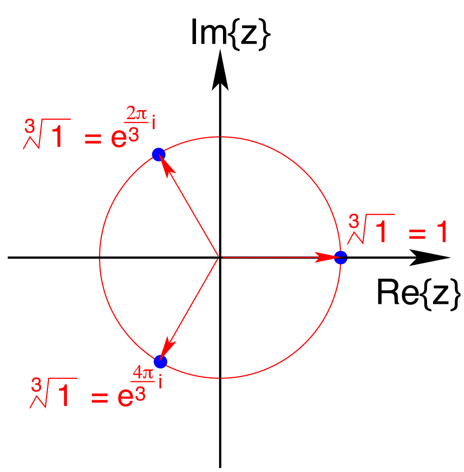


Figure 4: The cubic roots of number 1 in complex plane.

## 5.5 Polynomials of degree n must have n roots!

**Eg 5.5.1** Find all roots of $z^2 + 2z + 10 = 0$.

**Ans:** Notice that

$$z^2 + 2z + 10 = z^2 + 2z + 1 + 9 = (z+1)^2 + 9 = 0.$$

There is no real root! But there are two complex-valued roots forming a pair of complex conjugates.

$$(z+1)^2+9=0 \quad \Rightarrow \quad (z+1)^2 = -9 \quad \Rightarrow \quad z+1 = \pm\sqrt{-9} \quad \Rightarrow \quad z = -1 \pm 3i.$$

**Final remarks:**

(a) Any polynomial of degree $n$ can always be factored into the product of $n$ terms in which $z_i$, $(i = 1, \ldots, n)$ are the $n$ roots.

$$P_n(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 = a_n (z - z_n)(z - z_{n-1}) \cdots (z - z_1).$$

(b) Complex roots of a polynomial always occur as a pair of complex conjugates: $z_{\pm} = a \pm bi$.